Nabla and Tors

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Tor groups

The Haiman/Bridgeland-King-Reid map

$$R\Gamma(P \otimes \_) : \{(\text{complexes of}) \text{ sheaves } E \text{ on } \mathrm{Hilb}_n \mathbb{C}^2\} \leftrightarrow$$

$$\{(\text{complexes of}) \ \mathbb{C}[x,y] \rtimes S_n\text{-modules } M\}$$

$$x, y = \{x_1, .., .x_n\}, \{y_1, ..., y_n\}$$

isomorphism at the level of derived categories. Interested in Tor-groups $\mathrm{Tor}_i^{\mathbb{C}[x,y]}(M, \mathbb{C})$ as bigraded S_n-representations. For instance,

$$\mathrm{Tor}_0^{\mathbb{C}[x,y]} \Gamma(P \otimes P) = DR_n = \mathbb{C}[x,y] / \langle \sum_{i=1}^{n} x_i^r y_i^s : (r,s) \neq (0,0) \rangle$$

This talk: formula for the character of the equivariant index $\chi = \sum_i (-1)^i \mathrm{Tor}_i$, second formula has to do with resolution.

Examples

Variety X	Bundle E	$\chi_X(E)$
Hilb_n	$P^* \otimes P \otimes \mathcal{O}(k)$	affine Springer fiber/GKM space
	$P^* \otimes P$	Cauchy product
	$P^* \otimes P^{k-1}$	mixed Hodge of $g = 0$ Character varieties / Higgs moduli space (HLV)
	$\mathbb{S}_\lambda(B)$	Hall-Littlewood (C)
	$\mathbb{S}_b$	Knot invariants
Z_n	$\mathcal{O}(1)$	(q, t)-Catalan numbers
	P	diag. coinv./Shuffle conjecture
	$\mathcal{O}(k)$, other sheaves	rational q, t Catalan, Homology of compactified Jacobian varieties (GOSRN, etc.)
	$P^{\otimes 2}$	possible new conjectures (other experts)
$\mathcal{M}_{r,n}$	Ext^1 bundles	AGT

Hilbert scheme

$$\mathrm{Hilb}_n = \{I \subset \mathbb{C}[x, y] : \dim(R/I) = n\},$$

smooth, dimension $2n$. There is an open subset

$$U_n = \left\{(p_1 \neq \cdots \neq p_n) \subset \left(\mathbb{C}^2\right)^n\right\}/S_n \subset \mathrm{Hilb}_n$$

and the punctual Hilbert scheme

$$Z_n = \pi^{-1}(0), \quad \pi : \mathrm{Hilb}_n \to (\mathbb{C}^2)^n/S_n$$

which is $\mathbb{CP}^1$ for $n = 2$, singular for $n > 2$. The standard torus action

$$T = \mathbb{C}^* \times \mathbb{C}^* \circlearrowright \mathrm{Hilb}_n$$

is induced from

$$(q, t) \cdot (x, y) = (q^{-1}x, t^{-1}y)$$

Procesi bundle

There is a bundle P on Hilb_n whose restriction to the open subset $U_n \subset \mathrm{Hilb}_n$ by

$$P\Big|_{U_n} = \{(p_1 \neq \cdots \neq p_n)\} \times_{S_n} \mathbb{C}[S_n]$$

- P can be extended to a vector bundle on all of Hilb_n.
- $S_n \circlearrowright P$, fibers $\cong$ regular representation.
- $P^{S_{n-1}\times S_1} = B$ where

$$rank(B) = n, \quad B|_I = \mathbb{C}[x,y]/I$$

- $P^{\mathrm{sign}} = \mathcal{O}(1) = \det(B)$
- Procesi bundles are more general objects in the categorical McKay correspondence (see Loseu).

Garsia-Haiman module

Torus-fixed points of Hilb_n are monomial ideals, spanned by all $x^i y^j$ for i, j outside a Young diagram μ. Fibers of P over torus-fixed points μ are the Garsia-Haiman modules: Given μ, order the boxes $(a_1, b_1)...(a_n, b_n)$ in some way, i.e.

$$\mu = [3,3,1] \longrightarrow \{(a_i, b_i)\} =$$

$$\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0)\}$$

Then we set

$$A_\mu = (a_{ij}), \quad a_{i,j} = x_i^{a_j} y_i^{b_j}$$

$$P_\mu = \langle f(\partial x_i, \partial y_j) \det(A_\mu) \rangle$$

If $\mu = [2]$ then

$$A_{[2]} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}, \quad P_\mu = \langle 1, x_1 - x_2 \rangle$$

Modified Macdonald polynomials

The Frobenius character of P_μ is the modified Macdonald polynomial (part of Haiman's proof of n! conjecture):

$$H_\mu = \mathcal{F} P_\mu = \sum_\lambda m_\lambda \operatorname{ch}_{q,t} P_\mu^{S_{\lambda_1} \times \cdots \times S_{\lambda_l}} =$$

$$\sum_\lambda s_\lambda \operatorname{ch}_{q,t} \langle P_\mu, \chi^\lambda \rangle$$

Here $P_\mu^{S_\lambda}$ is the invariant subspace of the Young subgroup, and $\langle P_\mu, \chi^\lambda \rangle$ is the multiplicity of the irreducible representation χ^λ of S_n. For instance,

$$H_{[2]} = m_{[2]} + (1+q) m_{[1,1]} = s_{[2]} + q s_{[1,1]}$$

so $P_{[2]}$ has one component of the trivial representation in degree $(0,0)$, one component of the sign rep in degree $(1,0)$.

Localization

The equivariant Euler characteristic of $E = P^* \otimes P^{l-1} \otimes \mathcal{O}(k)$ is given by

$$\chi_{\mathrm{Hilb}_n}(E) := \sum_{i \geqslant 0} (-1)^i \mathcal{F} H^i(E) =$$

$$\sum_{|\mu|=n} \frac{H_\mu[X^1] \cdots H_\mu[X^l](q^{n(\mu)} t^{n(\mu')})^k}{(H_\mu, H_\mu)_*} =$$

$$\nabla^k_{X^i} \sum_{|\mu|=n} \frac{H_\mu[X^1] \cdots H_\mu[X^l]}{(H_\mu, H_\mu)_*},$$

where

$$\nabla H_\mu = q^{n(\mu)} t^{n(\mu')} H_\mu.$$

Conjecture (Haiman,Bergeron,Garsia,Tesler)

$\langle \nabla^k s_\mu, s_\nu \rangle$ *is signed-positive*

For $l = 2$, the above sum is

$$\chi(P^* \otimes P \otimes \mathcal{O}(k)) = \nabla^k h_n \left[\frac{XY}{(1-q)(1-t)} \right]$$

where

$$f\left[\frac{XY}{(1-q)(1-t)} \right] = f\Big|_{p_k = p_k(X)p_k(Y)/(1-q^k)/(1-t^k)}$$

If $f(X) = \mathcal{F}(M)$ for a $\mathbb{C}[x] \rtimes S_n$ module-M, then

$$f[(1-q)X] = \sum_i (-1)^i \mathcal{F}\, \mathrm{Tor}_i^{\mathbb{C}[x]} M$$

Haiman gives $J =$space of sections as an explicit $S_n \times S_n$-equivariant module over $4n$ variables x, y, z, w. For $k = 1$

$$J = \mathbb{C}[x, y, z, w] / \bigcap_{\sigma \in S_n} \ker_{\mathbb{C}[x,y,z,w]}(f \mapsto f(x, y, z_\sigma, w_\sigma))$$

Fix N, and let

$$(m, a, b) \in \mathbb{Z}_{\geqslant 0}^{n} \times \{1, \dots, N\}^{n} \times \{1, \dots, N\}^{n}$$

be sorted so that $m_i \geqslant m_j$, $m_i = m_j \Rightarrow a_i \leqslant a_j$, $m_i = m_j, a_i = a_j \Rightarrow b_i \leqslant b_j$.

$$\mathrm{dinv}_k(m, a, b) = \sum_{i<j} \max\left(k - 1 + m_j - m_i + \delta(a_i > a_j) + \delta(b_i > b_j), 0\right),$$

Theorem (Mellit, C)

$$\nabla_X^k h_n \left[\frac{\pm XY}{(1-q)(1-t)}\right] =$$

$$\sum_{[m,a,b]} t^{|m|} q^{\mathrm{dinv}_k(m,a,b^\pm)} h_{\mu(m,a,b)} \left[\frac{1}{1-q}\right] X_a Y_b$$

Theorem (M,C)

$$\nabla_X^k h_n \left[\frac{-XY}{1-q}\right] = \sum_{[m,a,S\subset\{1,\dots,n\},b]} (-1)^{|S|} \begin{array}{l}\textit{above formula, but}\\ b_i \textit{ is ``odd'' if } i \in S\end{array}$$

Affine Springer fiber

Let

$$\mathcal{F}l_n = \{\mathbb{C}^n((t)) \cdots \supset \Lambda_i \supset \Lambda_{i+1} \supset \cdots :$$

$$\dim(\Lambda_i/\Lambda_{i+1}) = 1,\ \Lambda_{i+n} = t\Lambda_i,\ ind(\Lambda_0) = 0\} = SL_n((t))/I$$

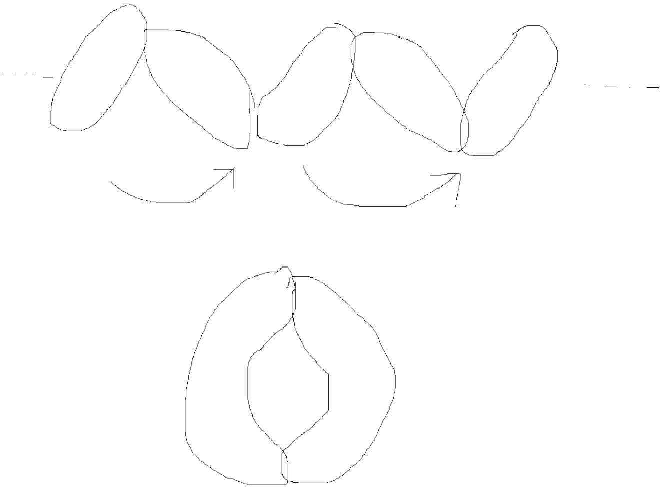
$$X_\gamma = \left\{g \cdot I \in \mathcal{F}l_n : g^{-1}\gamma g \in Lie(I)\right\},$$

where in our case, $\gamma = \operatorname{diag}(a_1 t^k, ..., a_n t^k)$.

- The torus action by $T \subset SL_n \subset SL((t))$ on $\mathcal{F}l_n$ preserves X_γ, and the fixed set is the entire affine Weyl group W.
- The equivariant homology is is characterized by relations in $i_*^{-1}(H_*(X_\gamma)) \subset \mathbb{C}(x_1, ..., x_n) \cdot W$ by GKM.
- There is a left and right action of W, $\mathbb{C}[x]$, and $\mathbb{C}[z]$ on homology.
- Homology is free over $\mathbb{C}[x]$, there is an affine paving, Schubert type basis.

Lattice quotient

The lattice $\mathbb{Z}^n \subset W$ acts space-level on X_γ. GKM study the quotient $H_*(\mathbb{Z}^n \backslash X_\gamma)$. No longer has affine paving but is compact. The $n = 2$ case is here:



GKM construction

Theorem (GKM)

$$H_m(\mathbb{Z}^n \backslash X_\gamma) = \bigoplus_{p+q=m} \mathrm{Tor}_p^{\mathbb{Z}^n}(H_q(X_\gamma))$$

Proof.
The Cartan-Leray spectral sequence collapses. $\square$

Relation with J

We have

$$J = \mathrm{Im}\left(\pi : \mathbb{C}[x,y,z,w] \to \bigoplus_{\sigma \in S_n} \mathbb{C}[x,y]\sigma\right)$$

$$\pi(f) = \sum_{\sigma} f(x,y,z_\sigma,w_\sigma)\sigma$$

since the kernel is the intersection. There's a map

$$J[y^{-1}] \to \bigoplus_{w \in W} \mathbb{C}(x)w, \quad f(x)y^a\sigma \mapsto \frac{f(x)}{\prod_{i<j}(x_i - x_j)}(a \cdot \sigma).$$

Claim: the image is precisely $H_*^T(X_\gamma)$. Oscar Kivinen showed that the GKM relations are satisfied by J for the case of the sign rep (Grassmannian case), showing $J \subset H_*^T(X_\gamma)$. Other inclusion is difficult, essentially follows from Haiman's papers.

First formula

Main points:

- $\mathrm{dinv}_k(m, a)$ is the dimension of the cell corresponding to $w(m, a)$.
- J is free over both $\mathbb{C}[x]$ and $\mathbb{C}[y]$, but not $\mathbb{C}[x, y]$. Freeness over x corresponds to equivariant formality.
- Tensoring out over x with $\mathbb{C}$ is like passing from equivariant to non-equivariant homology.
- Tensoring over $\mathbb{C}[y]$ with $\mathbb{C}$ is more subtle. In the GKM formula, you have that $y \in \mathbb{Z}^n$ acts by 1 on $\mathbb{C}$, not 0. This means you lose the t grading, which does not exist in GKM Theorem.

Second formula

Replace $H_*(X_\gamma)$ on the right hand side with J, and $\mathbb{Z}^n$ with $\mathbb{C}[y]$ on the right-hand side.

$$\sum_i x^i \operatorname{Tor}_i^{C[x,y]} J_2 = a_2 b_2 + (q+t) a_2 b_{1,1} + tq a_{1,1} b_{1,1} x \to 1 + x + 2x^2$$

at $f \mapsto x^{\dim(X_\gamma)}(f\big|_{t=1, q=1/x^2})$, (forgetting S_n-action).
Observation: this agrees with the Betti numbers of two spheres glued together.

$n = 3$ case

$$\sum_i x^i \operatorname{Tor}_i^{C[x,y]} J_3 = \big(q^3 a_3 b_3 + q^2 t a_3 b_3 + q t^2 a_{1,1,1} b_3 + t^3 a_3 b_3 + q^2 a_3 b_{2,1} + q t a_{1,1,1} b_3 + q t a_3 b_{2,1} + t^2 a_3 b_{2,1} + q a_{1,1,1} b_{2,1} + t a_3 b_{2,1} + a_3 b_{1,1,1}\big) + \big(q^3 t a_{2,1} b_3 + q^2 t^2 a_{2,1} b_3 + q t^3 a_{2,1} b_3 + q^2 t a_{2,1} b_{2,1} + q t^2 a_{2,1} b_{2,1}\big) x + \big(q^3 t^2 a_{1,1,1} b_3 + q^2 t^3 a_{1,1,1} b_3 + q^2 t^2 a_{1,1,1} b_{2,1}\big) x^2 \to 6x^6 + 6x^5 + 9x^4 + 6x^3 + 4x^2 + 2x + 1$$

Proofs of formulas

- First formula: current proof, modified vector bundle counting method discovered earlier by Mellit.
- Once formula is discovered, should be many proofs. Potential second proof, first formula can be taken as definition. Other conjectures show that it satisfies defining properties.
- Proof of second formula: first formula.
- Idea behind second formula: tensor over $\mathbb{C}[y]$ first.
- Like taking equivariant homology of $\mathbb{Z}^n \backslash X_\gamma$.

Garsia-Stanton descent order

Method for getting "dinv" type formulas. Suppose you have a module J over $\mathbb{C}[x,y] \rtimes S_n$. Define a filtration on J by

$$F_a J = \langle y^b : b \leqslant_{des} a \rangle_{\mathbb{C}[x]}$$

where for $a, b \in \mathbb{Z}^n_{\geqslant 0}$, we have $a \leqslant_{des} b$ if

1. $sort(a, >) <_{lex} sort(b, >)$
2. $sort(a, >) = sort(b, >)$ and $a \leqslant_{lex} b$.

Conjecture

In many situations, $\mathrm{ch}_{q,t}\, F_a J / F_{a-1} J$ *has a combinatorial dinv type formula.*

Can be used to predict combinatorial formulas, but might be harder to prove the conjecture than the formula.

Example: Diagonal coinvariants

Theorem (C,Oblomkov)

Let $J = DR_n$, $F_a = F_a J$. Then either F_a/F_{a-1} is nonzero for only $n!$ different choices of a, namely

$$a = a(\tau), \quad y^{a(\tau)} := \prod_{\tau_i > \tau_{i+1}} y_{\tau_1} \cdots y_{\tau_i}.$$

In this case

$$\mathrm{ch}_{q,t} F_a/F_{a-1} = \prod_i [w_i(\tau)]_q$$

where $w_i(\tau)$ is the number of τ_j greater than τ_i in the "run" containing τ_i, plus the number of τ_j less than τ_i in the next run. Moreover, F_a/F_{a-1} has an explicit description as modules over $\mathbb{C}[x]$ in terms of the homology of certain Hessenberg varieties.

Proof.
Relate the subquotients to a different affine Springer fiber, descent order to Bruhat order. The combinatorial description was known to several authors, including Gorsky, Mazin, Hikita,...Missing part of the argument was to define the t grading. □

Example: $w_4(2,9,6,4,5,8,1,3,7) = 2+2 = 4$. We have

$$\mathrm{ch}_{q,t}\, DR_3 = (1+q)(1+q+q^2)+t^2+t(1+q)+t^2(1+q)+t(1+q)^2+t^3$$

Example: Hall-Littlewood polynomials

Theorem (C)

Let $J = \Gamma_{\mathrm{Hilb}_n}\mathbb{S}_\lambda$. *Then*

$$\mathrm{ch}_{q,t} J = \sum_a t^{|a|} \sum_\mu q^* \langle HL_a, s_\lambda HL_\mu \rangle$$

The inner product is the matrix element of multiplication by s_λ in the Hall-Littlewood basis. It's positive valued.

Conjecture

$t^{|a|}\,\mathrm{ch}_q F_a J/F_{a-1}$ *is the summand*

Conjecture

The summand in the first theorem is $t^{|m|} \operatorname{ch}_q F_m J / F_{m-1} J$ *where*

$$F_m J = \langle y^{\sigma(m)} \sigma \rangle.$$

Conjecture

The summand in the second theorem is $t^{|m|} \operatorname{ch}_q F_m y^S J / F_{m-1} y^S J$ *where*

Conjecture

The summand in the second formula

$$\chi_k(m,a) = \sum_{S,b} (-1)^{|S|} q^{\operatorname{dinv}_k(m,a,\tilde{b})} h_{\mu(m,a,b)} \left[\frac{1}{1-q} \right] X_a Y_b$$

is Schur-positive. The substitution $\chi_k(m,a)[Y(1-q)]$ *is signed-positive.*

Happy birthday Andrei and Pavel!