Algebraic Fourier bases and probability

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Rational Schur symmetric functions

$$S_\lambda(z_1,\dots,z_N) = \frac{\det\left[z_i^{\lambda_j+N-j}\right]_{i,j=1}^N}{\det\left[z_i^{N-j}\right]_{i,j=1}^N} \in \mathbb{C}\left[z_1^{\pm1},\dots,z_N^{\pm1}\right]^{\mathrm{Symm}}, \qquad \lambda = (\lambda_1 \geq \dots \geq \lambda_N) \in \mathbb{Z}^N.$$

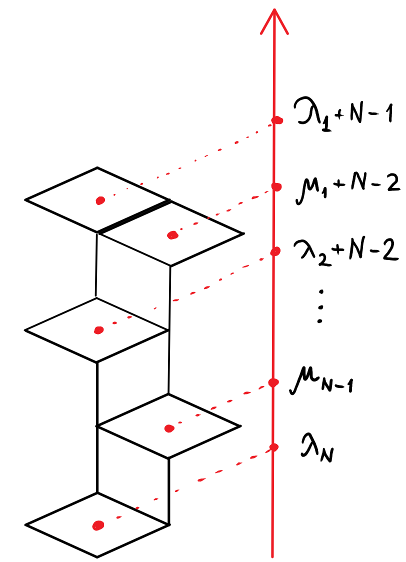
Two orthogonality relations :

$$\frac{1}{N!}\frac{1}{(2\pi i)^N} \oint_{|z_j|=1}\cdots\oint S_\lambda(z) S_\mu(z^{-1}) \prod_{i<j} |z_i - z_j|^2 \frac{dz_1\cdots dz_N}{z_1\cdots z_N} = \mathbb{1}_{\lambda=\mu}$$

$$\sum_{-\infty<\lambda_N\leq\dots\leq\lambda_1<+\infty} S_\lambda(z)\, S_\lambda(w^{-1}) \cdot \prod_{i<j}(z_i-z_j)(w_i^{-1}-w_j^{-1}) \cdot \frac{1}{w_1\cdots w_N} = \det\left[\delta(w_i - z_j)\right]_{i,j=1}^N$$

The Schur functions are characters of the (complex) irreducible representations of $GL(N,\mathbb{C})$ (or $U(N)$).

Rational Schur symmetric functions



Branching rule (restriction from U(N) to U(N-1))

$$S_\lambda(z_1,\dots,z_{N-1},c) = \sum_{\lambda_1\geq\mu_1\geq\lambda_2\geq\dots\geq\mu_{N-1}\geq\mu_N} c^{\sum\lambda_i-\sum\mu_i}\cdot S_\mu(z_1,\dots,z_{N-1})$$

Cauchy identity (reproducing kernel)

$$\sum_{\lambda_1\geq\dots\geq\lambda_N\geq 0} S_\lambda(z_1,\dots,z_N)\,S_\lambda(w_1,\dots,w_N) = \prod_{i,j=1}^{N}\frac{1}{1-z_iw_j}$$

$$\mathrm{Pol}\Big(\underset{GL^1(N)\,\circlearrowleft}{{}_N\overset{N}{\boxed{\quad}}}\circlearrowleft GL^2(N)\Big) = \bigoplus_\lambda T^1_\lambda\otimes T^2_\lambda$$

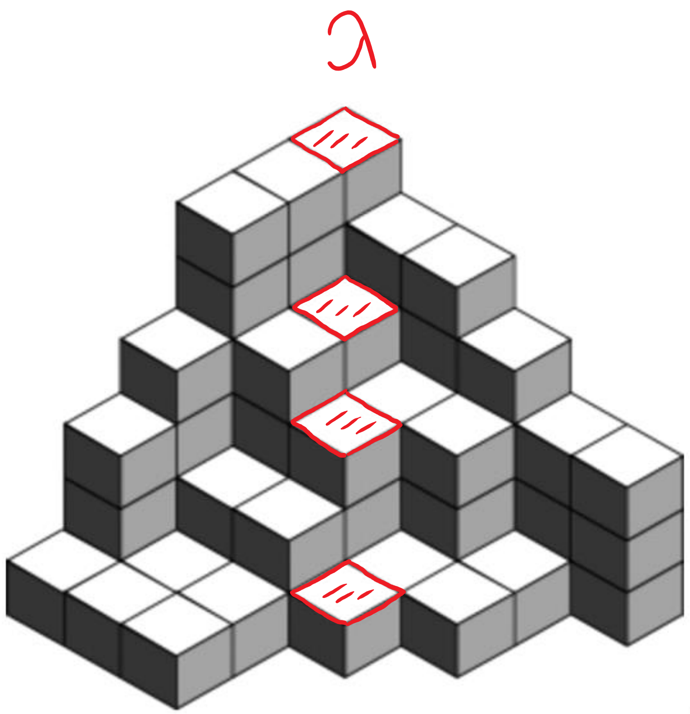
Difference operators

$$(z_1+\dots+z_N)\,S_\lambda(z) = \sum_{\mu=\lambda+\underset{(0,\dots,0,1,0,\dots,0)}{\vec{e}_r}} S_\mu(z)$$

$$\sum_{i=1}^{N}\prod_{j\neq i}\frac{z_j-qz_i}{z_j-z_i}\,S_\lambda(z_1,\dots,qz_i,\dots z_N) = \Big(\sum_{i=1}^{N} q^{\lambda_i+N-i}\Big)\cdot S_\lambda(z)$$

Eigenvalues

Random plane partitions



$$\text{weight(plane partition)} = Q^{\text{volume}} = \left(s_\lambda(Q^{1/2}, Q^{3/2}, Q^{5/2}, \dots)\right)^2$$

$$\sum_{\text{plane partitions}} Q^{\text{volume}} = \prod_{n\geq 1} \frac{1}{(1-Q^n)^n}$$

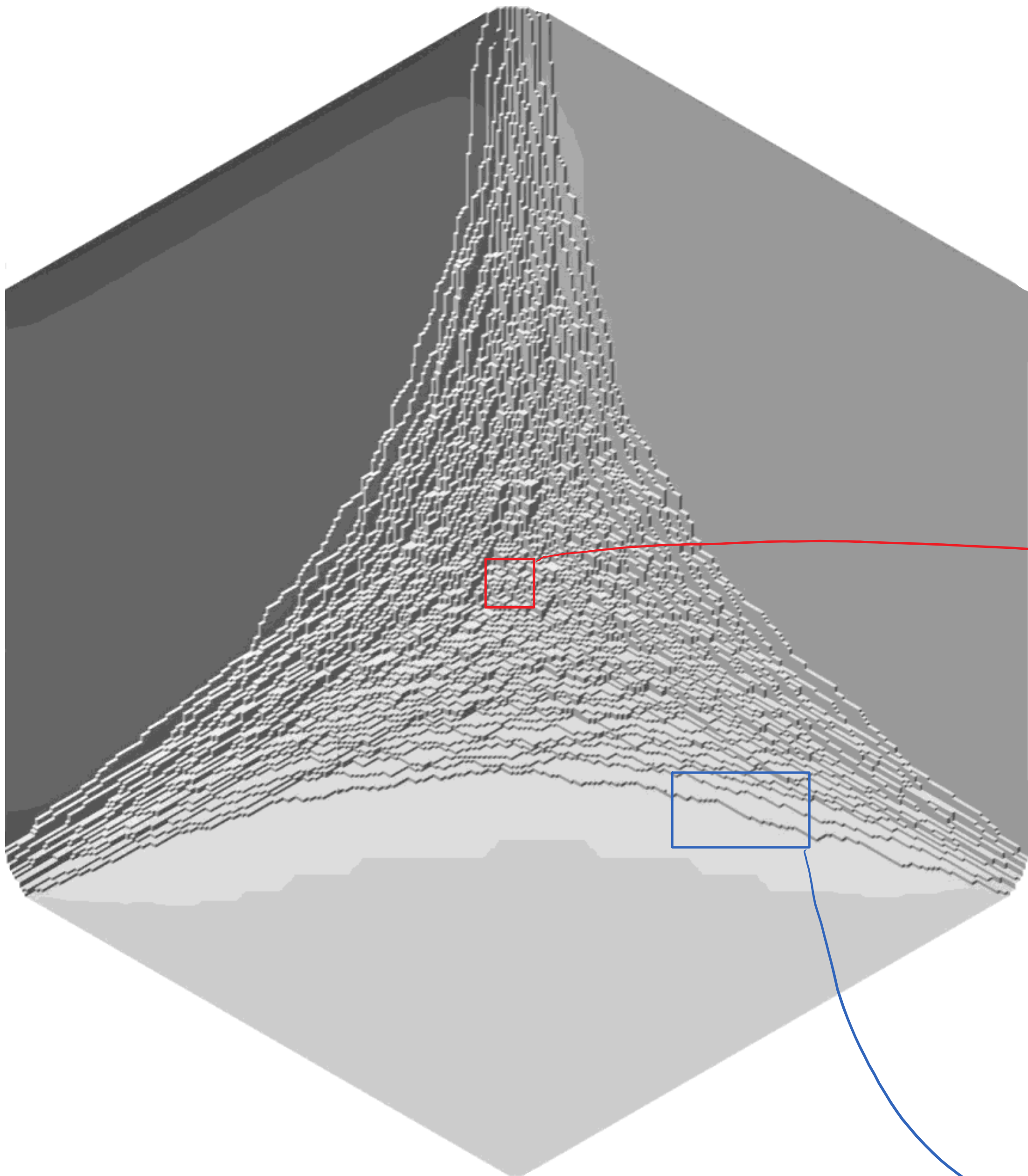
Cauchy/MacMahon identity

$$E\left(\sum_i q_1^{\lambda_i+N-i} \cdot \ldots \cdot \sum_i q_m^{\lambda_i+N-i}\right) = \left.\frac{\mathcal{D}_1^{(q_1)} \cdots \mathcal{D}_1^{(q_m)} \sum_\lambda s_\lambda(z)\, s_\lambda(w)}{\sum_\lambda s_\lambda(z)\, s_\lambda(w)}\right|_{z=w=(Q^{1/2}, Q^{3/2}, \dots)}$$

$$\mathcal{D}_1^{(q)} s_\lambda(z) := \sum_{i=1}^N \prod_{j\neq i} \frac{z_j - q z_i}{z_j - z_i}\, s_\lambda(z_1, \dots, q z_i, \dots z_N) = \left(\sum_{i=1}^N q^{\lambda_i+N-i}\right) \cdot s_\lambda(z)$$

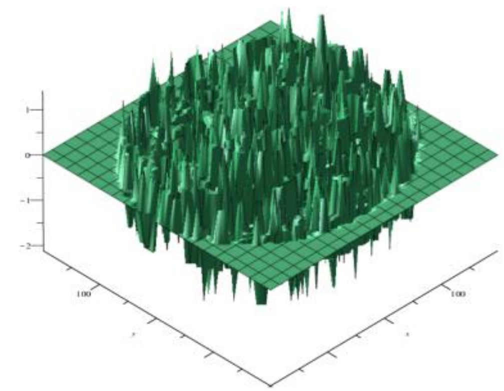
$$\sum_{\lambda_1 \geq \dots \geq \lambda_N \geq 0} s_\lambda(z_1, \dots, z_N)\, s_\lambda(w_1, \dots, w_N) = \prod_{i,j=1}^N \frac{1}{1 - z_i w_j}$$

Random plane partitions

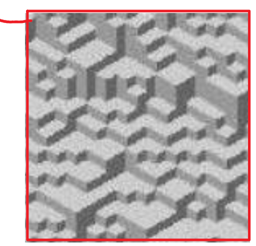


Global limit shape (Wulff droplet or 'crystal', Ronkin function of a complex line)

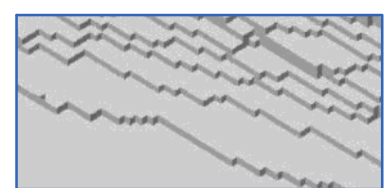
Global fluctuations (Gaussian Free Field)

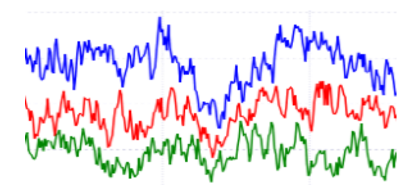


Local correlations (translation invariant Gibbs measures)

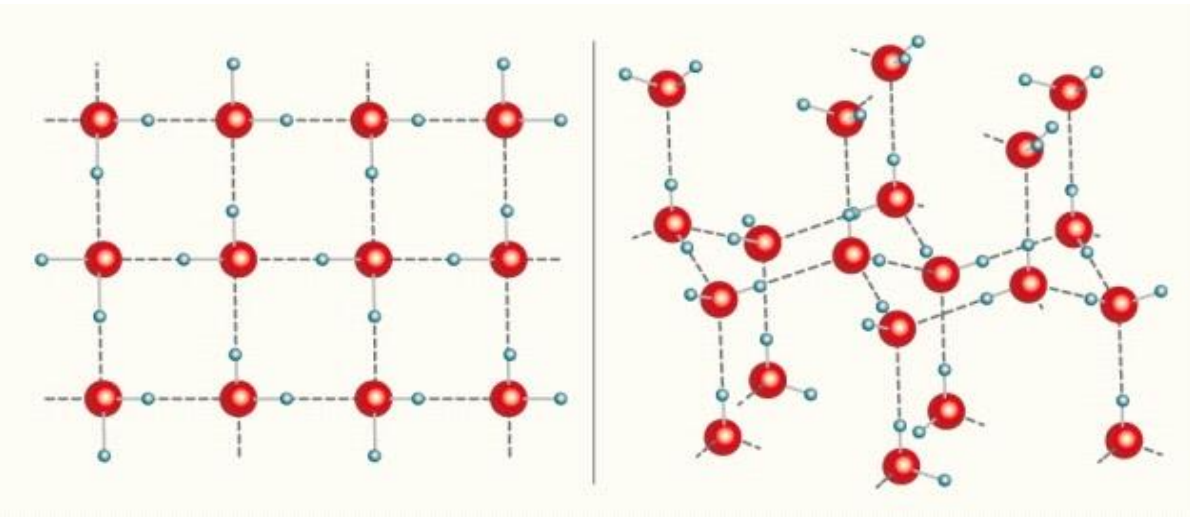


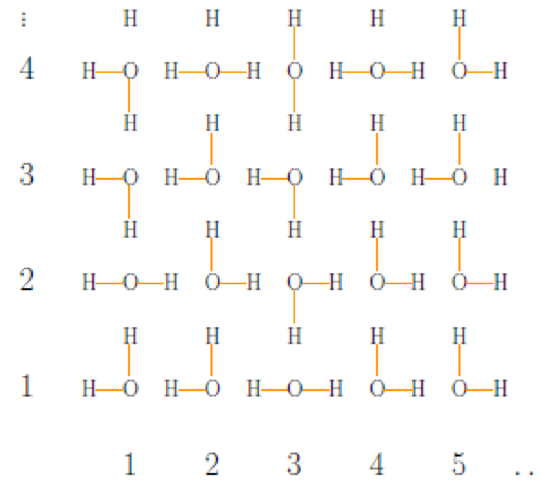
Edge fluctuations (Airy processes)



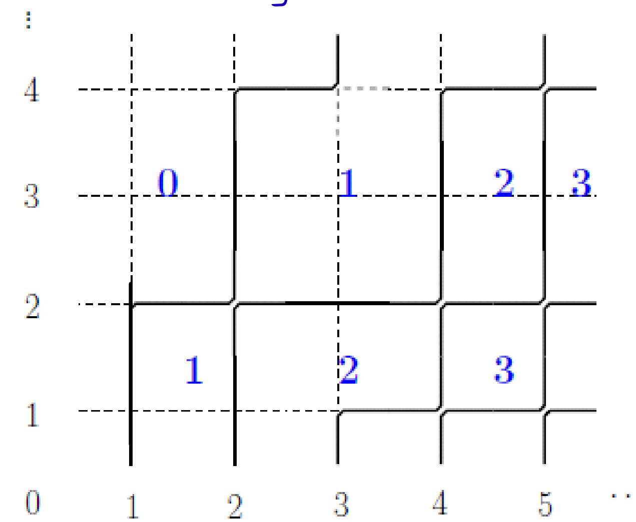


The six vertex model (Pauling, 1935)



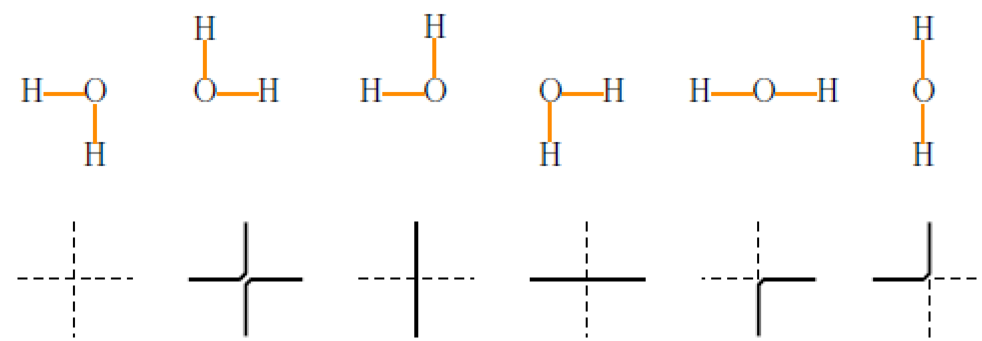


height function



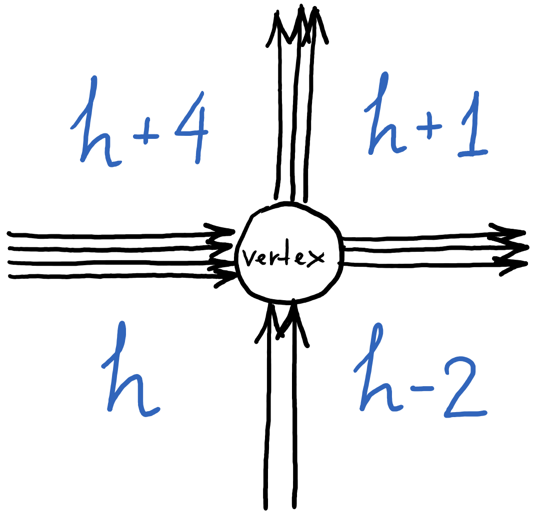
In 'square ice', which has been seen between graphene sheets, water molecules lock flat in a right-angled formation. The structure is strikingly different from familiar hexagonal ice (right).

From <http://www.nature.com/news/graphene-sandwich-makes-new-form-of-ice-1.17175>

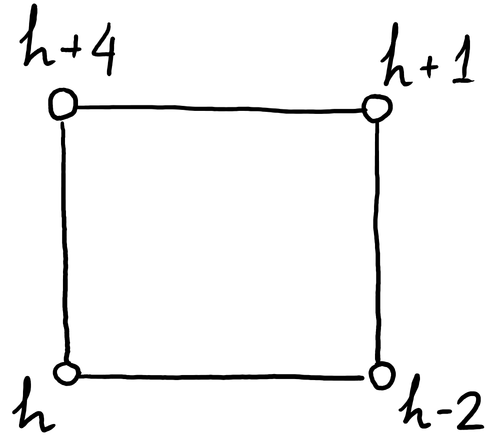


Lieb in 1967 computed the partition function of the square ice on a large torus – an estimate for the residual entropy of real ice.

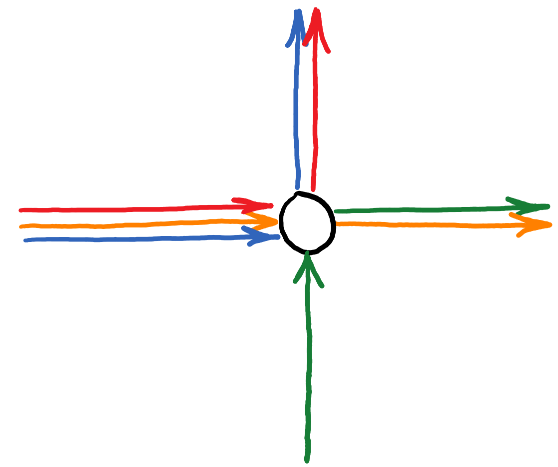
More general models



Higher spin vertex models
(only gradient of the height function matters)

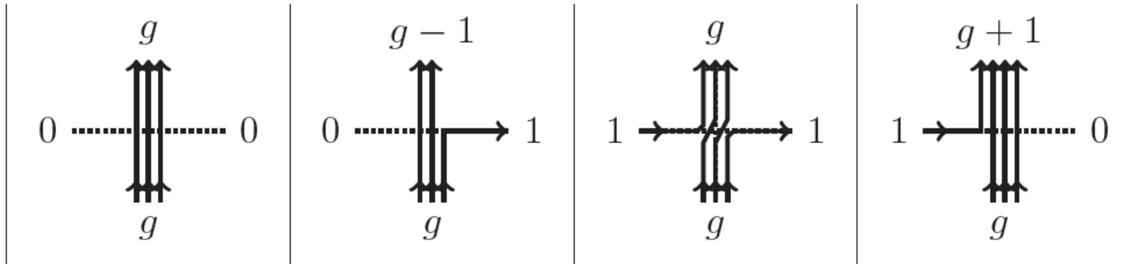


SOS (Solid-On-Solid)
or
IRF (Interaction-Round-a-Face)
models

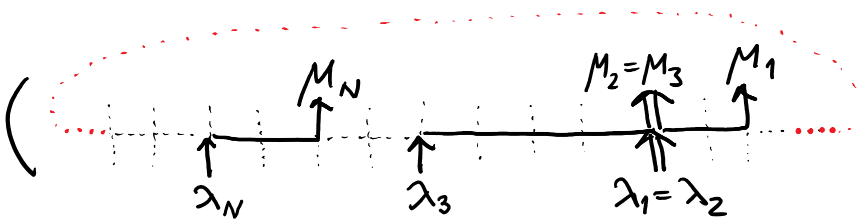


Colored (higher rank)
models

Key property: commutation of transfer-matrices

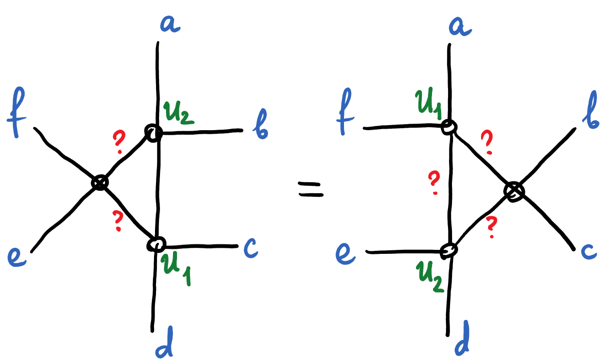


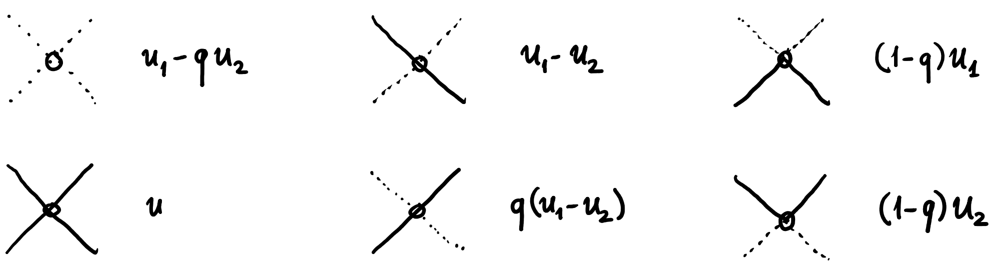
$w_{u,\mathsf{s}}$	$\dfrac{1-\mathsf{s}q^{g}u}{1-\mathsf{s}u}$	$\dfrac{(1-\mathsf{s}^2q^{g-1})u}{1-\mathsf{s}u}$	$\dfrac{u-\mathsf{s}q^{g}}{1-\mathsf{s}u}$	$\dfrac{1-q^{g+1}}{1-\mathsf{s}u}$

$X_{\lambda/\mu}(u) = ($  $)$

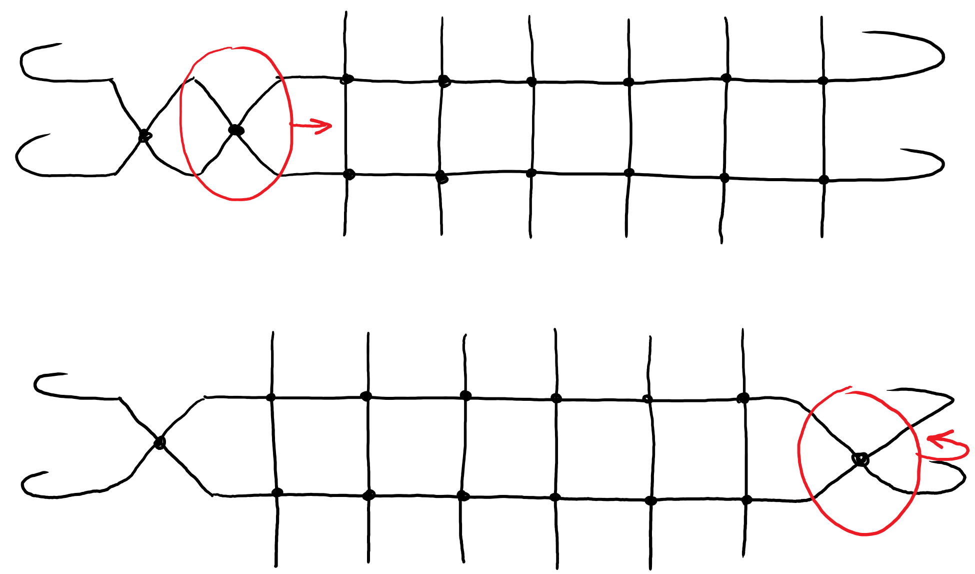
$$X_{\lambda/\mu}(u_1)\,X_{\lambda/\mu}(u_2) = X_{\lambda/\mu}(u_2)\,X_{\lambda/\mu}(u_1)$$

The Yang-Baxter (star-triangle) equation:

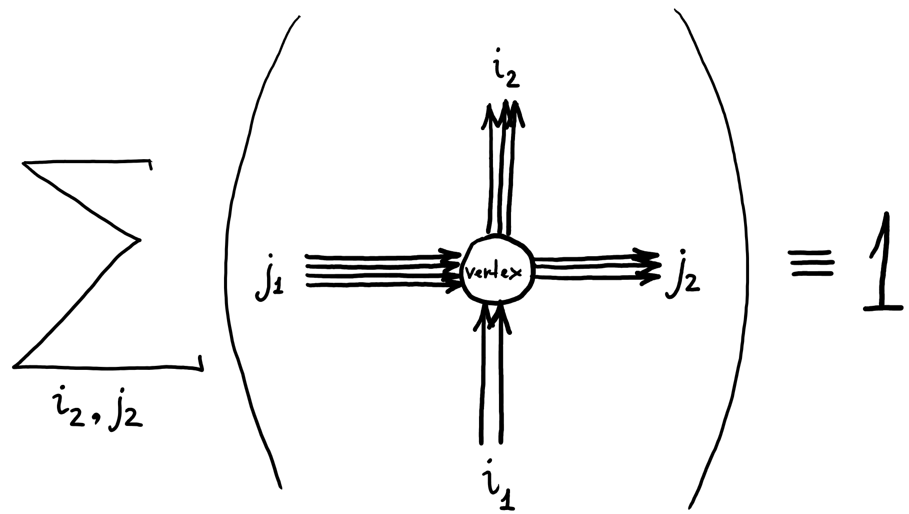




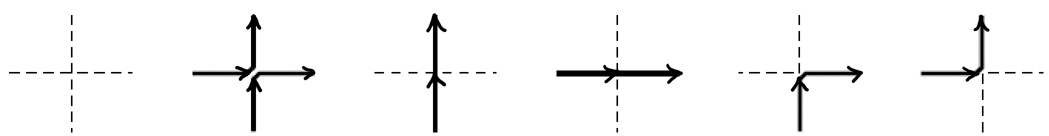
Zipper argument:



New ingredient: stochasticity



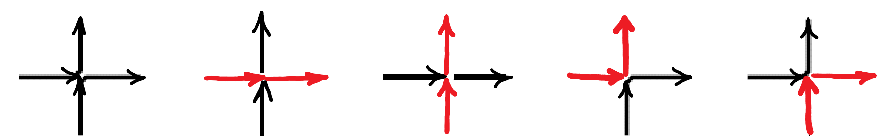
Example 1: stochastic six vertex model



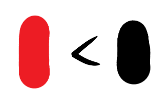
$1 \quad 1 \quad b_1 \quad b_2 \quad 1-b_1 \quad 1-b_2$

[Gwa-Spohn 1992]

Example 2: colored stochastic six vertex model

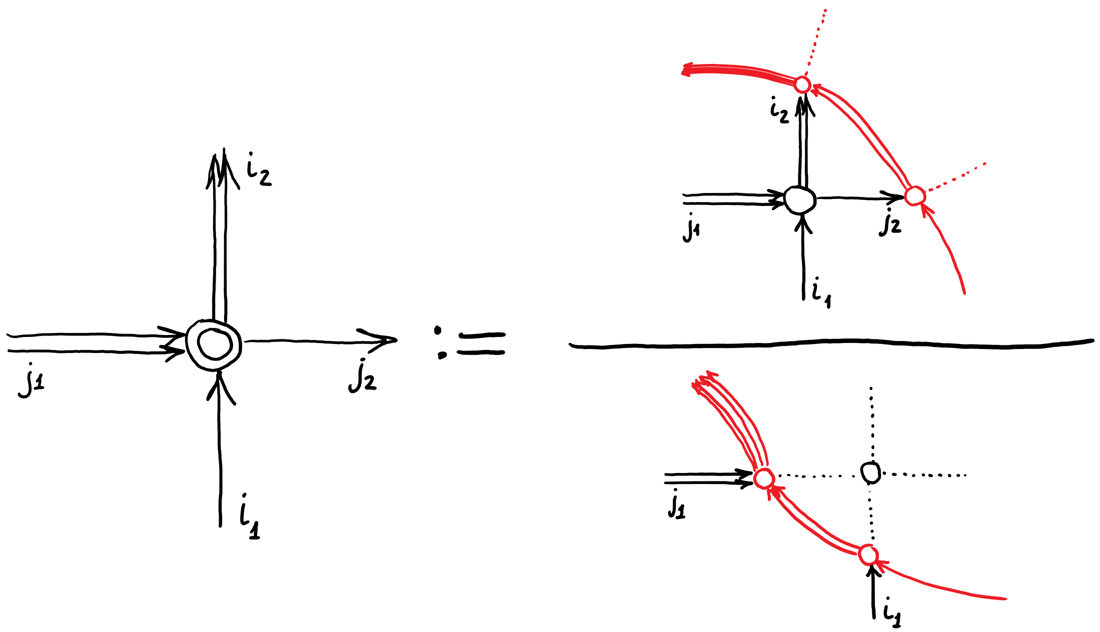


$1 \quad b_1 \quad b_2 \quad 1-b_1 \quad 1-b_2$

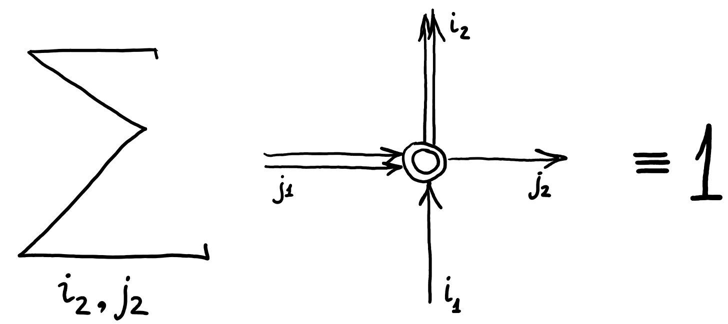
for colors 

[Kuniba-Mangazeev-Maruyama-Okado 2016]
[Kuan 2017]
[B-Wheeler 2018]

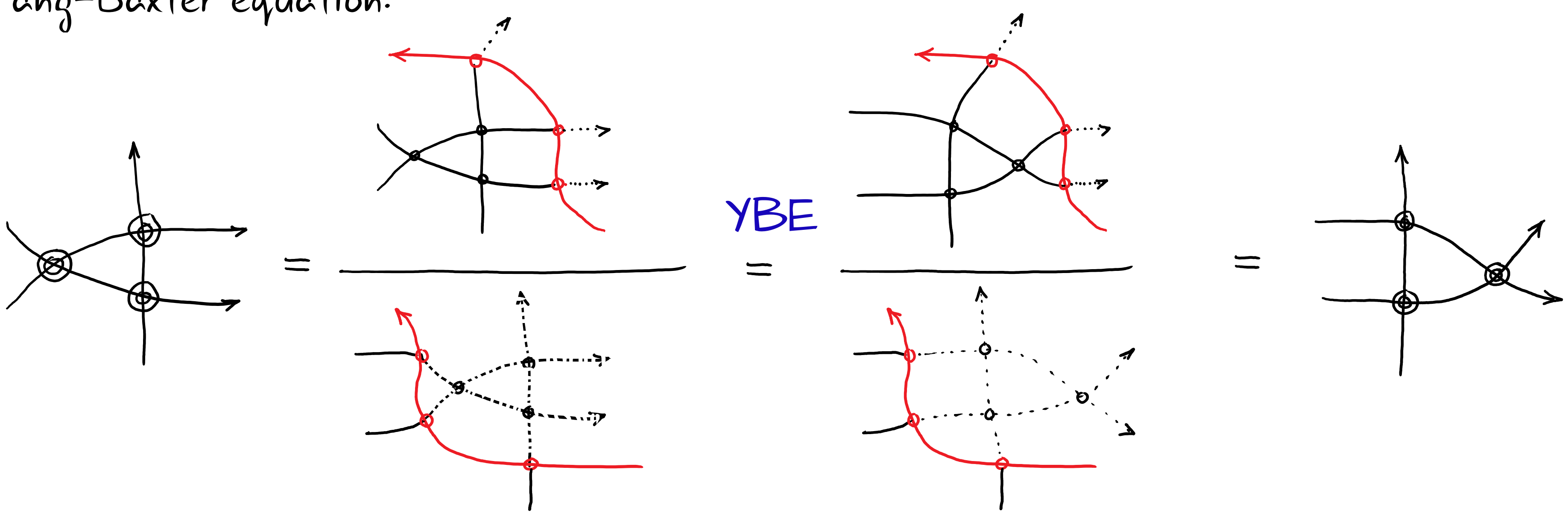
Stochastization



The Yang-Baxter equation implies



Stochastic Yang-Baxter equation:

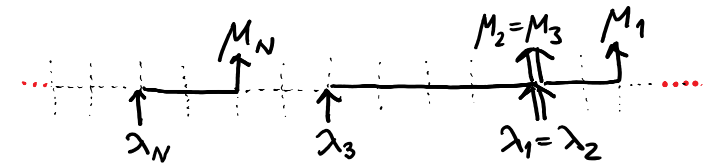


[B-Bufetov-Aggarwal 2018]

Higher spin stochastic six vertex model on $\mathbb{Z}$

$$\mathbb{P}\{\quad\} = \frac{1-sq^m u}{1-su} \qquad \mathbb{P}\{\quad\} = \frac{s^2q^m - su}{1-su}$$

$$\mathbb{P}\{\quad\} = \frac{(q^m-1)su}{1-su} \qquad \mathbb{P}\{\quad\} = \frac{-s^2q^m}{1-su}$$

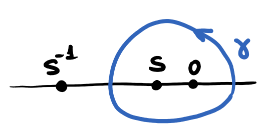
Complete basis of eigenfunctions of the transfer matrix $X_{\lambda/\mu}(u) = \Big($  $\Big)$

$$F_\lambda(u_1,\dots,u_N) = \frac{(1-q)^N}{\prod_{i=1}^N(1-su_i)} \cdot \sum_{\sigma\in S_N} \sigma\left(\prod_{i<j}\frac{u_i-qu_j}{u_i-u_j}\cdot\prod_{i=1}^N\left(\frac{u_i-s}{1-su_i}\right)^{\lambda_i}\right)$$

$\lambda = (\lambda_1 \geq \dots \geq \lambda_N) \in \mathbb{Z}^N$

Orthogonality

$$\frac{c(\lambda)}{(2\pi i)^N(1-q)^N N!}\oint_\gamma\cdots\oint_\gamma \prod_{1\le A\ne B\le N}\frac{u_A-u_B}{u_A-qu_B}\, F_\lambda(u_1,\dots,u_N)\,F_\mu(u_1^{-1},\dots,u_N^{-1})\prod_{i=1}^N\frac{du_i}{u_i} = \mathbb{1}_{\lambda=\mu}$$



[Tarasov-Varchenko 1997]
[Povolotsky 2013]
[B-Corwin-Petrov Sasamoto 2014-15]
[Corwin-Petrov 2014]
[B-Petrov 2016]

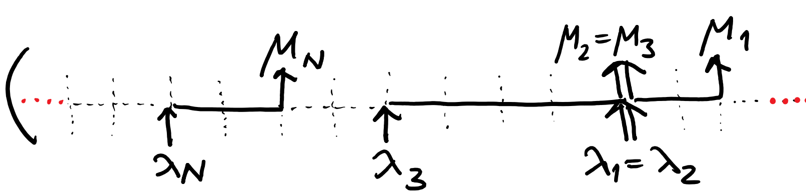
Spin Hall-Littlewood symmetric rational functions

$$F_\lambda(u_1,\dots,u_N) = \frac{(1-q)^N}{\prod_{i=1}^{N}(1-su_i)} \cdot \sum_{\sigma\in S_N} \sigma\left(\prod_{i<j} \frac{u_i-qu_j}{u_i-u_j} \cdot \prod_{i=1}^{N}\left(\frac{u_i-s}{1-su_i}\right)^{\lambda_i} \right)$$

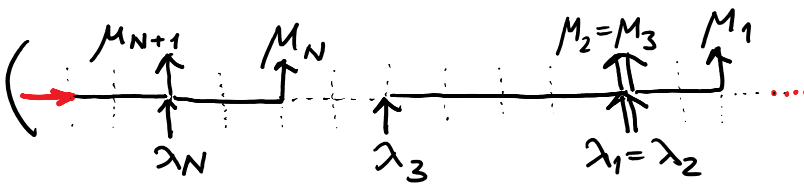
$$\lambda=(\lambda_1\geq\dots\geq\lambda_N)\in\mathbb{Z}^N$$

Specializing $s=q=0$ brings us back to the Schur, while setting $s=0$ yields the Hall-Littlewood polynomials that arise in connection with finite p-groups and representation theory of groups of p-adic type.

In $\mathrm{Span}\{e_\lambda : \lambda=(\lambda_1\geq\lambda_2\geq\dots\geq\lambda_N\geq 0)\}$ define

$A(u)e_\lambda = \sum_\mu \mathrm{weight}_u\Big($ 

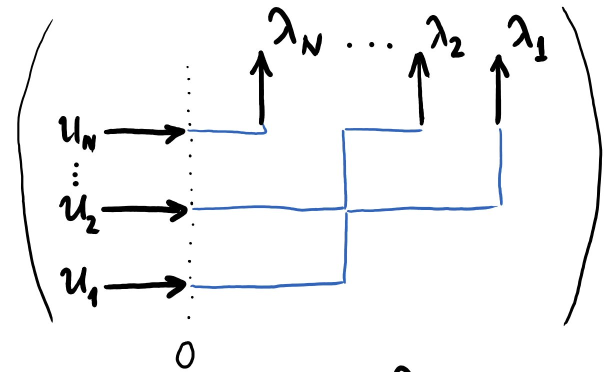
$\Big)\, e_\mu$

$B(u)e_\lambda = \sum_\mu \mathrm{weight}_u\Big($ 

$\Big)\, e_\mu$

Spin Hall-Littlewood symmetric rational functions

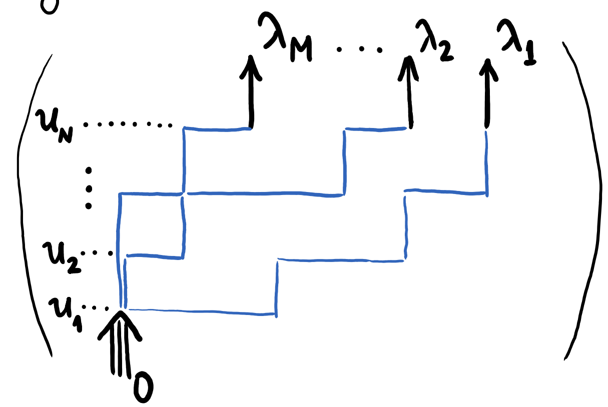
$$F_\lambda(u_1,\ldots,u_N) = \left\langle B(u_1)\cdots B(u_N)\, e_\varnothing , e_\lambda \right\rangle = \text{weight}\left(\right.$$



$$\left.\right)$$

$$= \frac{(1-q)^N}{\prod_{i=1}^N (1-su_i)} \cdot \sum_{\sigma \in S_N} \left(\prod \frac{u_i - q u_j}{u_i - u_j} \cdot \prod_{i=1}^N \left(\frac{u_i - s}{1 - s u_i}\right)^{\lambda_i} \right)$$

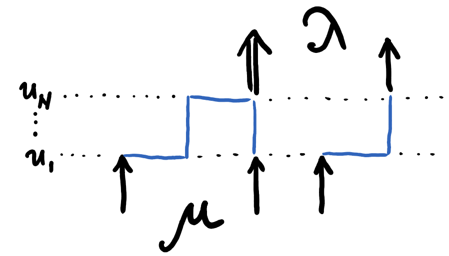
$$G_\lambda(u_1,\ldots,u_N) = \left\langle A(u_1)\cdots A(u_N)\, e_{(0^M)} , e_\lambda \right\rangle = \text{weight}\left(\right.$$



$$\left.\right)$$

More generally,

$$G_{\lambda/\mu}(u_1,\ldots,u_N) = \left\langle A(u_1)\cdots A(u_N)\, e_\mu , e_\lambda \right\rangle$$



Spin Hall-Littlewood symmetric rational functions

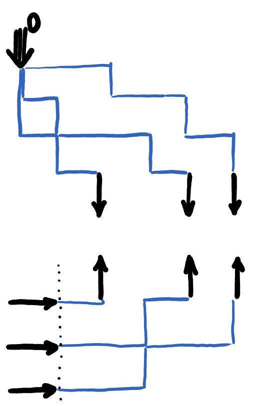
Difference operator (transfer-matrix)

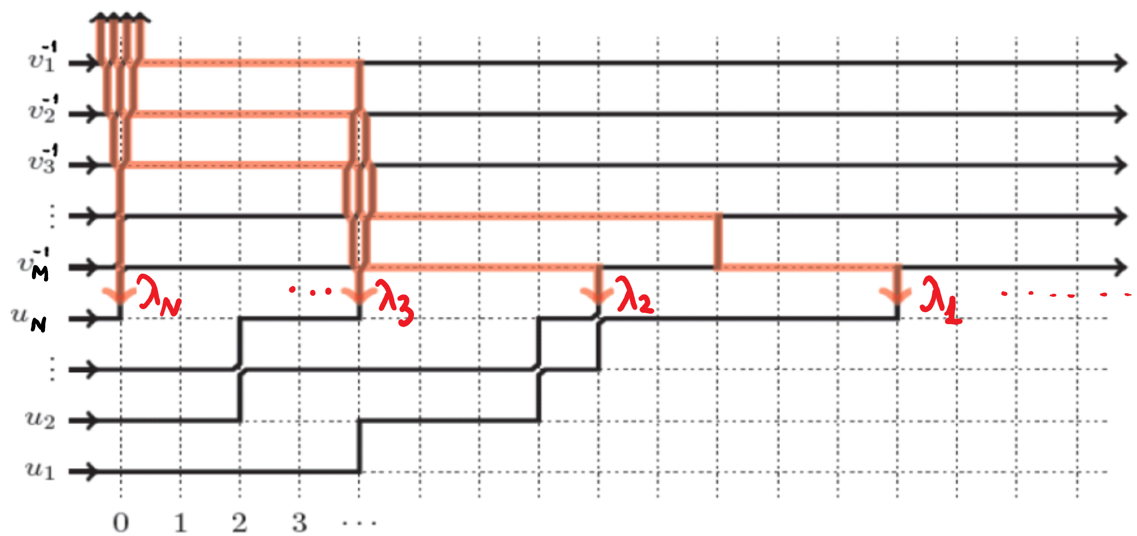
$$\sum_{\mu} \frac{c(\mu)}{c(\lambda)}\, G_{\mu/\lambda}(v)\cdot F_{\mu}(u_1,\dots,u_N) = \prod_{i=1}^{N} \frac{1-qu_iv}{1-u_iv}\cdot F_{\lambda}(u_1,\dots,u_N)$$

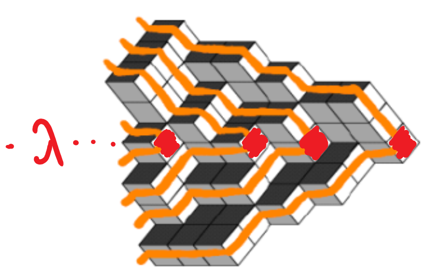
$$c(0^{m_0}1^{m_1}2^{m_2}\dots) = \prod_{k\geq 0} \frac{(s^2;q)_{m_k}}{(q;q)_{m_k}}$$

Cauchy identity [B.'14, B.-Petrov '16]

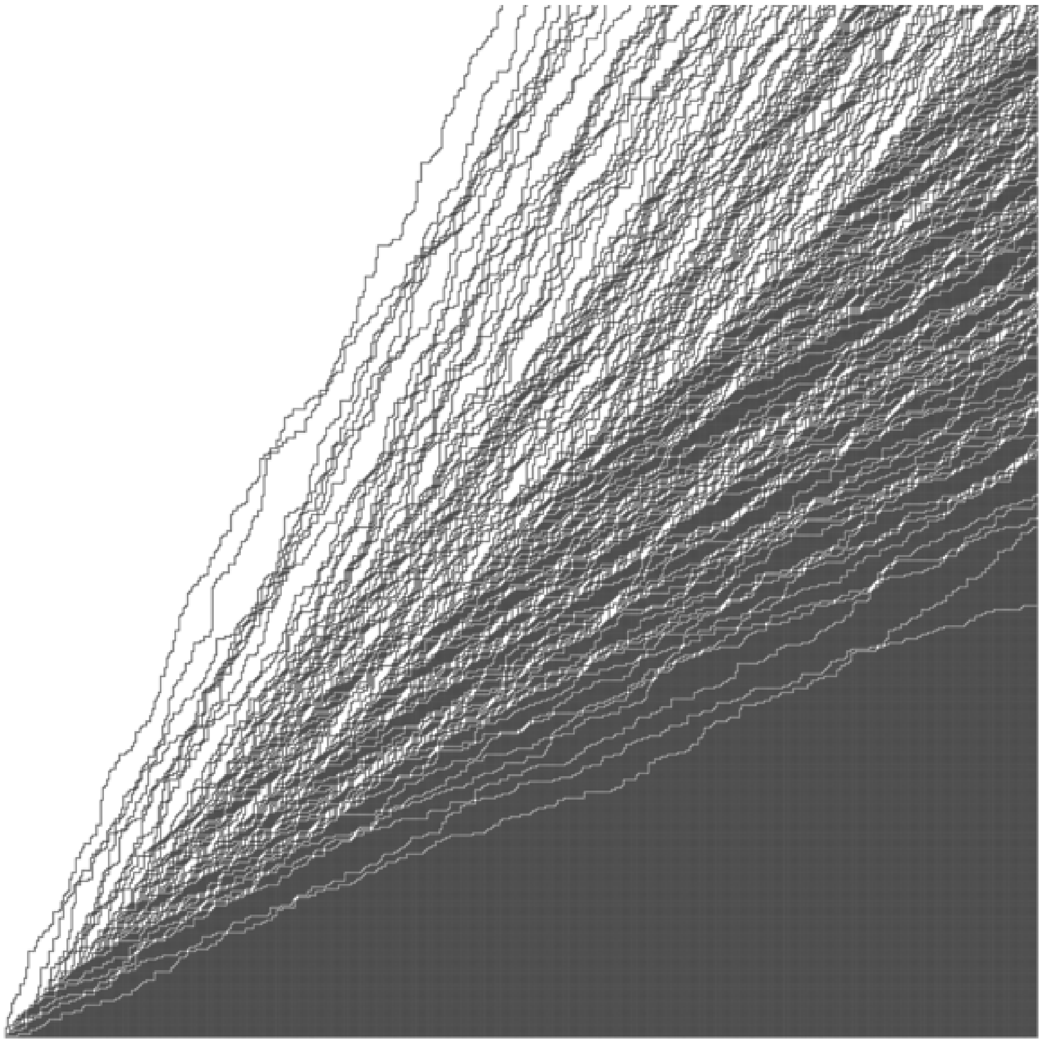
$$\sum_{\lambda=(\lambda_1\geq\dots\geq\lambda_N\geq 0)} F_{\lambda}(u_1,\dots u_N)\cdot \frac{c(\lambda)}{c(0^N)}\, G_{\lambda}(v_1,\dots,v_M) = \prod_{i=1}^{N} \frac{1-q^{i}}{1-su_i}\cdot \prod_{i,j} \frac{1-qu_iv_j}{1-u_iv_j}$$

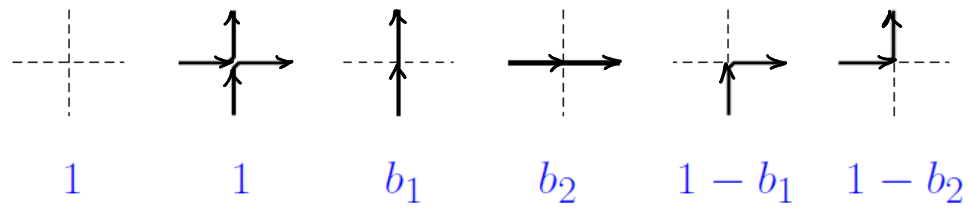


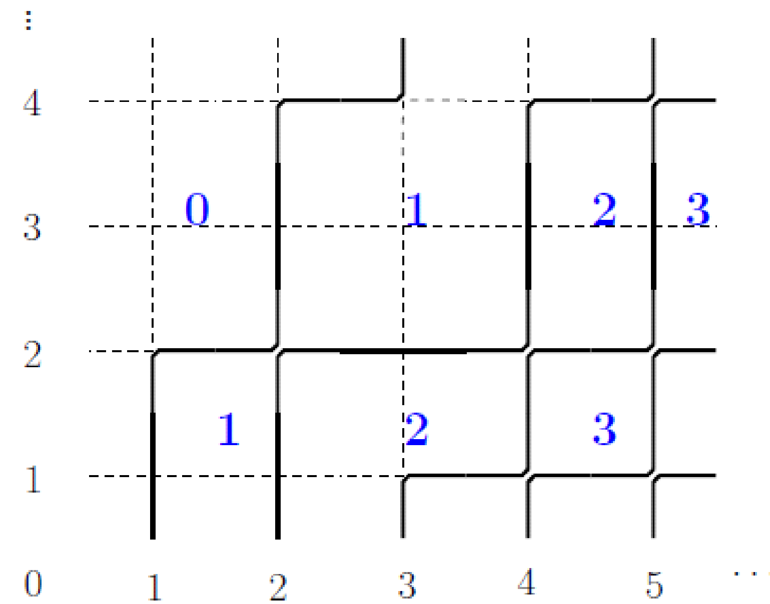




Stochastic six vertex model on $\mathbb{Z}$







Courtesy of Leo Petrov

Stochastic six vertex model on $\mathbb{Z}$

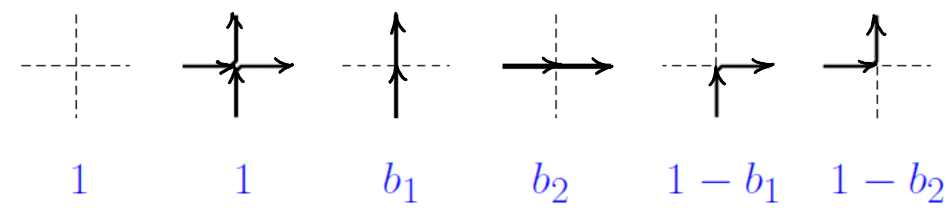
Theorem [B-Corwin-Gorin 2014]

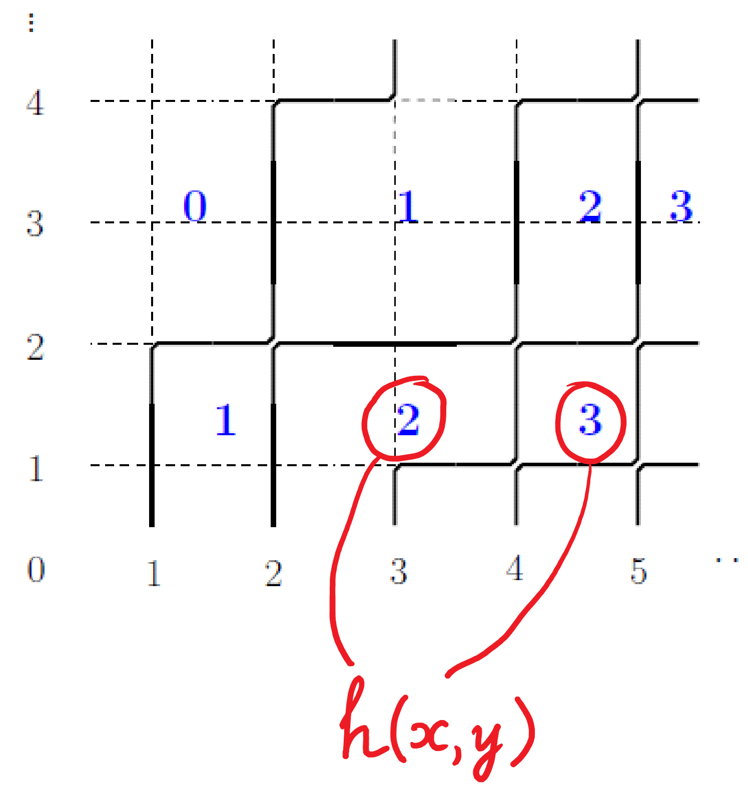
Assume $b_1 > b_2$. Then for $\frac{1-b_1}{1-b_2} < \frac{x}{y} < \frac{1-b_2}{1-b_1}$

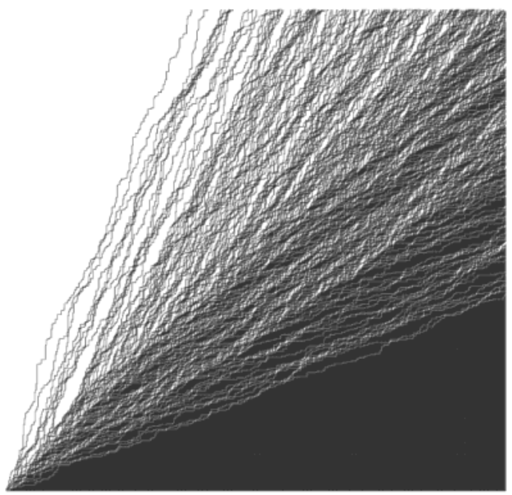
$$\lim_{L\to\infty} \mathbb{P}\left\{ \frac{h(Lx,Ly) - L\cdot H(x,y)}{L^{1/3}\cdot \sigma(x,y)} \le -s \right\} = F_{GUE}(s)$$

where $H(x,y) = \frac{\left(\sqrt{x(1-b_2)} - \sqrt{y(1-b_1)}\right)^2}{x-y}$, $\sigma(x,y)$ is explicit,

$F_{GUE}(s)$ is the GUE Tracy-Widom distribution.





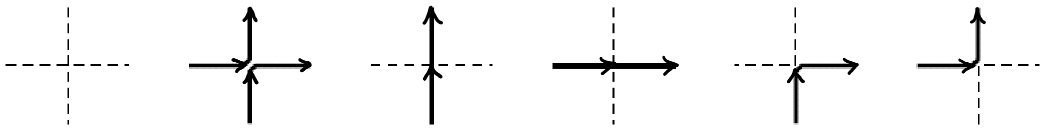


[Gwa-Spohn 1992]:

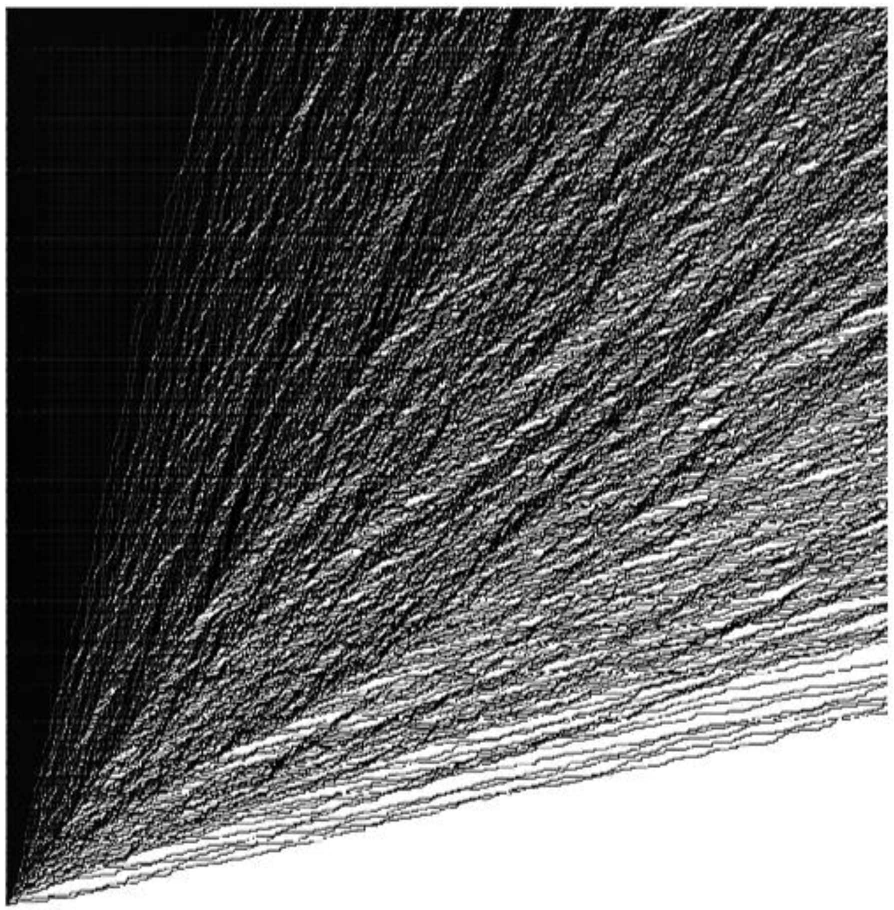
The stochastic six vertex model is a member of the KPZ universality class. This class was related to TW distributions in the late 1990's.

Colored (higher rank) models

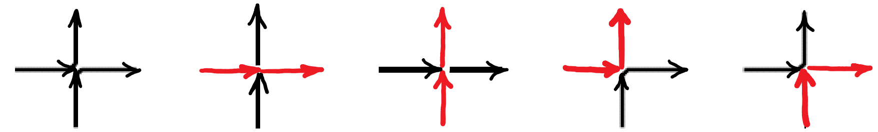
Stochastic six vertex model



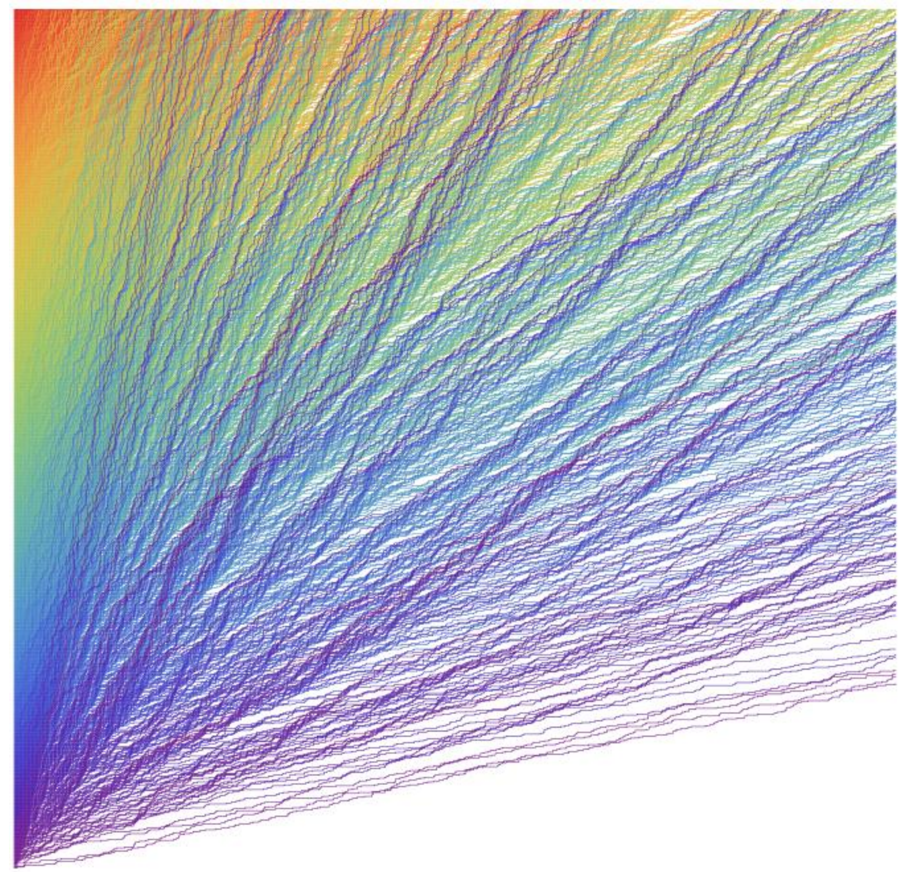
1 $\quad 1$ $\quad b_1$ $\quad b_2$ $\quad 1-b_1$ $\quad 1-b_2$



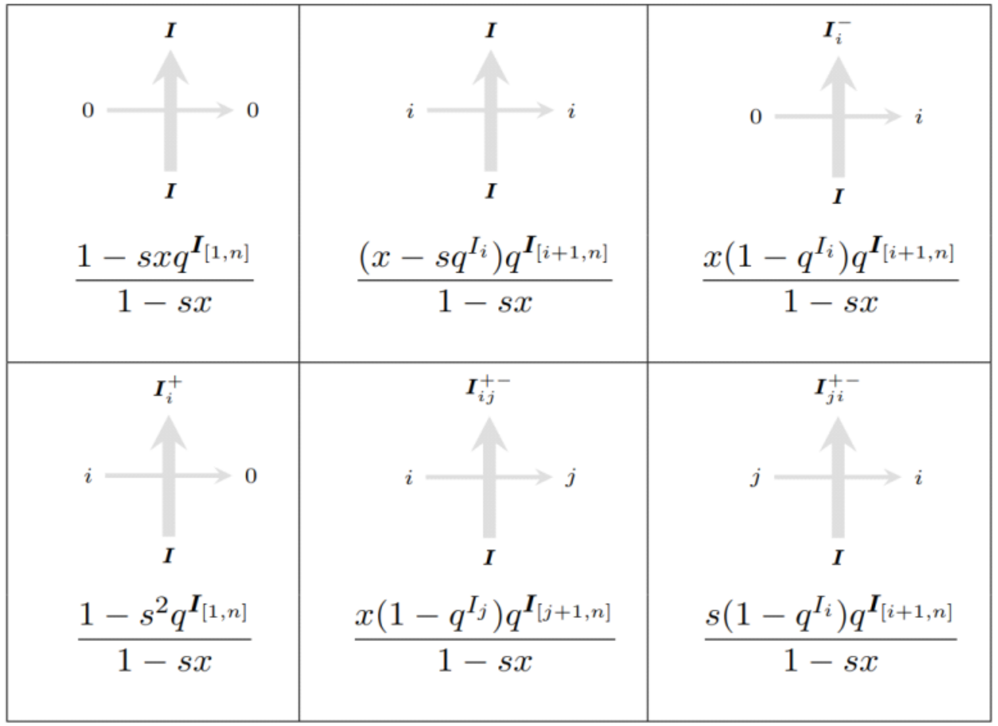
Colored stochastic six vertex model



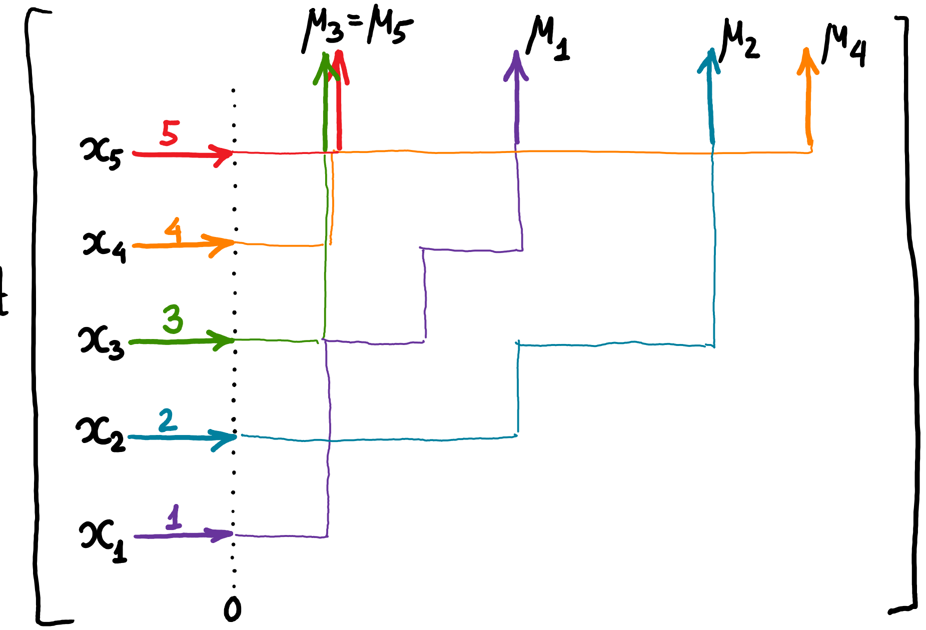
1 $\quad b_1$ $\quad b_2$ $\quad 1-b_1$ $\quad 1-b_2$



Nonsymmetric spin HL functions

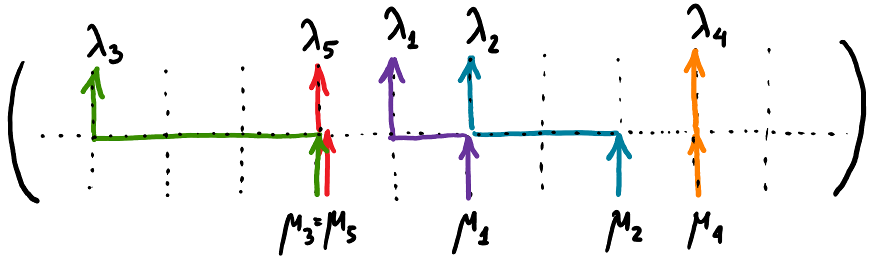


$f_\mu(x_1, \dots, x_5) = \text{weight}$



This is a complete basis of eigenfunctions of a transfer-matrix

$X_{\mu/\lambda}(u) = \text{weight}_u$

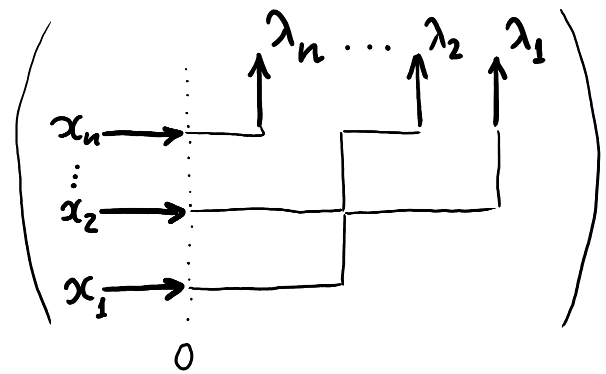


[B-Wheeler, 2018]

Nonsymmetric spin HL functions

Color-blindness

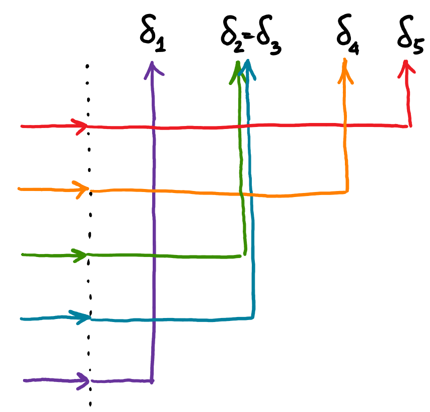
$$\sum_{\mu:\mu^+=\lambda} f_\mu(x_1,\dots,x_n) = \frac{c(\lambda)}{c(0)} \cdot F_\lambda(x_1,\dots,x_n) \sim \text{weight}$$



$(\ )^+$ is ordering the coordinates

Factorization for anti-dominant indices $\delta = (\delta_1 \leq \dots \leq \delta_n)$

$$f_\delta(x_1,\dots,x_n) = \frac{\prod_{j\geq 0}(s^2;q)_{m_j(\delta)}}{\prod_{i=1}^{n}(1-sx_i)} \cdot \prod_{i=1}^{n}\left(\frac{x_i - s}{1-sx_i}\right)^{\delta_i}$$



Unique path configuration

AHA exchange relations

$$T_i f_\mu(x_1,\dots,x_n) = f_{s_i\circ\mu}(x_1,\dots,x_n)$$

for $\mu_i < \mu_{i+1}$, $s_i = (i\ i{+}1)$

$$T_i = q - \frac{x_i - q x_{i+1}}{x_i - x_{i+1}}(1-s_i), \qquad 1 \leq i \leq n-1$$

$$(T_i - q)(T_i + 1) = 0$$

[B-Wheeler, 2018]

Nonsymmetric spin HL functions

Relation to off-shell nested Bethe vectors

Under the specialization

$$x_1^{(1)} = x_1^{(2)} = \ldots = x_1^{(n)} = x_1$$

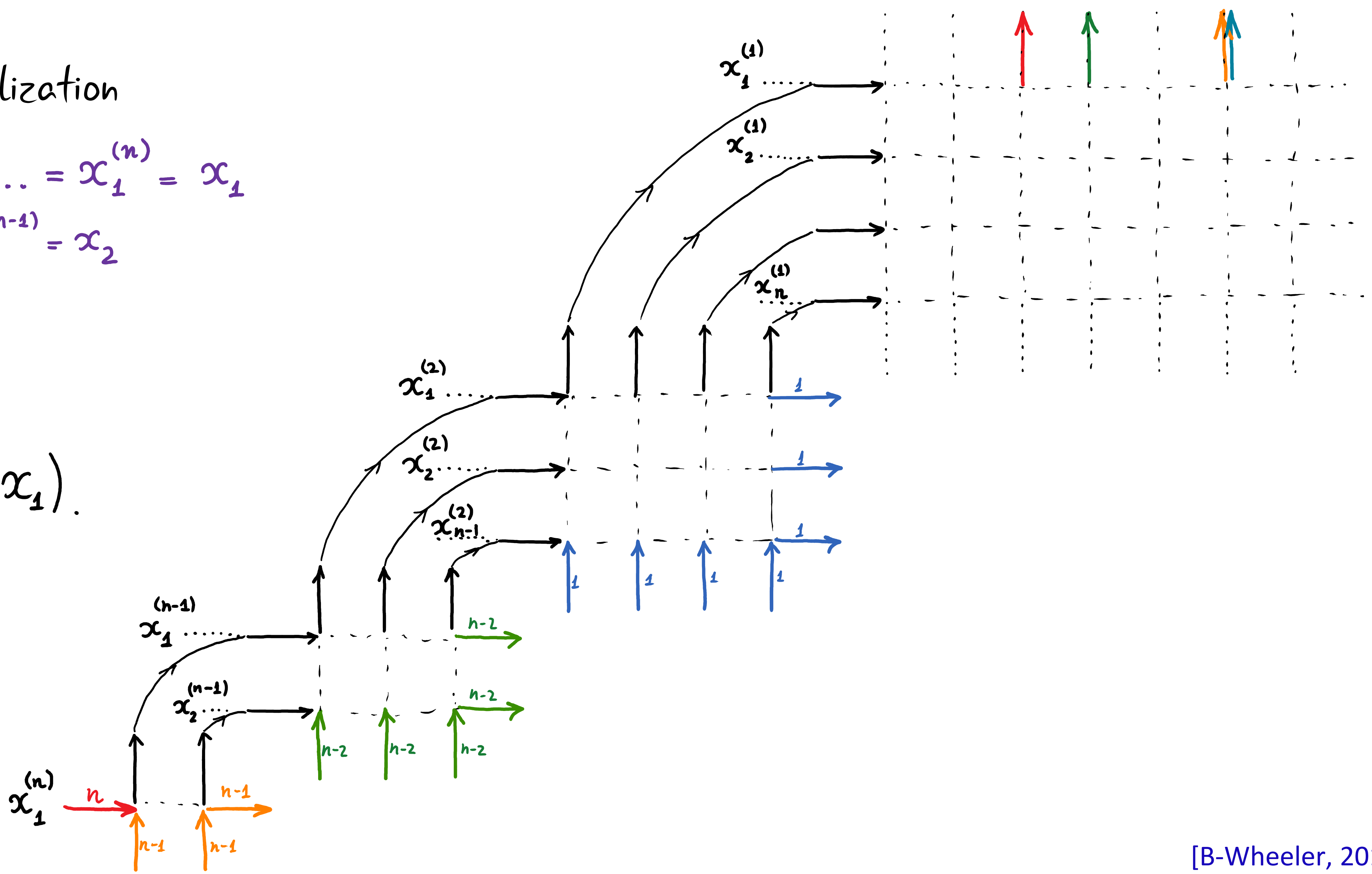
$$x_2^{(1)} \quad \ldots = x_2^{(n-1)} = x_2$$

$$\ldots$$

$$x_n^{(1)} = x_n$$

one obtains

$$f_\mu(x_n, \ldots, x_1).$$



[B-Wheeler, 2018]

Nonsymmetric spin HL functions

Cauchy type summation identity

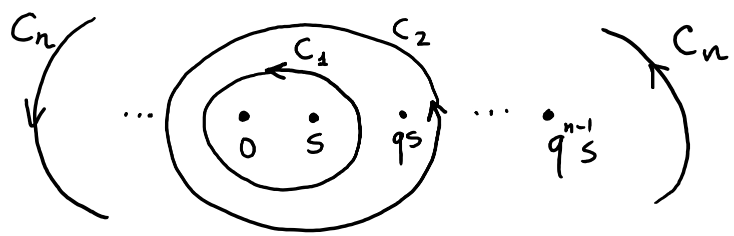
$$\sum_{\mu\geq 0} f_\mu(x_1,\dots,x_n)\, g^*_\mu(x_1,\dots,x_n) = \prod_{i=1}^{n}\frac{1}{1-x_i y_i}\cdot\prod_{i>j}\frac{1-q x_i y_j}{1-x_i y_j}$$

$$g^*_\mu(x_1,\dots,x_n;s,q) = const\cdot f_{\tilde{\mu}}(x_n^{-1},\dots,x_1^{-1};s^{-1},q^{-1})$$

$\tilde{\mu}=(\mu_n,\dots,\mu_1)$

Orthogonality

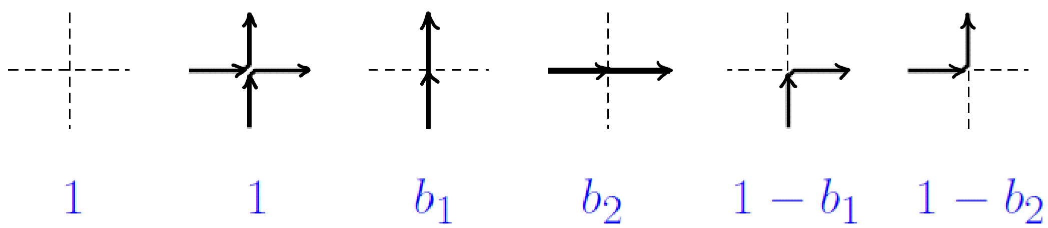
$$\frac{1}{(2\pi i)^n}\oint_{C_1}\cdots\oint_{C_n}\prod_{i<j}\frac{x_j-x_i}{x_j-q x_i}\, f_\mu(x_1^{-1},\dots,x_n^{-1})\, g^*_\nu(x_1,\dots,x_n)\, dx = \mathbb{1}_{\mu=\nu}$$

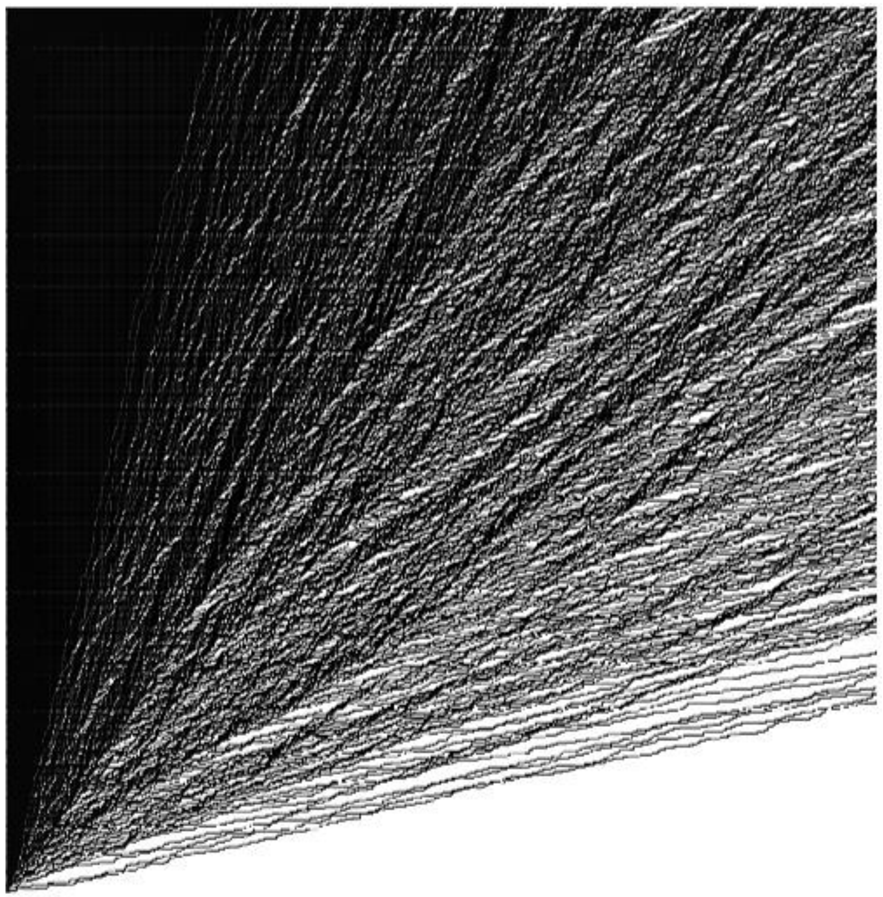


[B-Wheeler, 2018]

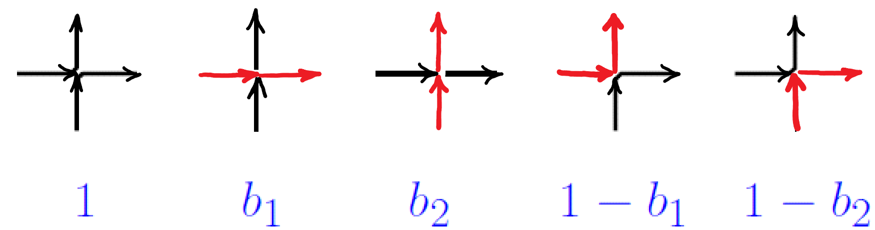
A result about colored stochastic vertex models

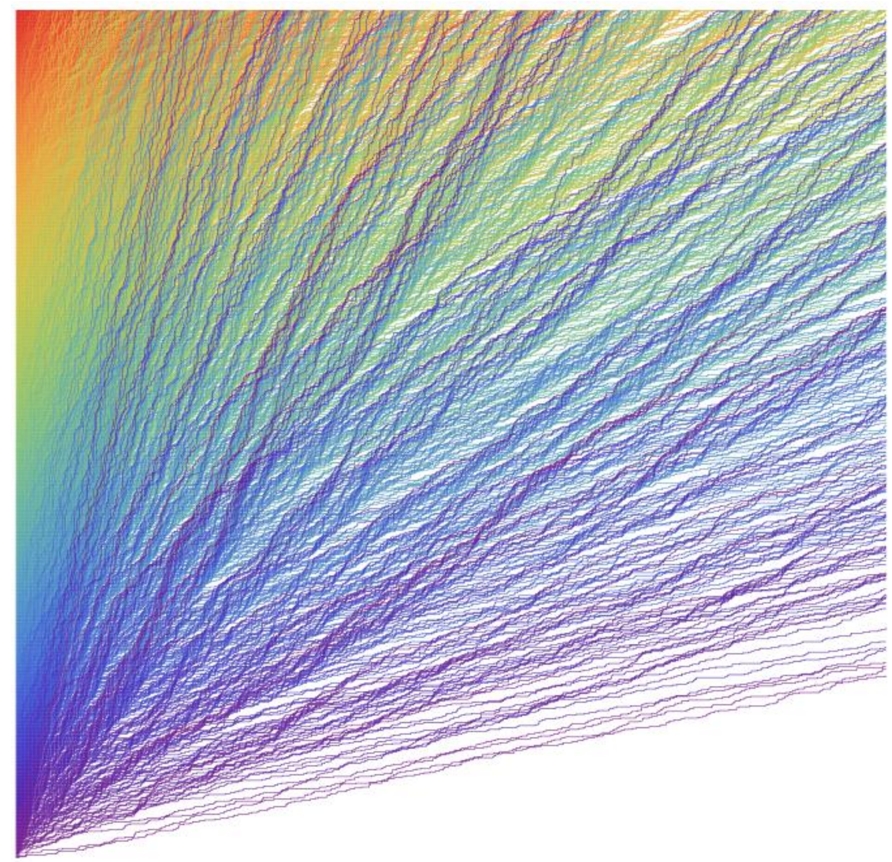
Stochastic six vertex model





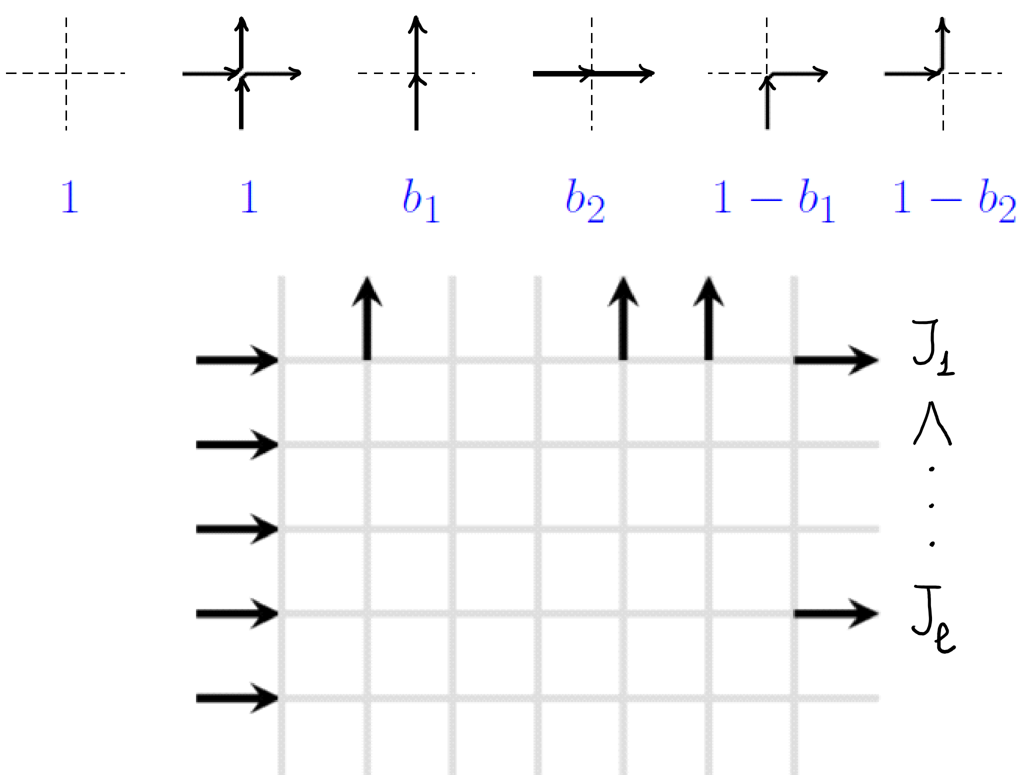
Colored stochastic six vertex model



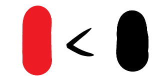


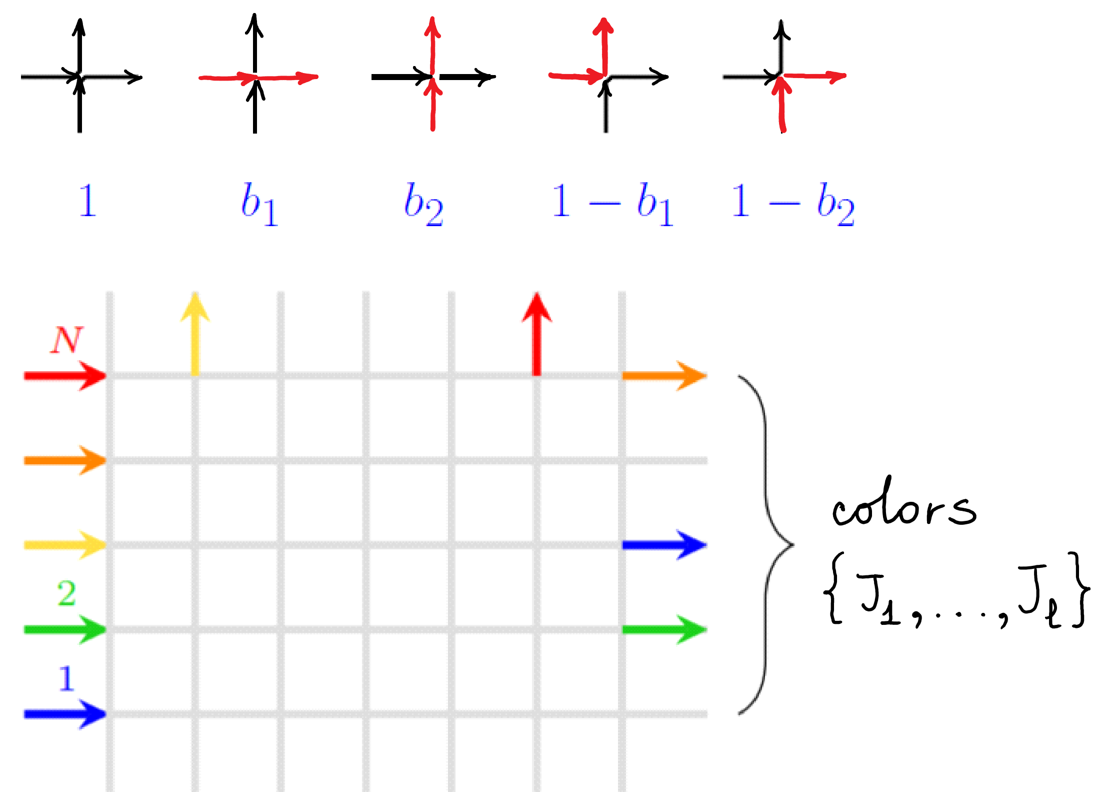
A result about colored stochastic vertex models

Stochastic six vertex model



Colored stochastic six vertex model





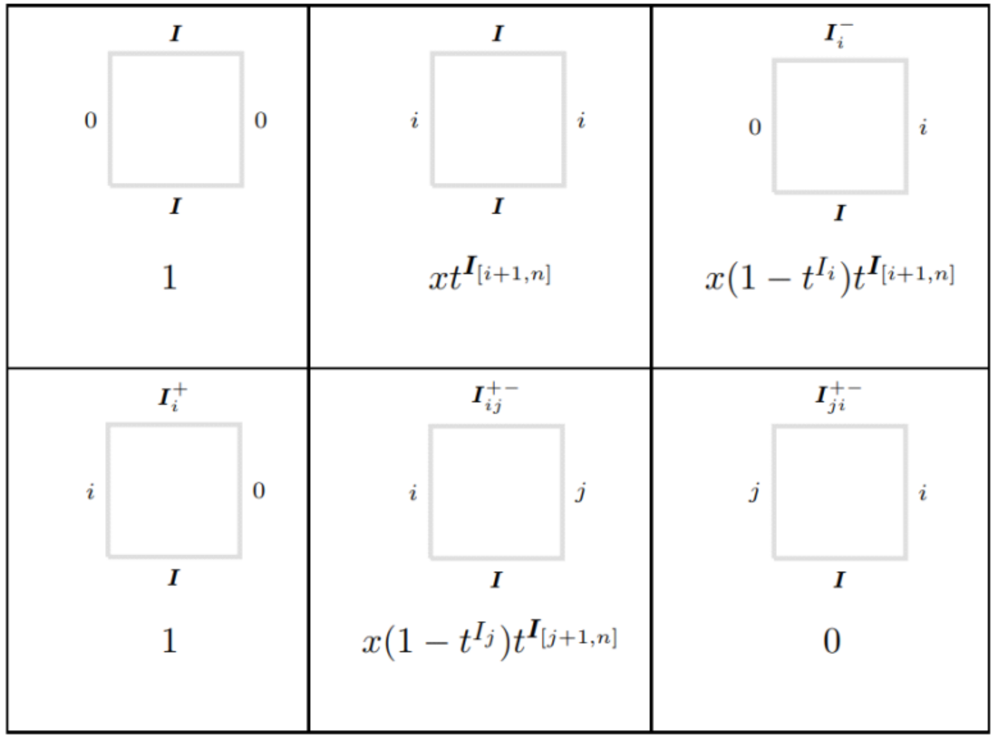
<u>Theorem</u> For any set $\{J_1,\ldots,J_\ell\}$ the following two probabilities coincide:

(a) In the color-blind model, paths exit on the right exactly at those positions;

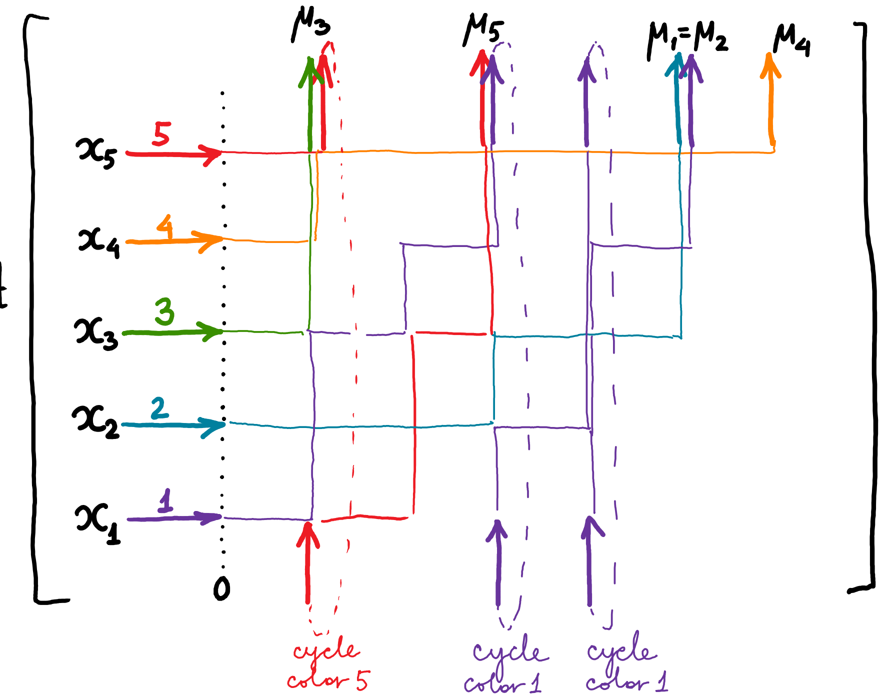
(b) In the colored model, paths exiting on the right have exactly these colors.

Also works for inhomogeneous and fused models. [B-Wheeler 2018]

Nonsymmetric Macdonald polynomials



$f_\mu(x_1, \ldots, x_5) = \text{weight}$



These are the same vertex weights with s=0 and q replaced by t.

<u>Theorem</u> [B-Wheeler, 2019] If each cycle of color i at position j carries the additional factor of

$$q^{\mu_i - j} \, t^{-\#\{k<i \,:\, \mu_k > \mu_i\} + \#\{k>i \,:\, j \le \mu_k < \mu_i\}}$$

then the partition function equals the nonsymmetric Macdonald polynomial indexed by μ, up to an explicit multiplicative constant.