Atlas of Lie Groups and Representations

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L-homomorphisms and lowest K-types

Jeffrey Adams and Alexandre Afgoustidis
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Overview

G: connected, complex reductive group

$G(\mathbb{R})$: real points

$W_{\mathbb{R}}$: Weil group of $\mathbb{R}$

$W_{\mathbb{R}} = \langle \mathbb{C}^\times, j\rangle$, $(jzj^{-1} = \overline{z}, j^2 = -1)$

${}^LG$: L-group of G

$\phi : W_{\mathbb{R}} \to {}^LG$: admissible homomorphism

$\Pi(\phi) = \{\pi_1, \dots, \pi_n\}$: L-packet of ϕ

Define: $W_{\mathbb{R},c} = \langle S^1, j\rangle$ be the (unique) maximal compact subgroup of $W_{\mathbb{R}}$.

Question: [J.K. Yu, ∼2000]: What does $\phi|_{W_{\mathbb{R},c}}$ tell you about the K-types of the representations in the L-packet $\Pi(\phi)$?

Answer : [Adams, ∼2000]: That's an excellent question! I don't know.

This talk: a better answer.

The same question makes sense over a p-adic field.

Given: G and $\delta \in \mathrm{Out}(G), \delta^2 = 1$.

$\delta \leftrightarrow$ an inner class of real forms

Fix once and for all $T \subset B$ (Cartan and Borel subgroups)

Fokko du Cloux: T is fixed, fixed fixed

$X^* = X^*(T), X_* = X_*(T)$ (character, co-character lattices)

$G^\vee$: connected, reductive complex group, dual to G

Comes with $(T^\vee, B^\vee)$, $X^*(T^\vee) = X_*(T)$, etc.

$\delta \mapsto \delta^\vee = -\delta^t$

Fix a pinning $(B, T, \{X_\alpha\})$, then

${}^\delta G : G \rtimes \langle\delta\rangle$

also:

${}^{\delta^\vee}G^\vee$: $\langle G^\vee, \delta^\vee\rangle$

Definition: a strong involution for G: $x \in G\delta, x^2 \in Z(G)$

$x \to \theta_x = \text{int}(x)$: $\theta_x(g) = xgx^{-1}$.

$K_x = G^{\theta_x}$

$x \to \theta_x$: $\{\text{strong involutions}\}/\sim \twoheadrightarrow \{\text{real forms}\}/\sim$

Definition: a representation of a strong involution x is a pair (x, π) - π an admissible $(\mathfrak{g}, K_x)$-module

Equivalance: $(x, \pi) \simeq (x', \pi')$ if there exists $g \in G, gxg^{-1} = x', \pi^g \simeq \pi'$

Langlands Parameters in Atlas: KGB

$X = KGB(G,\delta) = \{x \in \mathrm{Norm}_{G\delta}(T), x^2 \in Z(G)\}/\sim_T$

$x \in X \mapsto X[x] = \{x' \in KGB \mid x' \sim_G x\}$

Theorem: $x \in X$:

There is a canonical bijection

$$X[x] \leftrightarrow K_x \backslash G/B$$

Langlands Parameters in Atlas

Definition An Atlas Parameter is:

$p = (x, \lambda, \nu)$:

1) $x \in KGB(G, \delta)$

2) $\lambda \in (X^* + \rho)/(1 - \theta_x)X^*$

3) $\nu \in X^*_{\mathbb{C}}$

Definition: $\gamma(p) = \frac{1+\theta_x}{2}\lambda + \frac{1-\theta_x}{2}\nu$

Various conditions:

Always roots are for T (fixed) in G; "real, imaginary,…" are with respect to θ_x

1) Standard: α imaginary $\Rightarrow \langle \lambda, \alpha^\vee \rangle \geq 0$

2) Non-zero: α simple, imaginary, compact $\Rightarrow \langle \lambda, \alpha^\vee \rangle \neq 0$

3) Final: ν weakly dominant, α real-simple, $\langle \nu, \alpha^\vee \rangle = 0 \Rightarrow \langle \lambda, \alpha^\vee \rangle$ is odd

4) Normal: $\langle \gamma, \alpha^\vee \rangle = 0$, α simple θ_x-complex $\Rightarrow \theta_x(\alpha)$ is positive

Langlands Parameters in Atlas

Equivalence:

0) $(x, \lambda, \nu) \simeq (x, \lambda, \frac{1-\theta}{2}\nu)$

1) $(x, \lambda, \nu) \sim (s_\alpha x, s_\alpha \lambda, s_\alpha \nu)$ (α simple, θ_x-complex)

2) $(x, \lambda, \nu) \sim (x, w(\lambda + \rho_r) - \rho_r, w\nu)$ ($w \in W_r$)

Attached to $p = (x, \lambda, \nu)$ is a standard $(\mathfrak{g}, K_x)$-module $I(p)$, which has a unique irreducible quotient $J(p)$.

Theorem: The map $p \to J(p)$ is a bijection:

$\{\text{parameters}\}/\sim \leftrightarrow \{\text{irreducible representations of strong involutions}\}/\sim$

A few basic invariants

We say an infinitesimal character γ is *real* if $\gamma \in X^* \otimes \mathbb{R}$.

$p = (x, \lambda, \nu)$

1) The infinitesimal character of $J(p)$ is

$$\gamma(p) := \frac{1+\theta_x}{2}\lambda + \frac{1-\theta_x}{2}\nu$$

2) The central character of $J(x, \lambda, \nu)$ is: (R is the root lattice):

$$(\overline{\lambda}, \overline{\nu}) \in (X^* + \rho)/[(1-\theta_x)X^* + R], (X^*_{\mathbb{C}})^{-\theta_x})$$

3) $J(p)$ has real infinitesimal character $\Leftrightarrow \nu \in X^*_{\mathbb{R}}$

4) $J(p)$ is tempered $\Leftrightarrow \nu \in X^*_{i\mathbb{R}}$

Tempiric Representations

Definition:

1) A representation π is tempiric (temp-i-ric) if it is tempered, irreducible, with real infinitesimal character.

2) A (standard, final, non-zero) parameter $p = (x, \lambda, \nu)$ is tempiric if $J(p)$ is tempiric.

In other words

(x, λ, ν) is tempiric if and only if (it is standard, final, non-zero, and) $\nu = 0$.

[Note: suggestions of better terminology are welcome]

Langlands Parameters in Atlas: restriction to K

Theorem (Vogan):

$G(\mathbb{R})$: real form, $K(\mathbb{R})$ maximal compact subgroup, with complexification K.

1) If π is tempiric it has a unique lowest K-type $LKT(\pi)$

2) The map $\pi \mapsto LKT(\pi)$ is a bijection:

$$\{\text{tempiric representations}\} \leftrightarrow \widehat{K}$$

Note: This miraculously takes care of the problem parametrizing the representations of the possibly disconnected group K

This is the starting point to understanding the K-structure of representations, in particular their lowest K-types

Example: $PGL(2, \mathbb{R})$

$G(\mathbb{R}) = PGL(2, \mathbb{R})$, $K = O(2)$

Tempiric π and their LKTs:

1) $\pi = \mathrm{Ind}_B^G(1)$: spherical principal series $\mapsto$ trivial representation of K

2) $\pi = \mathrm{Ind}_B^G(\mathrm{sgn})$: non-spherical principal series $\mapsto$ sgn representation of K

3) $\pi(\lambda)$ discrete series, $\lambda = k + \rho$ $(k = 0, 2, 4 \dots)$ $\mapsto$ two-dimensional irreducible of $SO(2)$ weights $\pm(k+1)$

Tempiric representations

Basic Fact: $I(x,\lambda,\nu)$ and $I(x,\lambda,\nu')$ have same restriction to K

and $J(x,\lambda,\nu)$ and $J(x,\lambda,\nu')$ have same lowest K-type

Suppose (x,λ,ν) is a (non-zero,standard) final parameter

$(x,\lambda,\nu)\rightarrow(x,\lambda,0)$

$I(x,\lambda,\nu)$ and $I(x,\lambda,0)$ have the same K-types...

However: $(x,\lambda,0)$ may NOT be final (and/or normal)

Example: $\mathbf{SL}(2,\mathbb{R})$

$G = SL(2,\mathbb{R})$

x : open orbit on G/B

$p = (x, [0], [\nu])$: $Ind_B^G(\text{sgn} \otimes \nu)$

K-types: $2\mathbb{Z}+1$

Final condition:

$(\nu \geq 0)$: $\langle \nu, \alpha^\vee \rangle = 0 \Rightarrow \langle \lambda, \alpha^\vee \rangle$ is odd

condition is empty if $\nu \neq 0$

If $\nu = 0$: $\langle \lambda, \alpha^\vee \rangle$ is odd (which is false since $\lambda = [0]$)

Well known limit of discrete series picture:

$\text{Ind}_B^G(\text{sgn}, 0)$ is the direct sum of two limits of discrete series, with lowest K-types ± 1

Finalize

There is a well defined algorithm to replace a standard, non-zero, but non-final parameter p with a set of final parameters $\{p_1, \dots, p_n\}$.

Inductive:

1) if p fails to be final because of a real-simple root α: replace p with the Cayley transform of p (1 or 2 terms)

2) if p fails to be normal because of a complex simple roots α: replace p with $s_\alpha \times p$ (a single parameter)

Atlas algorthm for computing lowest K-types:

$p = (x, \lambda, \nu) \mapsto (x, \lambda, 0) \mapsto \text{Finalize}(x, \lambda, 0) = \{(x_1, \lambda_1, 0), \dots, (x_n, \lambda_n, 0)\}$

Then $J(x, \lambda, \nu)$ has n LKTs: the lowest K-types of the tempiric representations $J(x_i, \lambda_i, 0)$

Example: $\mathbf{SL}(2,\mathbb{R})$

```
atlas> set G=SL(2,R)
atlas> set p=parameter(KGB(G,2),[0],[1])
atlas> p
Value: final parameter(x=2,lambda=[2]/1,nu=[1]/1)
atlas> set q=parameter(KGB(G,2),[0],[0])
atlas> q
Value: non-final parameter(x=2,lambda=[2]/1,nu=[0]/1)
atlas> finalize(q)
Value:
1*parameter(x=1,lambda=[0]/1,nu=[0]/1) [0]
1*parameter(x=0,lambda=[0]/1,nu=[0]/1) [0]

atlas> print_branch_irr_long (p,KGB(G,1),10)
m  x  lambda    hw       dim  height
1  1  [ 0 ]/1   [ 1 ]    1    0
1  0  [ 0 ]/1   [ -1 ]   1    0
1  1  [ 2 ]/1   [ 3 ]    1    2
1  0  [ 2 ]/1   [ -3 ]   1    2
1  1  [ 4 ]/1   [ 5 ]    1    4
1  0  [ 4 ]/1   [ -5 ]   1    4
```

Examples

```
atlas> set G=Spin(4,4)
atlas> set p=all_parameters_gamma (G,G.rho)[2]
atlas> p
Value: final parameter(x=108,lambda=[1,2,1,1]/1,nu=[1,1,1,1]/1)
atlas> G.trivial
Value: final parameter(x=108,lambda=[1,1,1,1]/1,nu=[1,1,1,1]/1)
atlas> finalize(p*0)

1*parameter(x=7,lambda=[0,0,0,0]/1,nu=[0,0,0,0]/1) [0]
1*parameter(x=6,lambda=[0,0,0,0]/1,nu=[0,0,0,0]/1) [0]
1*parameter(x=5,lambda=[0,0,0,0]/1,nu=[0,0,0,0]/1) [0]
1*parameter(x=0,lambda=[0,0,0,0]/1,nu=[0,0,0,0]/1) [0]

atlas> for mu in LKTs(p) do prints(highest_weight(mu,KGB(G,0)), " "
((),KGB element #0,[  1,  1, -1, -1 ]) 3
((),KGB element #0,[ -1,  1,  1, -1 ]) 3
((),KGB element #0,[ -1,  1, -1,  1 ]) 3
((),KGB element #0,[  1, -1,  1,  1 ]) 3
```

What about Langlands parameters?

$W_{\mathbb{R}} = \langle \mathbb{C}^\times, j \rangle$

Definition: $\Phi_0(G) = \{\phi : W_{\mathbb{R}} \to {}^L G\}$

$\Pi_0 \ni \phi \mapsto \Pi(\phi) = \{\pi_1, \dots, \pi_n\}$

Complete Langlands parameters:

Roughly speaking the representations in $\Pi(\phi)$ are parametrized by characters of

$$\mathbb{S}_\phi = \mathrm{Cent}_{G^\vee}(\phi)/\mathrm{Cent}_{G^\vee}(\phi)^0$$

$\widetilde{\mathbb{S}}_\phi$: a certain cover of $\mathbb{S}_\phi$.

Langlands parameters

Definition:

$X(\widetilde{\mathbb{S}}_\phi)$: set of characters of $\widetilde{\mathbb{S}}_\phi$

Definition:

$\Phi_0(G,\delta) = \{\phi : W_{\mathbb{R}} \to {}^L G\}$

$\Phi(G,\delta) = \{(\phi,\chi) \mid \phi \in \Phi_0, \chi \in X(\widetilde{\mathbb{S}}_\phi)\}$

Langlands Classification

Theorem: (Adams/Barbasch/Vogan 1992):

There is a canonical bijection between:

$$\Phi(G,\delta)/G^\vee \longleftrightarrow \{(x,\pi)\}/\sim$$

π an irreducible representation of the strong involution x

Note: $(\phi,\chi=1)\mapsto$ a generic representation of the quasisplit group.

Note: The classical result for a fixed real form $G(\mathbb{R})$ is:

Fix $x_0, K=K_{x_0}\leftrightarrow G(\mathbb{R})$.

$\{(x,\pi)\mid x\sim x_0\}/\sim\leftrightarrow$ {irreducible admissible representations of $G(\mathbb{R})$}

Note: Replace $\widetilde{\mathbb{S}}_\phi$ with $\mathbb{S}_\phi$, restrict to subset of *pure* strong real forms (the quasisplit form is always pure).

Tempiric Parameters

Basic fact: ϕ is tempiric $\Leftrightarrow \phi|_{\mathbb{R}^+} = 1$

Example: $G = PGL(2,\mathbb{R})$, ${}^L G = SL(2,\mathbb{C})$,

Spherical principal series:

$\phi(z) = \text{diag}(|z|^\nu, |z|^{-\nu})$

Tempered: $\nu \in i\mathbb{R}$

Real infinitesimal character: $\nu \in \mathbb{R}$

Both: $\nu = 0$.

$$1 \to W_{\mathbb{R},c} \to W_{\mathbb{R}} \to \mathbb{R}^+ \to 1$$

(canonically split)

$W_{\mathbb{R},c}$ is the unique maximal compact subgroup of $W_{\mathbb{R}}$

Tempiric Parameters

Recall:

$\Phi(G,\delta)=\{(\phi,\chi)\}$ where $\phi: W_{\mathbb{R}}\to {}^LG$, χ is a character of $\mathbb{S}_\phi$ (not $\widetilde{\mathbb{S}}_\phi$).

Definition:

$\Phi_c(G,\delta)=\{(\phi_c,\chi)\}$ where $\phi_c: W_{\mathbb{R},c}\to {}^LG$, χ is a character of $\mathbb{S}_{\phi_c}$

$(\phi_c,\chi)\mapsto \mu(\phi_c,\chi)$: lowest K-type of $\pi(\phi_c,\chi)$

Definition: $\widehat{K}_{\mathrm{all}}=\{(x,\mu)\mid x\in X,\mu\in \hat{K}_x\}/G$

Corollary of the preceding Theorem:

The map $(\phi_c,\chi)\mapsto \mu(\phi_c,\chi)$ gives a bijection:

$$\Phi_c/G^\vee \longleftrightarrow \widehat{K}_{\mathrm{all}}$$

RHS: x is a (pure) strong involution, μ is an irreducible finite dimensional representation of K_x.

Restricting to K

There is an obvious map (restriction): $\Phi_0(G) \to \Phi_{c,0}(G)$:

$\phi \to \phi_c = \phi|_{W_{\mathbb{R},c}}$

So, given ϕ:

$$\phi \mapsto \{\pi(\phi, \chi) \mid \chi \in X(\mathbb{S}_\phi)\}$$

$$\phi \to \phi_c \mapsto \{\mu(\phi_c, \tau) \mid \tau \in X(\mathbb{S}_{\phi_c})\}$$

What is the relationship?

Restricting to K

$\phi : W_{\mathbb{R}} \rightarrow {}^L G, \mapsto \phi_c : W_{\mathbb{R},c} \rightarrow {}^L G$

$$\phi_c(W_{\mathbb{R},c}) \subset \phi(W_{\mathbb{R}})$$

$$\text{Cent}_{G^\vee}(\phi(W_{\mathbb{R}})) \subset \text{Cent}_{G^\vee}(\phi_c(W_{\mathbb{R},c}))$$

$$\mathbb{S}(\phi) \rightarrow \mathbb{S}(\phi_c)$$

Proposition: The map $\mathbb{S}(\phi) \rightarrow \mathbb{S}(\phi_c)$ is injective.

(Shelstad proves a closely related statement)

The Main Result

Given $\phi : W_{\mathbb{R}} \to {}^L G$

$\phi \to \phi_c = \phi|_{W_{\mathbb{R},c}}$

$$\mathbb{S}_\phi \hookrightarrow \mathbb{S}_{\phi_c}$$

$$\text{induces } \Gamma : X(\mathbb{S}_{\phi_c}) \twoheadrightarrow X(S_\phi)$$

Theorem: (Adams/Afgoustidis):

Suppose $\phi : W_{\mathbb{R}} \to {}^L G$ is tempered, and $\chi \in X(\mathbb{S}_\phi)$.

$(\phi, \chi) \mapsto \pi = \pi(\phi, \chi)$ irreducible, tempered

Then the lowest K-types for $\pi(\phi, \chi)$ are parametrized by the fiber of Γ:

$$LKTs(\pi(\phi, \chi) = \{\mu(\phi_c, \tau) \mid \Gamma(\tau) = \chi\}$$

Sketch of proof

The proof is by induction: we follow the steps in the finalize algorithm applied to ϕ_c. We prove injectivity $\mathbb{S}_\phi \hookrightarrow \mathbb{S}_c$ and the main Theorem at the same time.

The key step is a single Cayley transform.

Involutions of $T^\vee$

Key technical point: ϕ goes to an involution of $T^\vee$.

$\phi(j)$ is such an involution. This is *not* then one we need.

When γ is singular there is a choice:

Definition: $\phi \mapsto \tau^\vee$, an involution of $T^\vee$:
take the *most split* choice (corresponding to the most compact choice on the group side)

Cayley Transform

With this choice, let τ be the dual involution of T.

Suppose ϕ' is obtained from ϕ by a single real Cayley transform c_α.

Suppose τ is a (twisted) involution of $T^\vee$ ($\tau \in W\delta^\vee$, $\tau^2 = 1$)

$\mathbb{S}_{\tau^\vee}$: the component group of $(T^\vee)^{\tau^\vee}$

[Note: $\mathbb{S}_\phi = \mathbb{S}_{\tau^\vee}/\{m_\beta \mid \langle\gamma, \alpha^\vee\rangle = 0\}$]
Then $X(\mathbb{S}_{\tau^\vee})$ acts simply transitively on X_τ (to be precise $X_\tau(z_{\rho^\vee})$).
Easy fact: given ϕ, $\tau^\vee$ as above:

$$X(\mathbb{S}_\phi) \hookrightarrow X(\mathbb{S}_{\tau^\vee}) \leftrightarrow X_\tau$$

$$\begin{array}{ccc} X(\mathbb{S}_{\phi'}) & \longrightarrow\!\!\!\!\rightarrow & X(\mathbb{S}_\phi) \\ \big\downarrow & & \big\downarrow \\ X_{\tau'}(z_{\rho^\vee}, \alpha) & \xrightarrow{c_\alpha} & X_\tau(z_{\rho^\vee}) \end{array}$$

$X_\tau(z) = \{x \in X, x^2 = z, p(x) = \tau \in W\delta\}$

Example: $\mathbf{SL}(2,\mathbb{R})$

$G = SL(2,\mathbb{R})$

$G^\vee = PGL(2) = SO(3)$

$\phi(z) = \mathrm{diag}(|z|^\nu, |z|^{-\nu}, 1)$

$\phi(j) = \mathrm{diag}(-1,-1,1)$
$\nu \neq 0$: $\mathrm{Cent}(\phi) = \mathbb{C}^\times, \mathbb{S}_\phi = 1$

$\phi_c(\nu = 0)$: $\mathrm{Cent}(\phi_c) = O(2)$, $\mathbb{S}_{\phi_c} = \mathbb{Z}/2\mathbb{Z}$.

$\nu \neq 0 : \tau^\vee = 1$ (no choice)

$\nu = 0$: ϕ is conjugate to

$$\phi'(z) = 1, \phi'_c(j) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So $\tau^\vee = s_\alpha$

Example: $\mathbf{SL}(2,\mathbb{R})$

$$\begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} & \twoheadrightarrow & 1 \\ \downarrow & & \downarrow \\ \{x_0, x_1\} & \twoheadrightarrow & x_2 \end{array}$$

$$\begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} & \twoheadrightarrow & 1 \\ \downarrow & & \downarrow \\ \{x_0, x_1\} & \twoheadrightarrow & x_2 \end{array}$$

The Literature

Knapp-Stein: $R_{\sigma,\nu}$: defined in terms of the Plancherel measure, intertwining operators; reducibility of tempered representations

Vogan: algebraic definition: reduction to the quasisplit case, R_δ

Shelstad/Langlands: The group R_ϕ defined on the dual side:

$$1 \to \mathbb{S}_{\phi_M} \to \mathbb{S}_\phi \to R_\phi \to 1$$

$R_\phi \simeq R(\sigma,\nu)$

Dictionary

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{S}_{\phi_M} & \longrightarrow & \mathbb{S}_{\phi} & \longrightarrow & R(\sigma,\nu) & \longrightarrow & 1 \\
 & & \downarrow \simeq & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{S}_{\phi_{c_M}} & \longrightarrow & \mathbb{S}_{\phi_c} & \longrightarrow & R(\sigma,1) & \longrightarrow & 1
\end{array}
$$

This diagram commutes (not obvious):

$$\mathbb{S}_{\phi_c}/\mathbb{S}_{\phi} \simeq R(\sigma,1)/R(\sigma,\nu)$$

Conclusion

$\phi : W_{\mathbb{R}} \to {}^L G$, tempered

$\phi \mapsto \phi_c$

$\Gamma : X(\mathbb{S}_{\phi_c}) \twoheadrightarrow X(S_\phi)$

$(\phi, \chi) \in \Pi(G, \delta) \mapsto \pi(\phi, \chi)$

Then:

$$LKTs(\pi(\phi, \chi) = \{\mu(\phi_c, \tau) \mid \Gamma(\tau) = \chi\}$$

Question/Hope: is this the "right" formulation: does it generalize to the p-adic case?

Thank you (again) David - for everything!