L-homomorphisms and lowest K-types

Jeffrey Adams and Alexandre Afgoustidis
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**Overview**

$G$: connected, complex reductive group

$G(\mathbb{R})$: real points

$W_\mathbb{R}$: Weil group of $\mathbb{R}$

$$W_\mathbb{R} = \langle \mathbb{C}^\times, j \rangle, \ (jzj^{-1} = \bar{z}, j^2 = -1)$$

$L^G$: L-group of $G$

$\phi : W_\mathbb{R} \rightarrow L^G$: admissible homomorphism

$\Pi(\phi) = \{\pi_1, \ldots, \pi_n\}$: L-packet of $\phi$

Define: $W_{\mathbb{R}, c} = \langle S^1, j \rangle$ be the (unique) maximal compact subgroup of $W_\mathbb{R}$.

**Question:** [J.K. Yu, \~2000]: What does $\phi|_{W_{\mathbb{R}, c}}$ tell you about the K-types of the representations in the L-packet $\Pi(\phi)$?

**Answer:** [Adams, \~2000]: That’s an excellent question! I don’t know.

This talk: a better answer.

The same question makes sense over a $p$-adic field.
Given: $G$ and $\delta \in \text{Out}(G), \delta^2 = 1.$

$\delta \leftrightarrow$ an inner class of real forms

Fix once and for all $T \subset B$ (Cartan and Borel subgroups)

Fokko du Cloux: $T$ is fixed, fixed fixed

$X^* = X^*(T), X_* = X_*(T)$ (character, co-character lattices)

$G^\vee$: connected, reductive complex group, dual to $G$

Comes with $(T^\vee, B^\vee), X^*(T^\vee) = X_*(T), \text{ etc.}$

$\delta \mapsto \delta^\vee = -\delta^t$

Fix a pinning $(B, T, \{X_\alpha\}),$ then

$\delta G : G \rtimes \langle \delta \rangle$

also:

$\delta^\vee G^\vee : \langle G^\vee, \delta^\vee \rangle$
Definition: a strong involution for $G$: $x \in G\delta, x^2 \in Z(G)$

$x \rightarrow \theta_x = \text{int}(x): \theta_x(g) = xgx^{-1}$.

$K_x = G^{\theta_x}$

$x \rightarrow \theta_x$: \{strong involutions\}/$\sim$ $\rightarrow$ \{real forms\}/$\sim$

Definition: a representation of a strong involution $x$ is a pair $(x, \pi)$ - $\pi$ an admissible $(g, K_x)$-module

Equivalence: $(x, \pi) \simeq (x', \pi')$ if there exists $g \in G, gxg^{-1} = x', \pi^g \simeq \pi'$
Theorem: \( x \in X: \)

There is a canonical bijection

\[
X[x] \leftrightarrow K_x \backslash G / B
\]
**Langlands Parameters in Atlas**

Definition An Atlas Parameter is:

\[ p = (x, \lambda, \nu) : \]

1) \( x \in KGB(G, \delta) \)

2) \( \lambda \in (X^* + \rho)/(1 - \theta_x)X^* \)

3) \( \nu \in X_C^* \)

Definition: \( \gamma(p) = \frac{1 + \theta_x}{2} \lambda + \frac{1 - \theta_x}{2} \nu \)

Various conditions:

Always roots are for \( T \) (fixed) in \( G \); “real, imaginary, . . . ” are with respect to \( \theta_x \)

1) Standard: \( \alpha \) imaginary \( \Rightarrow \langle \lambda, \alpha^\vee \rangle \geq 0 \)

2) Non-zero: \( \alpha \) simple, imaginary, compact \( \Rightarrow \langle \lambda, \alpha^\vee \rangle \neq 0 \)

3) Final: \( \nu \) weakly dominant, \( \alpha \) real-simple, \( \langle \nu, \alpha^\vee \rangle = 0 \Rightarrow \langle \lambda, \alpha^\vee \rangle \) is odd

4) Normal: \( \langle \gamma, \alpha^\vee \rangle = 0 \), \( \alpha \) simple \( \theta_x \)-complex \( \Rightarrow \theta_x(\alpha) \) is positive
Equivalence:

0) \((x, \lambda, \nu) \simeq (x, \lambda, \frac{1-\theta}{2} \nu)\)

1) \((x, \lambda, \nu) \sim (s_\alpha x, s_\alpha \lambda, s_\alpha \nu)\) (\(\alpha\) simple, \(\theta_x\)-complex)

2) \((x, \lambda, \nu) \sim (x, w(\lambda + \rho_r) - \rho_r, w\nu)\) (\(w \in W_r\))

Attached to \(p = (x, \lambda, \nu)\) is a standard \((g, K_x)\)-module \(I(p)\), which has a unique irreducible quotient \(J(p)\).

**Theorem**: The map \(p \rightarrow J(p)\) is a bijection:

\[
\{\text{parameters}\}/\sim \leftrightarrow \{\text{irreducible representations of strong involutions}\}/\sim
\]
We say an infinitesimal character $\gamma$ is *real* if $\gamma \in X^* \otimes \mathbb{R}$.

$p = (x, \lambda, \nu)$

1) The infinitesimal character of $J(p)$ is

$$\gamma(p) := \frac{1 + \theta_x}{2} \lambda + \frac{1 - \theta_x}{2} \nu$$

2) The central character of $J(x, \lambda, \nu)$ is: ($R$ is the root lattice):

$$(\overline{\lambda}, \nu) \in (X^* + \rho)/[(1 - \theta_x)X^* + R], (X^*_\mathbb{C})^{-\theta_x}$$

3) $J(p)$ has real infinitesimal character $\iff \nu \in X^*_\mathbb{R}$

4) $J(p)$ is tempered $\iff \nu \in X^*_i\mathbb{R}$
Definition:

1) A representation $\pi$ is tempiric (temp-i-ric) if it is tempered, irreducible, with real infinitesimal character.

2) A (standard, final, non-zero) parameter $p = (x, \lambda, \nu)$ is tempiric if $J(p)$ is tempiric.

In other words

$(x, \lambda, \nu)$ is tempiric if and only if (it is standard, final, non-zero, and) $\nu = 0$.

[Note: suggestions of better terminology are welcome]
Langlands Parameters in Atlas: restriction to $K$

Theorem (Vogan):

$G(\mathbb{R})$: real form, $K(\mathbb{R})$ maximal compact subgroup, with complexification $K$.

1) If $\pi$ is tempiric it has a unique lowest $K$-type $LKT(\pi)$

2) The map $\pi \mapsto LKT(\pi)$ is a bijection:

$$\{\text{tempiric representations}\} \leftrightarrow \hat{K}$$

Note: This miraculously takes care of the problem parametrizing the representations of the possibly disconnected group $K$

This is the starting point to understanding the $K$-structure of representations, in particular their lowest $K$-types
Example: \( PGL(2, \mathbb{R}) \)

\[ G(\mathbb{R}) = PGL(2, \mathbb{R}), \ K = O(2) \]

Tempiric \( \pi \) and their LKTs:

1) \( \pi = \text{Ind}^G_B(1) \): spherical principal series \( \mapsto \) trivial representation of \( K \)

2) \( \pi = \text{Ind}^G_B(\text{sgn}) \): non-spherical principal series \( \mapsto \) sgn representation of \( K \)

3) \( \pi(\lambda) \) discrete series, \( \lambda = k + \rho \) (\( k = 0, 2, 4 \ldots \)) \( \mapsto \) two-dimensional irreducible of \( SO(2) \) weights \( \pm(k + 1) \)
Basic Fact: $I(x, \lambda, \nu)$ and $I(x, \lambda, \nu')$ have same restriction to $K$

and $J(x, \lambda, \nu)$ and $J(x, \lambda, \nu')$ have same lowest $K$-type

Suppose $(x, \lambda, \nu)$ is a (non-zero, standard) final parameter

$(x, \lambda, \nu) \rightarrow (x, \lambda, 0)$

$I(x, \lambda, \nu)$ and $I(x, \lambda, 0)$ have the same $K$-types...

However: $(x, \lambda, 0)$ may NOT be final (and/or normal)
**Example:** $SL(2, \mathbb{R})$

$G = SL(2, \mathbb{R})$

$x :$ open orbit on $G/B$

$p = (x, [0], [\nu])$: $Ind_B^G(\text{sgn} \otimes \nu)$

K-types: $2\mathbb{Z} + 1$

Final condition:

$(\nu \geq 0): \langle \nu, \alpha^\vee \rangle = 0 \Rightarrow \langle \lambda, \alpha^\vee \rangle$ is odd

condition is empty if $\nu \neq 0$

If $\nu = 0$: $\langle \lambda, \alpha^\vee \rangle$ is odd (which is false since $\lambda = [0]$)

Well known limit of discrete series picture:

$Ind_B^G(\text{sgn}, 0)$ is the direct sum of two limits of discrete series, with lowest K-types $\pm 1$
There is a well defined algorithm to replace a standard, non-zero, but non-final parameter $p$ with a set of final parameters $\{p_1, \ldots, p_n\}$.

Inductive:

1) if $p$ fails to be final because of a real-simple root $\alpha$: replace $p$ with the Cayley transform of $p$ (1 or 2 terms)

2) if $p$ fails to be normal because of a complex simple roots $\alpha$: replace $p$ with $s_\alpha \times p$ (a single parameter)

Atlas algorithm for computing lowest K-types:

$p = (x, \lambda, \nu) \mapsto (x, \lambda, 0) \mapsto \text{Finalize}(x, \lambda, 0) = \{(x_1, \lambda_1, 0), \ldots, (x_n, \lambda_n, 0)\}$

Then $J(x, \lambda, \nu)$ has $n$ LKTs: the lowest K-types of the tempiric representations $J(x_i, \lambda_i, 0)$
EXAMPLE: $SL(2, \mathbb{R})$

atlas> set G=SL(2,R)

atlas> set p=parameter(KGB(G,2),[0],[1])

atlas> p
Value: final parameter(x=2,lambda=[2]/1,nu=[1]/1)

atlas> set q=parameter(KGB(G,2),[0],[0])

atlas> q
Value: non-final parameter(x=2,lambda=[2]/1,nu=[0]/1)

atlas> finalize(q)
Value:
 1*parameter(x=1,lambda=[0]/1,nu=[0]/1) [0]
 1*parameter(x=0,lambda=[0]/1,nu=[0]/1) [0]

atlas> print_branch_irr_long (p,KGB(G,1),10)

<table>
<thead>
<tr>
<th>m</th>
<th>x</th>
<th>lambda</th>
<th>hw</th>
<th>dim</th>
<th>height</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td>[ 0 ]/1</td>
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<td>1</td>
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<td>[ 0 ]/1</td>
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<td>1</td>
<td>1</td>
<td>[ 2 ]/1</td>
<td>[ 3 ]</td>
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<tr>
<td>1</td>
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<td>[ 2 ]/1</td>
<td>[ -3]</td>
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<td>1</td>
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<td>[ 4 ]/1</td>
<td>[ 5 ]</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>[ 4 ]/1</td>
<td>[ -5]</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>
atlas> set G=Spin(4,4)
atlas> set p=all_parameters_gamma (G,G.rho)[2]
atlas> p
Value: final parameter(x=108,lambda=[1,2,1,1]/1,nu=[1,1,1,1]/1)
atlas> G.trivial
Value: final parameter(x=108,lambda=[1,1,1,1]/1,nu=[1,1,1,1]/1)
atlas> finalize(p*0)

1*parameter(x=7,lambda=[0,0,0,0]/1,nu=[0,0,0,0]/1) [0]
1*parameter(x=6,lambda=[0,0,0,0]/1,nu=[0,0,0,0]/1) [0]
1*parameter(x=5,lambda=[0,0,0,0]/1,nu=[0,0,0,0]/1) [0]
1*parameter(x=0,lambda=[0,0,0,0]/1,nu=[0,0,0,0]/1) [0]

atlas> for mu in LKTs(p) do prints(highest_weight(mu,KGB(G,0)), " "
(((),KGB element #0,[  1,  1, -1, -1 ])) 3
(((),KGB element #0,[ -1,  1,  1, -1 ])) 3
(((),KGB element #0,[ -1,  1, -1,  1 ])) 3
(((),KGB element #0,[  1, -1,  1,  1 ])) 3
What about Langlands parameters?

\[ W_\mathbb{R} = \langle \mathbb{C}^\times, j \rangle \]

**Definition:** \( \Phi_0(G) = \{ \phi : W_\mathbb{R} \to \mathcal{L} G \} \)

\[ \Pi_0 \ni \phi \mapsto \Pi(\phi) = \{ \pi_1, \ldots, \pi_n \} \]

Complete Langlands parameters:

Roughly speaking the representations in \( \Pi(\phi) \) are parametrized by characters of

\[ S_\phi = \text{Cent}_G \phi \big/ \text{Cent}_G (\phi)^0 \]

\( \tilde{S}_\phi \): a certain cover of \( S_\phi \).
Definition:
$
X(\tilde{S}_\phi): \text{set of characters of } \tilde{S}_\phi
$

Definition:
$
\Phi_0(G, \delta) = \{ \phi : W_\mathbb{R} \to^L G \} \\
\Phi(G, \delta) = \{ (\phi, \chi) \mid \phi \in \Phi_0, \chi \in X(\tilde{S}_\phi) \}
$
Theorem: (Adams/Barbasch/Vogan 1992):

There is a canonical bijection between:

\[ \Phi(G, \delta)/G^\vee \leftrightarrow \{(x, \pi)\}/\sim \]

\( \pi \) an irreducible representation of the strong involution \( x \)

Note: \((\phi, \chi = 1) \mapsto \) a generic representation of the quasisplit group.

Note: The classical result for a fixed real form \( G(\mathbb{R}) \) is:

Fix \( x_0, K = K_{x_0} \leftrightarrow G(\mathbb{R}). \)

\[ \{(x, \pi) \mid x \sim x_0\}/\sim \leftrightarrow \{\text{irreducible admissible representations of } G(\mathbb{R})\} \]

Note: Replace \( \tilde{S}_\phi \) with \( S_\phi \), restrict to subset of \textit{pure} strong real forms (the quasisplit form is always pure).
**Tempiric Parameters**

**Basic fact:** $\phi$ is tempiric $\iff \phi|_{\mathbb{R}^+} = 1$

**Example:** $G = PGL(2, \mathbb{R})$, $^lG = SL(2, \mathbb{C})$.

**Spherical principal series:**

$$\phi(z) = \text{diag}(|z|^\nu, |z|^{-\nu})$$

**Tempered:** $\nu \in i\mathbb{R}$

**Real infinitesimal character:** $\nu \in \mathbb{R}$

**Both:** $\nu = 0$.

$$1 \rightarrow W_{\mathbb{R},c} \rightarrow W_{\mathbb{R}} \rightarrow \mathbb{R}^+ \rightarrow 1$$

(canonical split)

$W_{\mathbb{R},c}$ is the unique maximal compact subgroup of $W_{\mathbb{R}}$
Recall:
\[ \Phi(G, \delta) = \{(\phi, \chi)\} \text{ where } \phi : W_{\mathbb{R}} \to L^G, \chi \text{ is a character of } S_\phi \text{ (not } \tilde{S}_\phi). \]

Definition:
\[ \Phi_c(G, \delta) = \{(\phi_c, \chi)\} \text{ where } \phi_c : W_{\mathbb{R},c} \to L^G, \chi \text{ is a character of } S_{\phi_c} \]
\[ (\phi_c, \chi) \mapsto \mu(\phi_c, \chi): \text{ lowest } K\text{-type of } \pi(\phi_c, \chi) \]

Definition: \[ \hat{K}_{all} = \{(x, \mu) \mid x \in X, \mu \in \hat{K}_x\}/G \]

Corollary of the preceding Theorem:
The map \( (\phi_c, \chi) \mapsto \mu(\phi_c, \chi) \) gives a bijection:
\[ \Phi_c/G^\vee \longleftrightarrow \hat{K}_{all} \]

RHS: \( x \) is a (pure) strong involution, \( \mu \) is an irreducible finite dimensional representation of \( K_x \).
There is an obvious map (restriction): \( \Phi_0(G) \to \Phi_{c,0}(G) \):

\[
\phi \mapsto \phi_c = \phi|_{W_{R,c}}
\]

So, given \( \phi \):

\[
\phi \mapsto \{ \pi(\phi, \chi) \mid \chi \in X(S\phi) \}
\]

\[
\phi \to \phi_c \mapsto \{ \mu(\phi_c, \tau) \mid \tau \in X(S_{\phi_c}) \}
\]

What is the relationship?
Proposition: The map $\mathcal{S}(\phi) \rightarrow \mathcal{S}(\phi_c)$ is injective.

(Shelstad proves a closely related statement)
Given $\phi : \mathcal{W}_\mathbb{R} \to L^G$

$\phi \to \phi_c = \phi|_{\mathcal{W}_{\mathbb{R},c}}$

$S_\phi \hookrightarrow S_{\phi_c}$

induces $\Gamma : X(S_{\phi_c}) \to X(S_\phi)$

**Theorem:** (Adams/Afgoustidis):

Suppose $\phi : \mathcal{W}_\mathbb{R} \to L^G$ is tempered, and $\chi \in X(S_\phi)$.

$(\phi, \chi) \mapsto \pi = \pi(\phi, \chi)$ irreducible, tempered

Then the lowest $K$-types for $\pi(\phi, \chi)$ are parametrized by the fiber of $\Gamma$:

$LKTs(\pi(\phi, \chi) = \{\mu(\phi_c, \tau) \mid \Gamma(\tau) = \chi\}$
The proof is by induction: we follow the steps in the finalize algorithm applied to $\phi_c$. We prove injectivity $S_\phi \rightarrow S_c$ and the main Theorem at the same time.

The key step is a single Cayley transform.
Key technical point: $\phi$ goes to an involution of $T^\vee$.

$\phi(j)$ is such an involution. This is not then one we need.

When $\gamma$ is singular there is a choice:

Definition: $\phi \mapsto \tau^\vee$, an involution of $T^\vee$:

take the *most split* choice (corresponding to the most compact choice on the group side)
With this choice, let \( \tau \) be the dual involution of \( T \).

Suppose \( \phi' \) is obtained from \( \phi \) by a single real Cayley transform \( c_\alpha \).

Suppose \( \tau \) is a (twisted) involution of \( T^\vee \) \( (\tau \in W_\delta^\vee, \tau^2 = 1) \)

\( S_{\tau^\vee} \): the component group of \( (T^\vee)^{\tau^\vee} \)

[Note: \( S_\phi = S_{\tau^\vee} \setminus \{ m_\beta | \langle \gamma, \alpha^\vee \rangle = 0 \} \)]

Then \( X(S_{\tau^\vee}) \) acts simply transitively on \( X_\tau \) (to be precise \( X_\tau(z_{\rho^\vee}) \)).

Easy fact: given \( \phi, \tau^\vee \) as above:

\[
X(S_\phi) \hookrightarrow X(S_{\tau^\vee}) \leftrightarrow X_\tau
\]

\[
X(S_\phi') \twoheadrightarrow X(S_\phi)
\]

\[
X_\tau'(z_{\rho^\vee}, \alpha) \xrightarrow{c_\alpha} X_\tau(z_{\rho^\vee})
\]

\[
X_\tau(z) = \{ x \in X, x^2 = z, p(x) = \tau \in W_\delta \}\]
**Example: \( \text{SL}(2, \mathbb{R}) \)**

\[
G = \text{SL}(2, \mathbb{R})
\]

\[
G^\vee = \text{PGL}(2) = \text{SO}(3)
\]

\[
\phi(z) = \text{diag}(|z|^\nu, |z|^{-\nu}, 1)
\]

\[
\phi(j) = \text{diag}(-1, -1, 1)
\]

\(\nu \neq 0\): \(\text{Cent}(\phi) = \mathbb{C}^\times, S_\phi = 1\)

\(\phi_c(\nu = 0)\): \(\text{Cent}(\phi_c) = O(2), S_{\phi_c} = \mathbb{Z}/2\mathbb{Z}\).

\(\nu \neq 0\) : \(\tau^\vee = 1\) (no choice)

\(\nu = 0\): \(\phi\) is conjugate to

\[
\phi'(z) = 1, \phi'_c(j) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

So \(\tau^\vee = s_\alpha\).
**Example:** \(SL(2, \mathbb{R})\)

\[
\begin{align*}
\mathbb{Z}/2\mathbb{Z} & \rightarrow 1 \\
\downarrow & \downarrow \\
\{x_0, x_1\} & \rightarrow x_2
\end{align*}
\]

\[
\begin{align*}
\mathbb{Z}/2\mathbb{Z} & \rightarrow 1 \\
\downarrow & \downarrow \\
\{x_0, x_1\} & \rightarrow x_2
\end{align*}
\]
Knapp-Stein: $R_{\sigma,\nu}$: defined in terms of the Plancherel measure, intertwining operators; reducibility of tempered representations

Vogan: algebraic definition: reduction to the quasisplit case, $R_{\delta}$

Shelstad/Langlands: The group $R_{\phi}$ defined on the dual side:

$$1 \to S_{\phi_M} \to S_{\phi} \to R_{\phi} \to 1$$

$R_{\phi} \simeq R(\sigma, \nu)$
This diagram commutes (not obvious):

\[
\begin{array}{cccccc}
1 & \rightarrow & S_{\phi_M} & \rightarrow & S_{\phi} & \rightarrow & R(\sigma, \nu) & \rightarrow & 1 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
1 & \rightarrow & S_{\phi_{cM}} & \rightarrow & S_{\phi_c} & \rightarrow & R(\sigma, 1) & \rightarrow & 1
\end{array}
\]

\[S_{\phi_c}/S_{\phi} \simeq R(\sigma, 1)/R(\sigma, \nu)\]
\( \phi : W_\mathbb{R} \to \text{L}_G \), tempered

\( \phi \mapsto \phi_c \)

\( \Gamma : X(S_{\phi_c}) \to X(S_\phi) \)

\((\phi, \chi) \in \Pi(G, \delta) \mapsto \pi(\phi, \chi) \)

Then:

\[
\text{LKTs}(\pi(\phi, \chi)) = \{ \mu(\phi_c, \tau) \mid \Gamma(\tau) = \chi \}
\]

**Question/Hope:** is this the “right” formulation: does it generalize to the \( p \)-adic case?
Thank you (again) David - for everything!