Stable envelopes for A_n, $\widehat{A}_n$-quiver varieties

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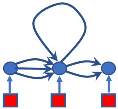
Elliptic sable envelopes

- Elliptic stable envelopes are introduced by M.Aganagic and A.Okounkov in "Elliptic stable envelope" arXiv: 1804.08779.
- It is a generalization of earlier constructions by D. Maulik and A. Okounkov "Quantum groups and quantum cohomology" arXiv:1211.1287 to the level of elliptic cohomology.
- Stabele envelopes in K-theory and Cohomology are limits of elliptic stable envelope.

Overview



$\big\downarrow \mathrm{Stab}^{Ell}$

$E_{\hbar,\tau}(\widehat{\mathfrak{g}})$ Elliptic dynamical quantum loop group

$\big\downarrow \mathrm{Stab}^{K-th}$

$\mathscr{U}_{\hbar}(\widehat{\mathfrak{g}})$ Quantum loop group

$\big\downarrow \mathrm{Stab}^{Coh}$

$Y_{\hbar}(\mathfrak{g})$ Yangian

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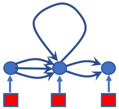

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monodromy of qKZ

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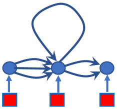

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Enumerative geometry, QH_{T} QK_{T}

quantum integrable system (N. Nekrasov, S. Shatashvily, A. Okounkov, D. Maulik,...)

Stable envelope = Bethe vectors

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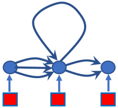

Symplectic duality

$X^\vee$

$$\mathrm{Stab}^{Ell}_X = \mathrm{Stab}^{Ell}_{X^\vee}$$

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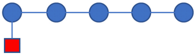
Known cases of elliptic stable envelope

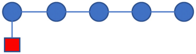


$\mathscr{U}_\hbar(\widehat{\mathfrak{gl}}_N)$

$\mathbb{C}^N(u_1) \otimes \cdots \otimes \mathbb{C}^N(u_n)$

[P. Etingof, A. Varchenko, H. Konno,....]
"Elliptic weight functions",
in integral representations of
solutions of qKZ-equations

New formulas



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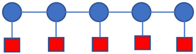
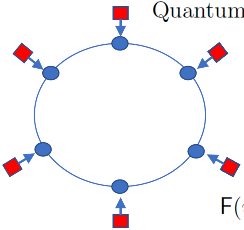
Goal of talk



$\mathscr{U}_\hbar(\widehat{\mathfrak{gl}}_N)$ $V_{\omega_1}(u_1)\otimes\cdots\otimes V_{\omega_m}(u_m)$



Quantum toroidal algebras

$\mathscr{U}_{t,q}(\widehat{\widehat{\mathfrak{gl}}}_N)$

Fock modules

$\mathsf{F}(u_1)\otimes\cdots\otimes\mathsf{F}(u_n)$

Plan

- Elliptic Cohomology
- Elliptic stable envelope
- Existence for quiver varieties
- Abelianization formula for $\widehat{A}_n$-type quivers
- Example: application to $\mathrm{Hilb}^n(\mathbb{C}^2)$.
  "Elliptic Macdonald polynomial")

Let us fix an elliptic curve

$$E = \mathbb{C}^\times / q^{\mathbb{Z}} \quad \textit{with} \quad 0 < |q| < 1.$$

For an algebraic torus T let (we assume X has no odd cohomology)

$$\mathrm{Ell}_{\mathsf{T}} : \{\mathsf{T} - \mathrm{spaces}\, X\} \longrightarrow \{\mathrm{schemes}\}$$

be the corresponding elliptic cohomology functor such that $\mathrm{Ell}_{\mathbb{C}^\times}(pt) = E$. the covarince in T implies that

$$\mathscr{E}_{\mathsf{T}} := \mathrm{Ell}_{\mathsf{T}}(\mathrm{pt}) = \mathsf{T}/q^{\mathrm{cochar}(\mathsf{T})} \cong E^{\,\mathsf{dim}(\mathsf{T})}.$$

Local description of $\mathrm{Ell}_{\mathsf{T}}(X)$

The canonical projection

$$X \to pt, \implies \pi : \mathrm{Ell}_{\mathsf{T}}(X) \to \mathscr{E}_{\mathsf{T}}$$

Let $t \in \mathscr{E}_{\mathsf{T}}$ and U_t be a small analytic neighborhood of t, which is isomorphic via exponential map to a small analytic neighborhood in $\mathrm{Lie}(\mathsf{T}) = \mathbb{C}^{\dim \mathsf{T}}$. Locally, the map π looks as follows:

$$\begin{tikzcd}
\mathrm{Spec} H^{\bullet}_{\mathsf{T}}(X^{\mathsf{T}_t}, \mathbb{C}) \arrow[d] & (\pi)^{-1}(U_t) \arrow[l] \arrow[r] \arrow[d] & \mathrm{Ell}_{\mathsf{T}}(X) \arrow[d, "\pi"] \\
\mathbb{C}^{\dim \mathsf{T}} & U_t \arrow[l] \arrow[r] & \mathscr{E}_{\mathsf{T}}.
\end{tikzcd}$$

for

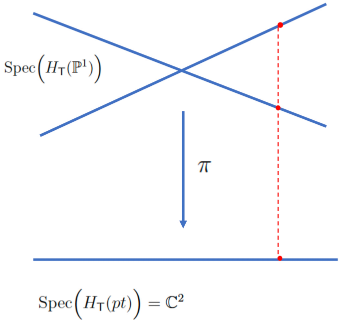
$$\mathsf{T}_t := \bigcap_{\substack{\chi \in \mathrm{char}(\mathsf{T}), \\ \chi(t)=0}} \ker \chi \subset \mathsf{T},$$

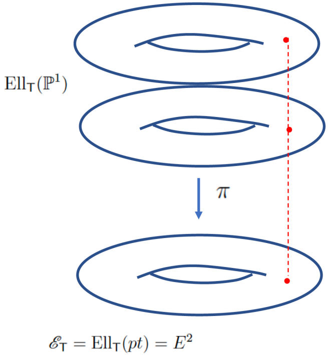
Example $\mathrm{Ell}_\mathsf{T}(\mathbb{P}^1)$

Take $\mathsf{T} = (\mathbb{C}^\times)^2$ act on $\mathbb{P}^1$ by $[x : y] \to [xa_1 : ya_2]$.

$$H_\mathsf{T}(\mathbb{P}^1) = \mathbb{C}[c, a_1, a_2]/(c - a_1)(c - a_2)$$

thus





Elliptic cohomology for finite X^{T}

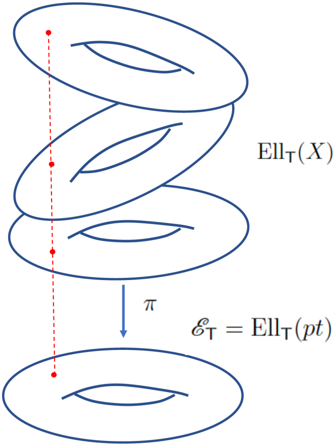
In general the elliptic cohomology is a union of $|X^{\mathsf{T}}|$ copies

$$\mathrm{Ell}_{\mathsf{T}}(X) = \left(\coprod_{p \in X^{\mathsf{T}}} \mathrm{O}_p \right) / \Delta$$

with intersection data Δ and

$$\mathrm{O}_p = \mathscr{E}_{\mathsf{T}} = \underbrace{E \times E \times \cdots \times E}_{\dim \mathsf{T}}$$

The coordinates in $\mathscr{E}_{\mathsf{T}}$ are called *equivariant parameters*.



Extension of $\mathrm{Ell}_{\mathsf{T}}(X)$ by Kähler parameters

We "extend" elliptic cohomology:

$$\mathsf{E}_{\mathsf{T}}(X) = \mathrm{Ell}_{\mathsf{T}}(X) \times \mathscr{E}_{\mathrm{Pic}_{\mathsf{T}}(X)},$$

by abelian variety:

$$\mathscr{E}_{\mathrm{Pic}_{\mathsf{T}}(X)} = \mathrm{Pic}_{\mathsf{T}}(X) \otimes E$$

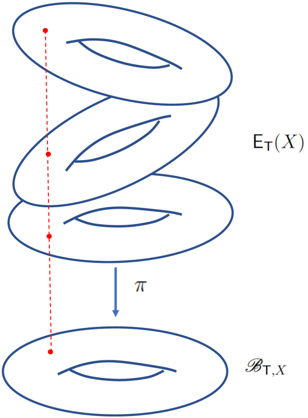
This scheme has the same description:

$$\mathsf{E}_{\mathsf{T}}(X) = \left(\coprod_{p \in X^{\mathsf{T}}} \widehat{\mathsf{O}}_p \right) / \Delta$$

with

$$\widehat{\mathsf{O}}_p = \mathscr{B}_{\mathsf{T},X} = \underbrace{E \times \cdots \times E}_{\dim \mathsf{T}} \times \underbrace{E \times \cdots \times E}_{\dim \mathrm{Pic}_{\mathsf{T}}(X)}$$

The coordinates in $\mathscr{E}_{\mathrm{Pic}_{\mathsf{T}}(X)}$ are called *Kähler parameters*.



Elliptic stable envelope

The elliptic stable envelope is a map of $\mathscr{O}_{\mathscr{B}_{X,T}}$-modules:

$$\underbrace{\Theta(T^{1/2}X^{A})}_{\text{line bundle on } \mathrm{Ell}_T(X^A)\otimes \mathscr{E}_{Pic_T(X)}} \otimes \mathscr{U}' \otimes \Theta(\hbar)^{-codim(X^A)/2} \longrightarrow \underbrace{\Theta(T^{1/2}X)}_{\text{line bundle on } E_T(X)=\mathrm{Ell}_T(X)\otimes \mathscr{E}_{Pic_T(X)}} \otimes \mathscr{U}$$

In the case of fine X^T the stable envelope of $p \in X^T$ is a section of a line bundle on $E_T(X)$:

$$Stab_\sigma(p) \text{ is a section of } \underbrace{\mathscr{L} = \Theta(T^{1/2}X) \otimes \mathscr{U} \otimes (\mathscr{U}')^{-1} \otimes \Theta(\hbar)^{dim/2}}_{\text{a line bundle over } E_T(X)}$$

Elliptic stable envelopes of fixed points

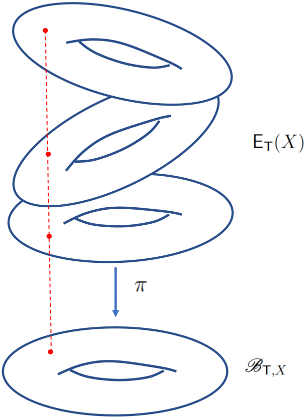
$Stab_\sigma(p)$ is a section of

$$\underbrace{\mathscr{L}_p = \Theta(T^{1/2}X) \otimes \mathscr{U} \otimes (\mathscr{U}')^{-1} \otimes \Theta(\hbar)^{dim/2}}_{\text{a line bundle over } E_{\mathsf{T}}(X)}$$

Thus, $Stab_\sigma(p)$ is a collection of sections

$$Stab_\sigma(p) = \{s_q(p), \quad q \in X^{\mathsf{T}}\}$$

of line bundles over abelian varieties $\widehat{\mathrm{O}}_q$ which agree on the intersection Δ.



Such sections can be described in terms of theta-functions associated with elliptic curve E:

$$\vartheta(x) = \prod_{i=1}^{\infty}(1 - q^i/x)(x^{1/2} - x^{-1/2}) \prod_{i=1}^{\infty}(1 - q^i x)$$

Uniqueness of elliptic envelope

A cocharacter σ of $\mathsf{A} = ker(\hbar) \subset \mathsf{T}$ provides decomposition of tangent spaces

$$T_pX = N_p^+ \oplus N_p^-$$

characters of N_p^+, N_p^- are positive and negative on σ respectively. Attracting subset set

$$\mathrm{Attr}_\sigma(p) = \{x : \lim_{z \to 0} \sigma(z) \cdot x = p\} \subset X$$

The section $Stab_\sigma(p)$ is uniquely defined by conditions:

- $Stab_\sigma(p)$ is supported on $Attr_\sigma^f(p)$.
- $Stab_\sigma(p)|_{\hat{O}_p} = \Theta(N_p^-) = \prod_{w \in \mathrm{char}(N_p^-)} \vartheta(w).$

Existence for quiver varieties

A quiver variety:

$$X = T^*V /\!/\!/\!/ G = \mu^{-1}_{\mathbb{C},G}(0) \cap \mu^{-1}_{\mathbb{R}}(\eta) /\!/ U$$

for maximal compact subgroup $U \subset G$.
The abelianization of X is a hypertoric variety:

$$AX = T^*V /\!/\!/\!/ T = \mu^{-1}_{\mathbb{C},T}(0) \cap \mu^{-1}_{\mathbb{R}}(\eta) /\!/ S$$

where T is a maximal torus of G and $S = T \cap U$.

Fact: the elliptic stable envelope for hypertoric varieties are simple to describe explicitly.

Stable envelopes for hypertoric varieties

Hypertoric variety:

$$X = T^*V /\!\!/\!\!/\!\!/ T, \quad T = (\mathbb{C}^\times)^n$$

X^{G} is finite set. A fixed point $\mathsf{t} \in X^{\mathsf{G}}$ is given by decomposition

$$V = V_0 \oplus V_1, \quad \dim V_0 = \dim T$$

so that $\mathsf{t} = T^*V_0 /\!\!/\!\!/\!\!/ T$. Tautological bundles $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ over X.

Theorem (Aganagic, Okounkov)

The elliptic stable envelopes of fixed points for hypertoric varieties equal:

$$Stab_\sigma(t) = \prod_{i=1}^{\dim V} \vartheta(x_i) \prod_{i=1}^{\dim V_0} \phi(x_i \alpha_i, z_i), \quad \phi(x,z) = \frac{\vartheta(xz)}{\vartheta(x)\vartheta(z)}$$

where α_i denote the G*-weights on V_0 and z_i Kähler parameters.*

Abelianization of quiver varieties

The abelianization fits into the diagram:

$$\begin{tikzcd} FL \arrow[r, "\mathsf{J}+"] \arrow[d, "\pi"] & \mu_{\mathbb{C}}^{-1}(b^{\perp})/\!\!/S \arrow[r, "\mathsf{J}-"] & AX \\ X=\mu_{\mathbb{C}}^{-1}(0)\cap\mu_{\mathbb{R}}^{-1}(\eta)/\!\!/U & & \end{tikzcd}$$

with $FL=\mu_{\mathbb{R}}^{-1}(0)/S$ so the fibers of π are flag varieties $G/B\cong U/S$.

There is similar diagram for abelianization of $\lambda\in X^{\mathsf{T}}$:

$$\begin{tikzcd} FL' \arrow[r, "\mathsf{J}'_+"] \arrow[d, "\pi'"] & \tilde{\mu}_{\mathbb{C}}^{-1}(b^{\perp})/\!\!/S \arrow[r, "\mathsf{J}'_-"] & A\{\lambda\} \\ \{\lambda\}=\tilde{\mu}_{\mathbb{C}}^{-1}(0)\cap\tilde{\mu}_{\mathbb{R}}^{-1}(\eta)/\!\!/U & & \end{tikzcd}$$

Abelianization of elliptic stable envelope

The map Stab_σ defined as a composition:

$$\begin{tikzcd} \mathscr{O}_{\mathscr{B}_{X,T}} \arrow[rr, "j'_{-*}\circ(j'^{*}_{+})^{-1}\circ\pi'^{-1}_{*}"] \arrow[d, "\mathrm{Stab}_\sigma"] & & \mathscr{L}(A\{\lambda\}) \arrow[d, "\mathrm{Stab}'_\sigma"] \\ \mathscr{L}(X) & & \mathscr{L}(AX) \arrow[ll, "\pi_*\circ j^*_+\circ(j_{-*})^{-1}"'] \end{tikzcd}$$

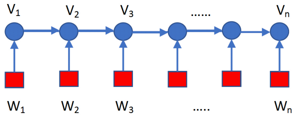
satisfies the defining properties of elliptic stable envelope.
Thus, Stab_σ exists for quiver varieties.
To compute Stab_σ one needs:
$(\star)$ find a section *Reprs* of $\mathscr{L}(A\{\lambda\})$ such that

$$\pi'_* \circ j'^{*}_{+} \circ (j'_{-*})^{-1}(Reprs) = 1.$$

$(\star,\star)$ Compute $\mathrm{Stab}'_\sigma(s)$.

Dimension vectors:

$$\mathbf{v} = (\mathrm{v}_1, \dots, \mathrm{v}_n), \quad \mathbf{w} = (\mathrm{w}_1, \dots, \mathrm{w}_n)$$



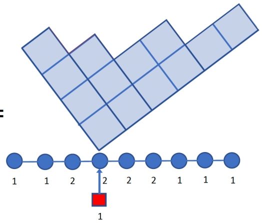
Then the fixed points are product of zero-dimensional quiver varieties with one-dimensional framings:

$$X(\mathbf{v}, \mathbf{w})^{\mathsf{T}} = \coprod_{\mathbf{v}_1 + \dots + \mathbf{v}_m = \mathbf{v}} X(\mathbf{v}_1, \delta_i) \times \cdots \times X(\mathbf{v}_m, \delta_i)$$

Zero-dimensional quiver varieties $X(\mathbf{v}_m, \delta_i)$ are labeled by Young diagrams λ.

Example: Zero-dimensional A_9-quiver varieties with $\mathrm{v} = (1, 1, 2, 2, 2, 2, 1, 1, 1)$ and $\mathrm{w} = \delta_4$:

$\{\lambda\} = X(\mathrm{v}, \delta_4) =$

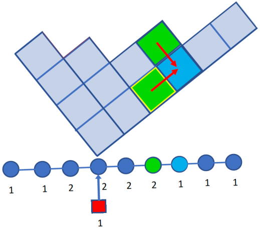


Abelianization of zero-dimensional quiver varieties

Example of zero-dimensional quiver variety:

$$\{\lambda\} = T^*\Big(V_0 \bigoplus_i Hom(V_i, V_{i+1})\Big) /\!\!/\!\!/\!\!/ \prod_i GL(V_i)$$

$$Hom({\color{green}V}, {\color{blue}V}) \cong \mathbb{C}^2$$



Abelianization of $\{\lambda\}$ is a hypertoric variety:

$$A\{\lambda\} = T^*\Big(V_0 \bigoplus_i Hom(V_i, V_{i+1})\Big) /\!\!/\!\!/\!\!/ \prod_i (\mathbb{C}^\times)^{\mathsf{v}_i}$$

In general it is nontrivial:

$$\dim A\{\lambda\} = 2(\#arrows - \#boxes) \geq 0$$

For a hypertoric variety

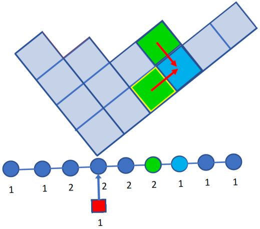
$$A\{\lambda\} = T^*V /\!\!/\!\!/\!\!/ S$$

A fixed point $p \in A\{\lambda\}^{\mathsf{T}}$ corresponds to a splitting:

$$V = V_0 \oplus V_1, \quad \dim V_0 = \dim S,$$

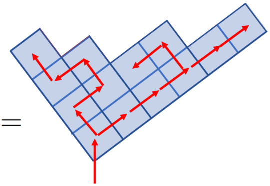
so that

$$p = T^*V_0 /\!\!/\!\!/\!\!/ S$$



There exist a set of fixed points in hypertoric variety $A\{\lambda\}^{\mathsf{T}}$ labeled by trees:

t =



Abelianization of stable envelope

Solution of problem $(\star)$:

Theorem (arXiv: 1804.08779)

Let $A\{\lambda\}$ be a hypertoric variety corresponding to abelianization of $\{\lambda\}$. Then there exists a subset of fixed points $\Upsilon_\lambda \subset A\{\lambda\}^{\mathsf{T}}$ corresponding to trees in the Young diagram λ and a cocharacter $\rho \in cochar(\mathsf{T})$ so that

$$Reprs := \alpha \sum_{tree \in \Upsilon_\lambda} Stab_\rho(tree)$$

satisfies the property

$$\pi'_* \circ \jmath'^*_+ \circ (\jmath'_{-*})^{-1}(Reprs) = 1$$

Solution of problem $(\star, \star)$ is by triangle Lemma:

$$Stab_\sigma(Reprs) = Stab_\sigma\Big(\sum_{\text{tree} \in \Upsilon_\lambda} Stab_\rho(\text{tree})\Big) = \sum_{\text{tree} \in \Upsilon_\lambda} Stab_{\rho+\sigma}(\text{tree})$$

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Theorem (Aganagic, Okounkov)

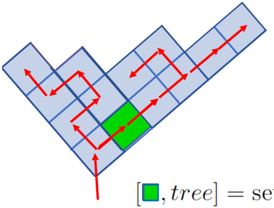
The elliptic stable envelopes of fixed points for hypertoric varieties equal:

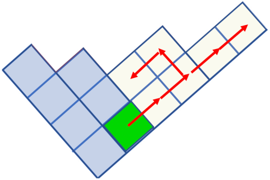
$$Stab_\rho(t) = \prod_{i=1}^{\dim V} \vartheta(x_i) \prod_{i=1}^{\dim V_0} \phi(x_i \alpha_i, z_i), \quad \phi(x,z) = \frac{\vartheta(xz)}{\vartheta(x)\vartheta(z)}$$

where α_i denote the G*-weights on V_0 and z_i Kähler parameters.*

Combinatorics of trees

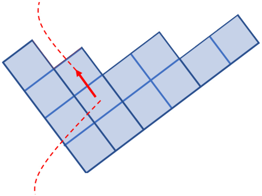
Trees in Young diagrams and subtrees:





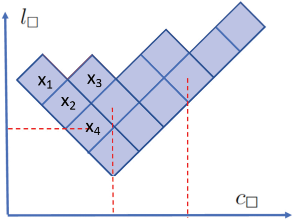
$[\blacksquare, tree] =$ set of boxes above $\blacksquare$ in the tree

$h(e)$ – "head" box of edge e



$t(\square)$ – "tail" box of edge e

Content and height of a box in a Young diagram



Stable envelope of a tree

Theorem (arXiv: 1804.08779)

The elliptic stable envelope of a **tree** $\in A\{\lambda\}^{\mathsf{T}}$ *in the hypertoric variety* $A\{\lambda\}$ *equals*

$$Stab_{\rho}(tree) = S_{tree} W_{tree}$$

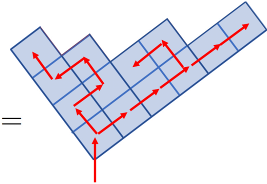
with

$$S_{tree} = \prod_{\substack{c_j = c_i + 1 \\ l_i > l_j}} \vartheta(x_i/x_j\hbar) \prod_{\substack{c_j = c_i + 1 \\ l_i < l_j}} \vartheta(x_j/x_i) \prod_{c_i = 0} \vartheta(x_i)$$

and

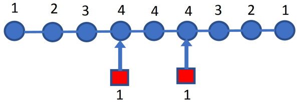
$$W_{tree} = \phi(x_r, \prod_{i=1}^{n} z_i) \prod_{e \in tree} \phi\Big(\frac{x_{h(e)}\varphi^{\lambda}_{t(e)}}{x_{t(e)}\varphi^{\lambda}_{h(e)}}, \prod_{i \in [h(e), tree]} z_i\Big)$$

where x_i *are the Grothendieck roots of tautological bundles.*

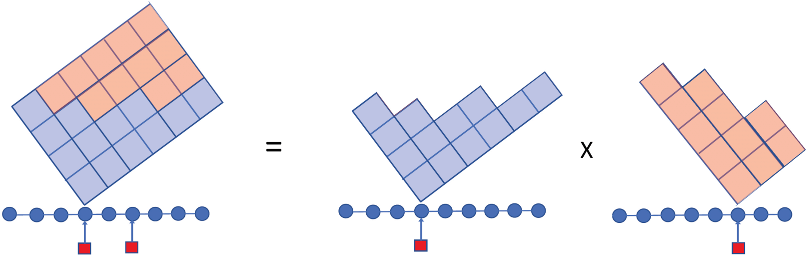
$\mathsf{t} =$ 

Summary: formula for stable envelope

For quiver variety of A_n or $\widehat{A}_n$-type:

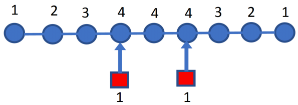


The fixed points are Young diagrams which fit into 4×6-box:

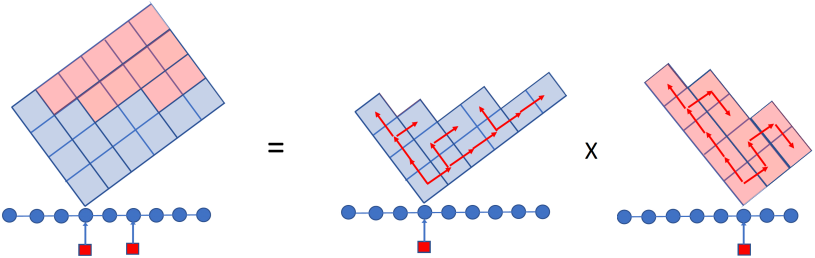


Thus $|X^{\mathsf{T}}| = \dfrac{6!}{4!2!}$

For quiver variety of A_n or $\widehat{A}_n$-type:



The fixed points are Young diagrams which fit into 4×6-box:



Thus $|X^{\mathrm{T}}| = \dfrac{6!}{4!2!}$

Summary: formula for stable envelope

The stable envelope of a fixed point then takes the form:

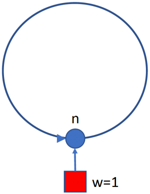
$$\boxed{Stab_\sigma(p) = Sym\left(\frac{\prod\limits_{\text{Young diagrams } \lambda}\left(\sum\limits_{\text{trees in } \lambda} Stab_\sigma(tree)\right)}{\prod\limits_{i<j}\vartheta(x_i/x_j)\vartheta(x_i\hbar/x_j)}\right)}$$

The symmetrization over Grothendieck roots and denominators comes from $\pi_* \circ \jmath_+^* \circ (\jmath_{-*})^{-1}$ in the abelianization diagram.

Application: Hilbert scheme of points in $\mathbb{C}^2$

The Hilbert scheme of n points in $\mathbb{C}^2$ parametrizes ideals in $\mathbb{C}[x,y]$ of codimension n:

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{I \subset \mathbb{C}[x,y] : \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n\}$$



A torus $\mathsf{T} = (\mathbb{C}^\times)^2$ acts on $\mathrm{Hilb}^n(\mathbb{C}^2)$ by scaling generators:

$$(x,y) \longrightarrow (xt_1, yt_2)$$

The fixed points $\mathrm{Hilb}^n(\mathbb{C}^2)^{\mathsf{T}}$ corresponds to monomial ideals.

Example: A monomial ideal

$$I = \langle y^3, x^4, x^2y^2 \rangle, \quad \dim \mathbb{C}[x,y]/I = n = 10$$

y^3	xy^3	x^2y^3	x^3y^3	x^4y^3
y^2	xy^2	x^2y^2	x^3y^2	x^4y^2
y	xy	x^2y	x^3y	x^4y
1	x	x^2	x^3	x^4

For $\mathrm{Hilb}^n(\mathbb{C}^2)$ we have:

$$\mathrm{Ell}_{\mathsf{T}}(pt) = \mathscr{E}_{\mathsf{T}} = E_{t_1} \times E_{t_2}, \quad \mathscr{E}_{\mathrm{Pic}} = E_z$$

Thus

$$Stab_\sigma(\lambda) \quad \text{is a section of } \mathscr{L} \text{ on} \quad \mathsf{E}_{\mathsf{T}}(\mathrm{Hilb}^n(\mathbb{C}^2)) = \left(\coprod_{|\lambda|=n} \widehat{\mathsf{O}}_\lambda \right) / \Delta$$

with abelian varieties $\widehat{\mathsf{O}}_\lambda = E_{t_1} \times E_{t_2} \times E_z$. These sections are explicitly described by the matrix of restrictions:

$$T_{\lambda,\mu} = Stab_\sigma(\lambda)|_{\widehat{\mathsf{O}}_\mu} = \begin{pmatrix} T_{1,1}(t_1,t_2) & T_{1,2}(t_1,t_2,z) & T_{1,3}(t_1,t_2,z) & \dots \\ 0 & T_{2,2}(t_1,t_2) & T_{2,3}(t_1,t_2,z) & \dots \\ 0 & 0 & T_{2,3}(t_1,t_2) & \dots \\ 0 & 0 & 0 & \dots \end{pmatrix}$$

with $T_{\lambda,\mu}$ - section of line bundles over $\widehat{\mathsf{O}}_\mu$.

Connection with symmetric polynomials

In case $n = 2$ the fixed set:

$\mathrm{Hilb}^2(\mathbb{C}^2)^{\mathsf{T}} = \{$  ,  $\}$

The restriction matrix takes the form

$$T_{\lambda,\mu} = \begin{bmatrix} \vartheta(t_2)\vartheta(t_2{}^2) & \dfrac{\vartheta(t_2)^2\vartheta(t_1t_2)}{\vartheta(t_1)\vartheta(z)}\vartheta\left(\dfrac{t_2 z}{t_1}\right) + \dfrac{\vartheta(t_2)\vartheta(t_1t_2)\vartheta(z^2t_2)\vartheta(t_2t_1z)}{\vartheta(t_1)\vartheta(z^2t_1t_2)\vartheta(z)}\vartheta\left(\dfrac{t_1}{t_2}\right) \\ 0 & -\vartheta(t_2)\vartheta\left(\frac{t_1}{t_2}\right) \end{bmatrix}$$

In general $T_{\lambda,\mu}$ is the elliptic “version” of transition matrix from Schur to Macdonald polynomials.

Fock representation of Heisenberg algebra

H.Nakajima defined a geometric action of Heisenberg algebra

$$Heis = \mathbb{C}\langle \alpha_1, \alpha_2, \dots, \rangle / [\alpha_m, \alpha_n] = n\delta_{n+m}$$

on the equivariant cohomologies of Hilbert schemes

$$\bigoplus_{n=0}^{\infty} H^{\bullet}_{T}(\mathrm{Hilb}^n(\mathbb{C}^2)) \cong \mathsf{Fock} \cong \mathbb{C}[p_1, p_2, \dots] \otimes \mathbb{C}[t_1, t_2]$$

$$\text{Classes of fixed points } \ [\lambda] \longrightarrow \text{Jack Polynomials } \ J_\lambda \in \mathsf{Fock}$$

$$\text{Stable envelope } \ Stab_\sigma(\lambda) \longrightarrow \text{Schur Polynomials } \ S_\lambda \in \mathsf{Fock}$$

$$S_\lambda = \sum_{\mu} T_{\lambda,\mu} J_\mu \qquad T_{\lambda,\mu} = \left(\begin{array}{cccc} * & * & * & \dots \\ 0 & * & * & \dots \\ 0 & 0 & * & \dots \\ \dots & \dots & \dots & \dots \end{array}\right), \quad * \in \mathbb{C}[t_1, t_2]$$

K-theoretic and cohomological limit

$$\mathscr{O}_{\mathrm{Ell}_{\mathsf{T}}(X)} \stackrel{q\to 0}{\longrightarrow} K_{\mathsf{T}}(X) \longrightarrow H^{\bullet}_{\mathsf{T}}(X)$$

The elliptic stable envelope degenerates to K-th and Coh versions

$$\mathrm{Stab}_{\sigma}(\lambda) \longrightarrow \mathrm{Stab}^{Kth}_{\sigma}(\lambda) \longrightarrow \mathrm{Stab}^{Coh}_{\sigma}(\lambda)$$

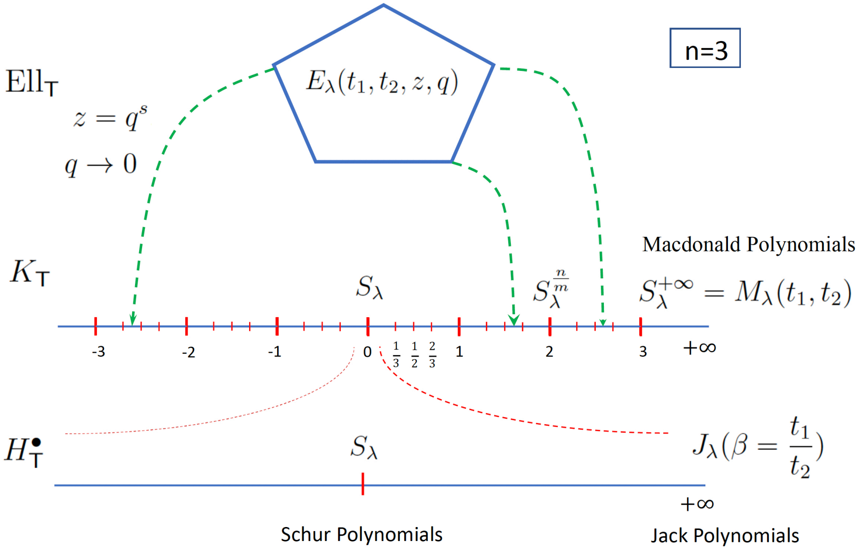
Let $z = q^s$ for $s \in \mathbb{R}$ and send $q \to 0$:

$$\lim_{q\to 0} \vartheta(x) = x^{1/2} - x^{-1/2}, \quad \lim_{q\to 0} \frac{\vartheta(xz^n)}{\vartheta(z^n)} = x^{1/2+\lfloor sn \rfloor}$$

locally constant function of "slope" s. Elliptic stable envelope has many K-theoretic limits depending on slope s:

$$Walls = \{\frac{a}{b} \in \mathbb{Q}, \quad |b| \leq n\} \subset \mathbb{R}$$

New symmetric polynomials



The Hilbert scheme is “self dual”:

$$Hilb^n(\mathbb{C}^2)^\vee \cong Hilb^n(\mathbb{C}^2)$$

this implies new hidden symmetry for restriction matrix

$$T_{\lambda,\mu} = Stab_\sigma(\lambda)|_{\widehat{O}_\mu} = \left(\begin{array}{cccc} T_{1,1}(t_1,t_2) & T_{1,2}(t_1,t_2,z) & T_{1,3}(t_1,t_2,z) & \dots \\ 0 & T_{2,2}(t_1,t_2) & T_{2,3}(t_1,t_2,z) & \dots \\ 0 & 0 & T_{2,3}(t_1,t_2) & \dots \\ 0 & 0 & 0 & \dots \end{array}\right)$$

Change of variables

$$a = \frac{t_1}{t_2}, \quad \hbar = t_1 t_2$$

Then $3d$ mirror symmetry implies that the stable envelope is symmetric invariant $a \leftrightarrow z$:

$$T(a, z, \hbar, q) = T(z, a, 1/\hbar, q)^{-1}$$

Checked on computer using explicit formula for stable envelope.

Happy Birthdays!