

Elliptic double affine Hecke algebras

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Quantum structures in algebra and geometry
Northeastern University, August 30, 2019

*Partially supported by NSF

Motivation 1

A while back, I was able to use a construction of Okounkov (the “binomial formula” for Koornwinder polynomials (the 5-parameter version of Macdonald polynomials for type C) in reverse to give new proofs of the Macdonald conjectures for such polynomials, and then generalize this to the elliptic level, giving biorthogonal families of C_n -invariant rational functions on E^n degenerating to both Koornwinder and (GL_n -type) Macdonald polynomials, and satisfying analogues of Macdonald’s conjectures. The original proof for Koornwinder and Macdonald polynomials used double affine Hecke algebras (DAHAs), suggesting that there should be an elliptic analogue of these algebras. (I originally was hoping such an extension would let me settle some additional conjectures about elliptic biorthogonal functions, but those have since been settled by other means.)

Motivation 2

Back in summer 2011, Etingof asked me* whether I could construct a flat noncommutative deformation of $\text{Sym}^n(\mathbb{P}^2 \setminus C)$ for C a smooth cubic curve. Though \mathbb{P}^2 was not obvious, it was clear that a recently developed interpretation of deformations of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus C$ (with C smooth biquadratic) should carry over to symmetric powers, and further work that fall (on sabbatical at MIT) extended to arbitrary blowups of Hirzebruch surfaces with smooth anticanonical curves. (\mathbb{P}^2 is actually the hardest case!)

Main difficulty: how to prove these are flat? Starting with a Hirzebruch surface lets us think of everything as sheaves on \mathbb{P}^n , so we just need to show that *those* are flat. Pavel suggested this might follow from an interpretation as a spherical algebra of a DAHA-like object.

*via Okounkov!

Further Motivation

A deformation of $\mathrm{Sym}^n(X \setminus C)$ is very close to a deformation of $\mathrm{Hilb}^n(X \setminus C)$, which in turn is birational to moduli spaces of sheaves on X with 1-dimensional support disjoint from C . For each such moduli space, there's an irreducible hypersurface in $\mathrm{Hilb}^n(X \setminus C)$ such that removing that hypersurface and some other codimension ≥ 2 stuff then adding back different codimension ≥ 2 stuff gives the moduli space. Geometric Langlands involves derived equivalences between noncommutative deformations of such moduli spaces (with highly singular anticanonical curve)...

In particular, removing the given hypersurface essentially makes $\text{Hilb}^n(X \setminus C)$ an abelian fibration, so should have derived autoequivalences, and those should extend to the deformations (acting nontrivially on the deformation parameters). (This works for $n = 1$, giving discrete versions of geometric Langlands for GL_2 on \mathbb{P}^1 minus 4 points.)

Main DAHA result: Given any abelian variety with a suitable action of a Coxeter group (not necessarily finite or affine!), suitable “twisting” data, and any choice of an effective divisor on each orbit of “coroot” curves (see below), there is an associated Hecke algebra; in the affine case, a special case degenerates to the usual double affine Hecke algebra. Under very mild additional conditions, the spherical algebra associated to any finite parabolic subgroup is still flat.

Main deformations of symmetric powers result: In the affine $C^\vee C_n$ case, if we assign a degree 1 divisor (t) to the D_n roots and a degree m divisor to the orbit of α_0 , then the family of spherical algebras is a two-parameter deformation of the family of m -point blowups of Hirzebruch surfaces with smooth anticanonical curve. (Also a conjectural way to extend to a deformation of $\text{Hilb}^n(X)$.)

Actions of Coxeter groups on abelian varieties

Given an abelian variety A , a *reflection* is an automorphism of order 2 that fixes a hypersurface pointwise. (Could also allow complex reflections, but I don't have a Hecke algebra theory.) A reflection r gives rise to a pair of (elliptic) curves: the “root curve” $\text{im}(1 - r)$ and the “coroot curve” $\text{coker}(1 + r)$. (Note that taking the dual of A swaps these)

An action of a Coxeter group by reflections on A is an action such that each s_i acts as a reflection. Note: can extend these notions to A -torsors X without too much difficulty.

Prototypical such action comes from an “elliptic root datum”: an elliptic curve E_i for each simple reflection, homomorphisms $\mu_{ij} : E_i \rightarrow E_j$, and positive integers r_i such that (a) $r_i \mu_{ji} = r_j \mu_{ij}^\vee$, (b) $\mu_{ij} \mu_{ji} = [4 \cos(\pi/m_{ij})^2]$, and (c) any composition around a loop in the Dynkin diagram is multiplication by a positive integer.

This gives rise to a faithful action of W on $\prod_i E_i$ by

$$s_i : z_i \mapsto -z_i + \sum_{j \neq i} \mu_{ji}(z_j).$$

(Why faithful? Replacing any product of μ_{ji} by the positive square root of its degree and rescaling by $\sqrt{r_i}$ gives the standard reflection representation! This also answers “why (c)” . . .)

May also consider action on $X = \prod_i E_i \times B$ for general abelian B , or image of such under an isogeny. Each orbit of roots gives isomorphic root curves, and fixing one gives a bijection between the orbit and corresponding “root maps”. Positivity around cycles lets us distinguish positive and negative root maps.

We’d actually prefer the coroot maps to be well-behaved, so take the dual. In the affine case, such an action of “coroot type” can be viewed as a 1-parameter (q) family of actions by affine reflections (i.e., like reflections but not fixing the identity); note however that for q torsion, this action is not faithful.

Master Hecke algebras (finite case)

For W finite, there's a natural sheaf of algebras on X/W , namely $\mathcal{E}nd(\pi_{W*}\mathcal{O}_X) =: \mathcal{H}_W(X)$. (Caution: $\pi_{W*}\mathcal{O}_X$ may not be a vector bundle!) Over $k(X/W)$, this may be identified with $k(X)[W]$, and thus $\mathcal{E}nd(\pi_{W*}\mathcal{O}_X) \subset k(X)[W]$, as those meromorphic operators that take (locally) holomorphic functions to (locally) holomorphic functions.

This makes sense for arbitrary groups: find that poles only occur along “reflection hypersurfaces” (hypersurfaces fixed pointwise by nonidentity elements of G). So algebra is generated by corresponding reflection subgroups, and general W reduces to A_1 .

Master Hecke algebras (rank 1 case)

Here $X = C$, a hyperelliptic curve of genus 1, and an operator $c_0 + c_1s$ preserves holomorphy iff the coefficients have poles bounded by C^s (the fixed subscheme) and their sum is holomorphic. Equivalently, operator is $f_0 + f_1(s - 1)$ where f_0 is holomorphic and f_1 has poles bounded by C^s . (Of course, these are just fancy ways of writing $\mathcal{E}nd(\pi_*\mathcal{O}_C) = \mathcal{E}nd(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))!$)

Important caveat: This description splits the rank 1 algebra as a left $\pi_*\mathcal{O}_C$ -module, and one can also split it as a right module, but these splittings are not compatible.

General Hecke algebras (finite case)

To put in parameters: in rank 1, replace the condition on f_1 in $f_0 + f_1(s-1)$ by saying that $\text{div}(f_1)$ is bounded by $C^s - T$ for some effective divisor T . (If $T = C^s$, this gives the usual holomorphic group algebra $\mathcal{O}_C[\langle s \rangle]$.) Note that there is a $C_{\deg(T)}$ symmetry: we can apply s to any subdivisor of T by conjugating by a suitable product of theta functions.

More generally, if we choose parameters for each simple root subject to the obvious compatibility condition for roots in the same orbit, we may consider the corresponding sheaf of algebras generated in rank 1; again, can obtain $\mathcal{O}_X[W]$ by appropriate choices of T_i .

Theorem: The sheaf of algebras $\mathcal{H}_{W;\vec{T}}(X)$ is locally free (as a left \mathcal{O}_X -module) of rank $|W|$ with Hilbert polynomial independent of X and \vec{T} .

Basic idea: the Bruhat order on W induces a natural filtration on $\mathcal{H}_{W;\vec{T}}(X)$, so it's enough to prove the subquotients are invertible sheaves. For \vec{T} in general position, this is straightforward, and semicontinuity extends to arbitrary \vec{T} . One can also see that the associated graded has a presentation of the form we'd expect (generators for each simple, plus analogues of quadratic and braid relations).

(Note: This generalizes algebras constructed by Ginzburg, Kapranov, and Vasserot, mainly by allowing more general systems of parameters \vec{T})

The infinite case

This construction clearly extends to the infinite case, except for one piffling issue: it doesn't make sense to talk about a sheaf of algebras on X/W when W is infinite and the quotient doesn't exist!

One approach to fix this: for W finite, we can think about a sheaf of algebras on X/W in terms of X alone: we have an algebra on every W -invariant affine open subset of X , subject to the usual sheaf conditions. So clearly we should just do this, again subject only to nitpicking from those who note that there aren't any W -invariant affine open subsets when W is infinite.

Luckily, when trying to glue together a sheaf from what happens on affine open subsets, “affine” is far more important than “open”. In particular, you can specify a quasicoherent \mathcal{O}_X -module using a covering by *intersections* of open subsets. (This is a very special case of fpqc descent.) There’s no difficulty in finding nonempty such intersections, and usually no difficulty in finding a covering.

So we could define $\mathcal{H}_{W;\vec{T}}(X)$ via an abstract gluing operation from its restriction to W -invariant affine “localizations”. (This also gives us some freedom to twist: we can also include a “twisting datum”: a W -equivariant structure on the trivial gerbe on X along with compatible explicit expressions of the restrictions to rank 1 as coboundaries.) The $C_{\deg(T)}$ symmetry extends, though now we must conjugate by an infinite formal product of theta functions (and this changes the twisting nontrivially).

This approach is somewhat less workable for spherical algebras (where the operators involve correspondences rather than automorphisms); one can still deal with this by generalizing “invariant”, but there’s another approach.

Gluing along affine localizations means that we may view $\mathcal{H}_{W;\vec{T}}(X)$ as a sheaf on X , but in fact we can say more: since $\mathcal{H}_{W;\vec{T}}(X)$ contains \mathcal{O}_X , it is a *bimodule* over \mathcal{O}_X , and thus we may glue up to obtain a quasicohherent sheaf on $X \times X$. Moreover, the algebra structure may be phrased in terms of a suitable notion of tensor products on such “sheaf bimodules”, making $\mathcal{H}_{W;\vec{T}}(X)$ a “sheaf algebra”. (This notion was originally introduced by Artin and Van den Bergh.)

Theorem. The restriction of $\mathcal{H}_{W;\vec{T}}(X)$ to any Bruhat interval in W is a locally free sheaf on X of rank equal to the size of the Bruhat interval. Moreover, the morphism $\mathcal{H}_{W;\vec{T}}(X) \rightarrow k(X)[W]$ is injective on fibers.

Theorem. If W_I, W_J are finite parabolic subgroups, then

$$\text{Hom}(\text{Ind}_{W_I}^{W;\vec{T}} \mathcal{O}_X, \text{Ind}_{W_J}^{W;\vec{T}} \mathcal{O}_X)$$

can be embedded in $k(X)[W/W_I]^{W_J}$ and the map is again injective on fibers and locally free on Bruhat intervals.

Note that for the second result (proved using, among other things, a Mackey-type result in which the direct sum is replaced by a filtration), we would normally use symmetric idempotents to compute this, but those don't exist if \vec{T} has too large a degree.

For the (DAHA!) case when W is affine, the above results are not quite what we want, as we want to view q (determining the action of s_0) as a parameter. Luckily it is not too hard to do the appropriate base change, so the theorems carry over.

The spherical algebra w.r.to the finite Weyl group has a natural interpretation as a sheaf algebra (on X/W) of difference operators preserving W -invariant holomorphic functions (and satisfying vanishing conditions associated to \vec{T}). (In particular, it's a domain!) One can show that for generic \vec{T} , the DAHA is Morita-equivalent to not only its spherical algebra, but the spherical algebras in which some of the parameters have been shifted by q (and thus to the DAHA in which some of the parameters have been shifted by q).

If q is torsion, then \tilde{W} does not act faithfully, and $\mathcal{H}_{\tilde{W};\vec{T}}(X)$ has a large center. If $\vec{T} = 0$, we can write it as a (twisted!) holomorphy-preserving algebra on a $2n$ -dimensional scheme (a relative affine blow up of a torus bundle over X), and thus easily write down the center. For general \vec{T} , all I can say is that the center is isomorphic to a $q = 0$ spherical algebra living on an isogenous abelian variety.

Conjecture: the center is Noetherian and $\mathcal{H}_{\tilde{W};\vec{T}}(X)$ is finite over its center. (This would imply $\mathcal{H}_{\tilde{W};\vec{T}}(X)$ Noetherian for q torsion, and probably for general q (via a trick of Artin/Tate/Van den Bergh).)

For $C^\vee C_n$, the action is $s_0 : z_1 \mapsto q - z_1$, $s_n : z_n \mapsto -z_n$ with s_1, \dots, s_{n-1} acting by permutations. If we choose $q/2$, we also have a diagram automorphism ω which we can incorporate into the algebra.

This gives a new feature: if only s_i has parameters (think just one point t), then the DAHA is generated by the parabolic subalgebra corresponding to W along with the diagram automorphism, and we can filter by the number of times we used the diagram automorphism. (This is a coarser version of the usual Bruhat filtration). We can then use this filtration to “compactify” the sheaf algebra to a graded object (which since things are non-commutative is actually a “sheaf category”).

Passing to a category means we don't need the diagram automorphism to respect the parameters, so we can include parameters on α_0 , and then extend to include bimodules corresponding to the Morita equivalences shifting those parameters.

If $t = 0$, the resulting spherical algebra is the symmetric power of the category for $n = 1$, and if $q = 0$, the category for $n = 1$ is essentially the category of line bundles on some rational surface with smooth anticanonical curve. So this is the desired deformation. Taking global sections and restricting to line bundles pulled back from \mathbb{P}^2 gives a flat deformation of $\text{Sym}^n(\mathbb{P}^2)$ (modulo a codimension ≥ 2 set of bad parameter values, probably empty).

For Hilbert schemes: we can include the Morita equivalences that shift t to get a slightly larger category. I can no longer prove flatness in general, but for $n = 2$, flatness holds, and the Euler characteristics agree with $\text{Hilb}^2(X)$ when we would expect them to (i.e., when the line bundle is acyclic over $\text{Sym}^2(\mathbb{P}^1)$).

In general, the line bundles for which I can prove flatness also have Euler characteristics agreeing with the Hilbert scheme, and for $q = t = 0$ one can interpret the sections as giving a rational map to a Grassmannian and show that the closure of the image is the same as that of a natural map from the Hilbert scheme to the same Grassmannian.

So it is natural to conjecture that this indeed gives a flat deformation of $\text{Hilb}^n(X)$.

Recall that we expect to get abelian fibrations by contracting suitable divisors. In the simplest case (corresponding to GL_2), contracting the relevant divisor simply gives the analogous algebra with no t parameter. So I conjecture that if $\mathcal{H}^{(n)}$ is the family of spherical DAHAs of type C_n with $2n + 6$ parameters assigned to α_0 , then there is an action of a congruence subgroup of $SL_2(\mathbb{Z})$ acting on the base of the family such that fibers in the same orbit are derived equivalent. (The case $n = 1$ is known, and in fact the derived equivalences extend to the compactified spherical DAHAs.)