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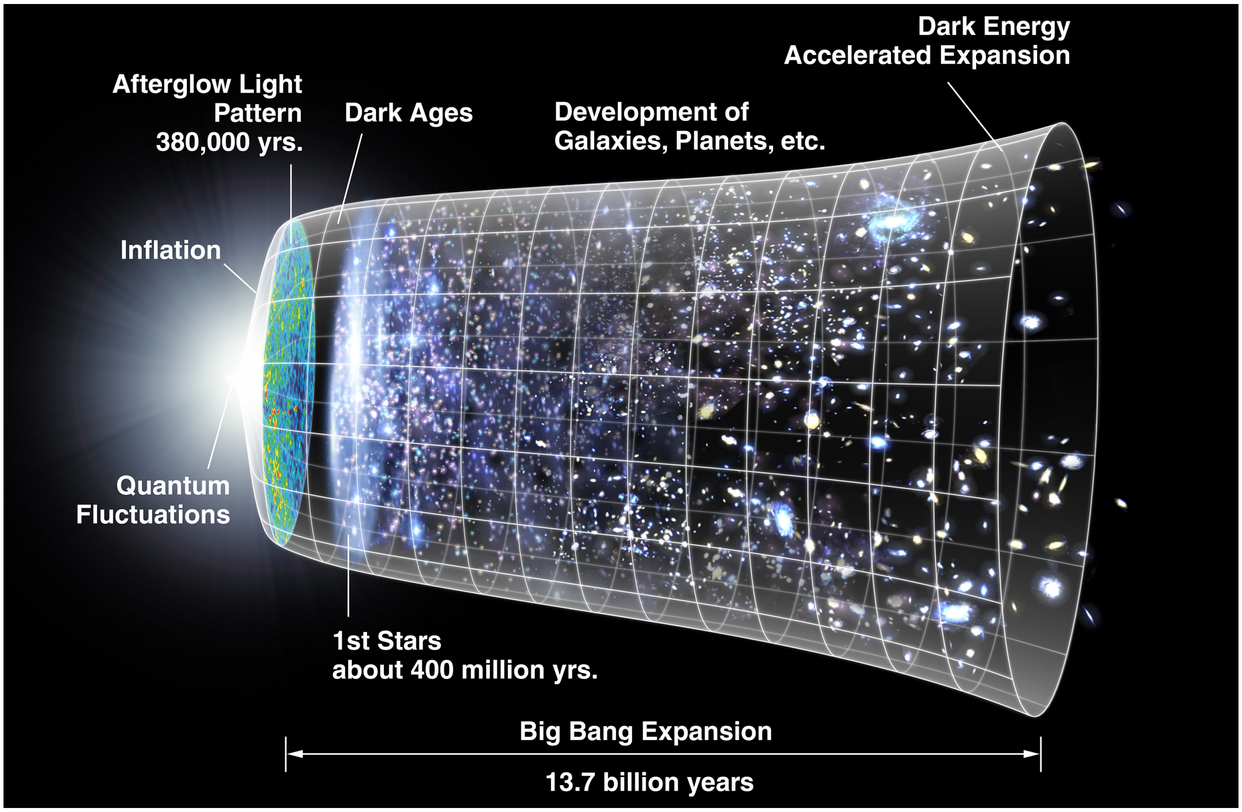


Symmetric Matrix

$$A = \left[a_{ij}\right]_{1 \le i,j < \infty}, a_{ij} = a_{ji} \in \mathbb{R}.$$

$$A_n = \left[a_{ij}\right]_{1 \le i,j < n}$$

If $\lambda_{n,1} \geq \lambda_{n,2} \geq ... \geq \lambda_{n,n}$ is the spectrum of A_n counting multiplicities then we have the interlacing property

$$\lambda_{n+1,1} \geq \lambda_{n,1} \geq \lambda_{n+1,2} \geq \lambda_{n,2} \geq \lambda_{n+1,3} \geq \cdots$$

Orthogonal Polynomials

Let $-\infty \leq a < b \leq \infty$ and let μ be a positive Borel measure on (a, b) such that

$$\int_a^b x^{2k} d\mu(x) < \infty$$

for all $k = 1, 2, ...$

$\{f_j\}_{j \geq 0}$ the Gram-Schmidt orthonormalization in $L^2((a, b), \mu)$ of $1, x, x^2, ...$ Or any family of nonzero multiples of such.

The zeros of f_n are all real and distinct and those of f_n and f_{n+1} strictly interlace.

$a = -\infty, b = \infty$ and $d\mu = e^{-x^2}dx$. Hermite Polynomials $H_n, n \geq 1$:

$$x, x^2 - \frac{1}{2}, x^3 - \frac{3}{2}x, x^4 - 3x^2 + \frac{3}{4}, ...$$

$$A = \begin{bmatrix} 0 & \sqrt{\frac{1}{2}} & 0 & 0 & 0 & \cdots \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{2}{2}} & 0 & 0 & \cdots \\ 0 & \sqrt{\frac{2}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 & \cdots \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{4}{2}} & \cdots \\ 0 & 0 & 0 & \sqrt{\frac{4}{2}} & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$H_n(x) = \det[x - A_n]$$

Theorem 1 *Suppose that we have* $n \geq 1$, $\lambda_{n,1} > \lambda_{n,2} > ... > \lambda_{n,n}$ *strictly interlacing. Then there exists a at most one (up to conjugation by a diagonal matrix with entries* ± 1*) tridiagonal* $\infty \times \infty$ *real, symmetric matrix,* A,*such* *that the eigenvalues of A_n are $\{\lambda_{n,i}|i = 1, ..., n\}$. Furthermore, if A is a tridiagonal $\infty \times \infty$ real, symmetric matrix whose truncated spectra have the interlacing property then the functions $1, \det(xI - A_1), \det(xI - A_2), ..., \det(xI - A_k), ...$ form a family of orthogonal polynomials for an appropriate measure.*

Theorem 2 *Let for each $1 \leq n \leq m$, $\lambda_{n,1} > \lambda_{n,2} > ... > \lambda_{n,n}$ be given with*

$$\lambda_{n+1,1} > \lambda_{n,1} > \lambda_{n+1,2} > \lambda_{n,2} > ...\lambda_{n,n} > \lambda_{n+1,n+1}$$

for $n+1 \leq m$. Then there exist exactly $2^{\binom{m}{2}}$ real symmetric $m \times m$ matrices whose truncations have this given spectrum and at most one (up to conjugation by a diagonal $m \times m$ matrix with entries ± 1) is tridiagonal.

Theorem 3 *Let for each $1 \leq n \leq m$, $\{\lambda_{n,1}, ..., \lambda_{n,n}\} = \Lambda_n$ be distinct complex numbers and assume that $\Lambda_{n+1} \cap \Lambda_n = \emptyset$ then the set of all $X \in M_m(\mathbb{C})$ such that the spectrum of X_n is Λ_n is an subvariety isomorphic with $(\mathbb{C}^{\times})^{\binom{m}{2}}$.*

$\mathfrak{g}$ a Lie algebra over $\mathbb{C}$ with a nondegenerate, symmetric, bilinear, $\mathfrak{g}$–invariant form.

Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and thereby $S(\mathfrak{g})$ with $S(\mathfrak{g}^*) = P(\mathfrak{g})$. $U(\mathfrak{g})$ filtered by the standard filtration $U^j \subset U^{j+1}$ one has

$$GrU(\mathfrak{g}) \cong S(\mathfrak{g})$$

as $P(\mathfrak{g})$. If $f, g \in P(\mathfrak{g})$ are homogeneous of degree j, k respectively then there are $u \in U^j, v \in U^k$ such that

$$u + U^{j-1} \to f, v + U^{k-1} \to g$$

Then $[u, v] \in U^{j+k-1}$ and modulo U^{j+k-2} we get an element of P^{j+k-1}. Independent of all of the choices, $\{f, g\}$.The Poisson-Bracket.

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

This implies that there is a vector field, X_f. on $\mathfrak{g}$ with polynomial coefficients such that $X_f g = \{f, g\}$.

$\mathfrak{g}_n = M_n(\mathbb{C})$ imbedded in $\mathfrak{g}_{n+1}$ as the upper left corner.

Take $f_{j,k,n}(X) = \mathrm{tr} X_k^j, j = 1, 2, ..., k, k = 1, ..., n$. These polynomials Poisson commute and are functionally independent.

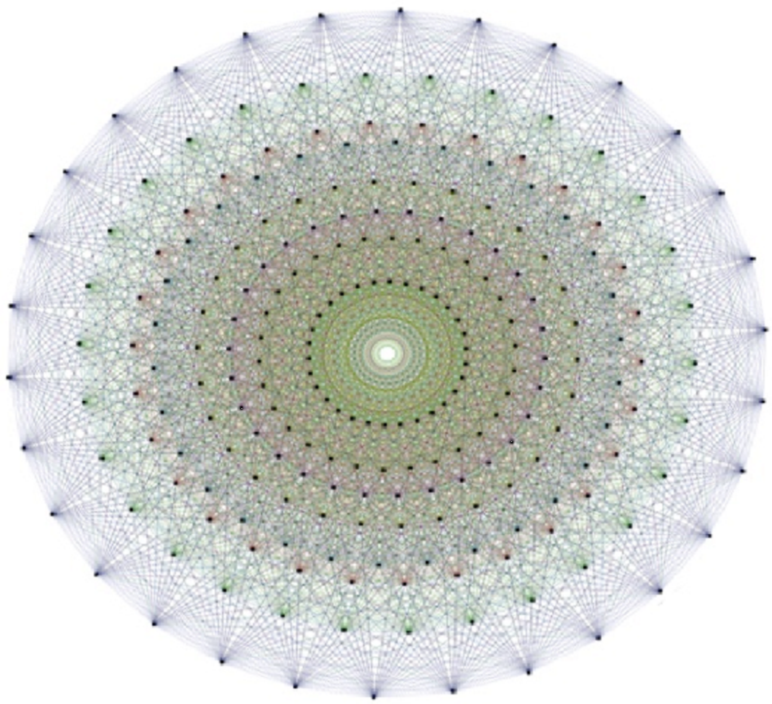
GZ_n is the algebra generated by $f_{j,k,n}(X) = \mathrm{tr} X_k^j, j = 1, 2, ..., k, k = 1, ..., n-1$.

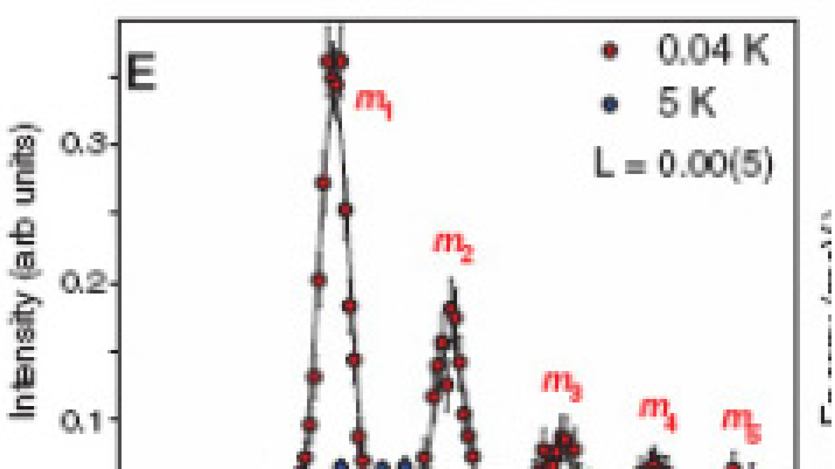
Theorem 4 *If $f \in GZ_m$ then X_f generates a global one parameter group of holomorphic transformations. Thus, in particular, product of the groups generated by the $X_{f_{j,k,n}}$ acts on $M_n(\mathbb{C})$ holomorphically yielding holomorphic action of the additive group of $\mathbb{C}^{\binom{n}{2}}$ on $M_n(\mathbb{C})$.*

Theorem 5 *If $X \in \Omega_m$ then $AX \subset \Omega_m$ and the orbit, AX is the set of all $Y \in M_m(\mathbb{C})$ such that the spectra of the truncations of Y are the same as those of X. Furthermore, the restriction of the action of A to Ω_n yields an algebraic action of $(\mathbb{C}^\times)^{\binom{m}{2}}$.*

The last part of this theorem involves work form Mark Colarusso's thesis. He and Sam Evens have a series of papers that give a detailed study of the properties of these flows with applications.

The analogous flow on Hermitian matrices was studied by Guilleman and Sternberg.




E
0.04 K
5 K
L = 0.00(5)
m_1
m_2
m_3
m_4
m_5
Intensity (arb units)
0.3
0.2
0.1
0.0
1.0
1.2
1.4
1.6
Energy (meV)

Bertram Kostant, The Principal Three-Dimensional Subgroup and the Betti Numbers of a Complex Simple Group, American Journal of Mathematics, 81(1959),973-1032.

Bertram Kostant , Lie Group Representations on Polynomial Rings, American Journal of Mathematics, 85(1963), 327-404.

Bertram Kostant, Eigenvalues of a Laplacian and Commutative Lie Subalgebras, Topology, 3(1965), 147-159.

$\mathfrak{g}$ simple Lie algebra over $\mathbb{C}$ of rank l. $\mathfrak{b}$ a Borel subalgebra.

$\wedge^l\mathfrak{g}$ as a $\mathfrak{g}$-module in the usual way.

C the Casimir operator corresponding to the Killing form of $\mathfrak{g}$.

The C eigenvalues on $\wedge^l\mathfrak{g}$ are all less than or equal to l

Theorem 6 *The largest eigenvalue of* C *on* $\wedge^l\mathfrak{g}$ *is* l. *The* $\mathfrak{b}$ *the highest weight spaces of the* l*-eigenspace,* V_l*, in* $\wedge^l\mathfrak{g}$ *are the lines* $\wedge^l\mathfrak{a}$ *with* $\mathfrak{a}$ *an abelian ideal in* $\mathfrak{b}$ *of dimension* l. *Let* V_l *be this eigenspace.*

Ranee Brylinski:

$G_l(\mathfrak{g})$ the Grassmannian of l dimensional subspaces of $\mathfrak{g}$.

$\mathfrak{h}$ a Cartan subalgebra.

G the adjoint group of $\mathfrak{g}$.

The closure of $G\mathfrak{h}$ contains all abelian ideals in $\mathfrak{b}$ of dimension l.

We proved.

e a regular nilpotent of $\mathfrak{g}$.

Theorem 7 *If x is a regular element of $\mathfrak{g}$ contained in $\mathfrak{b}$ then $\mathfrak{g}^e$ is contained in the closure of $B\mathfrak{g}^x$.*

Theorem 8 *The closure of $B\mathfrak{g}^e$ contains all abelian ideals of $\mathfrak{b}$ of dimension l.*

Considering the action of G on $G_l(\mathfrak{g})$ then each orbit GV contains a closed orbit GW in its closure. The stabilizer of W is a parabolic subgroup of G. Hence it contains a Borel subgroup. One checks that if $\dim W = l$ and $BW = W$ then $W \subset [\mathfrak{b}, \mathfrak{b}]$. Thus it is an ideal in $\mathfrak{b}$.

Theorem 9 *l–dimensional ideals in $\mathfrak{b}$ are abelian.*

Critical to the proofs are Kostant's cyclic elements that appear in two of the three papers listed. Let $f_1, ..., f_l$ be a minimum set of homogeneous generators for $P(\mathfrak{g})^G$ ordered by degree. An element, $x \in \mathfrak{g}$ is called cyclic if

$$f_i(x) = 0, i < l, f_l(x) \neq 0.$$

Note that if c is a Coxeter element of $W(\mathfrak{h})$ then the order of c is the Coxeter number

$$h = d_l = \deg f_l > d_j = \deg f_j, j < l.$$

Let ζ be a promitive h–th root of 1.Then if $z \in \mathfrak{h}, z \neq 0$ is such that $cz = \zeta z$ then

$$f_i(z) = f_i(cz) = \zeta^{d_i} f_i(z).$$

Thus z is a cyclic element.

The singular set X the set of $x \in \mathfrak{g}$, $\dim \mathfrak{g}^x > l$.

The zero set of the $l \times l$ minors of

$$\mathrm{Jac}(f_1, .., f_l).$$

$$\mathcal{I}(X) = \{f \in P(\mathfrak{g}) | f(X) = 0\}.$$

Do the $l \times l$ minors of $\mathrm{Jac}(f_1, .., f_l)$ generate $\mathcal{I}(X)$?

$(..., ...)$ the Killing form.

If $f \in P(\mathfrak{g})$ then define $\nabla f(x)$ for $x \in \mathfrak{g}$ by

$$(\nabla f(x), y) = df_x(y), y \in \mathfrak{g}.$$

$$\{\nabla f_1(x), ..., \nabla_l f(x)\}$$

is a basis of $\mathfrak{g}^x$ if x is regular. Noting that if $f \in P(\mathfrak{g})^G$ then

$$g\nabla f(x) = \nabla f(gx).$$

If x is regular semisimple then

$$C\left(\nabla f_1(x) \wedge \cdots \wedge \nabla_l f(x)\right) = l\nabla f_1(x) \wedge \cdots \wedge \nabla_l f(x).$$

Thus the result is true for all x. If M is the span of the $l \times l$ minors of Jac$(f_1, .., f_l)$ then $Cf = lf$ for $f \in M$.

Theorem 10 *M is equivalent with V_l as a G–module.*

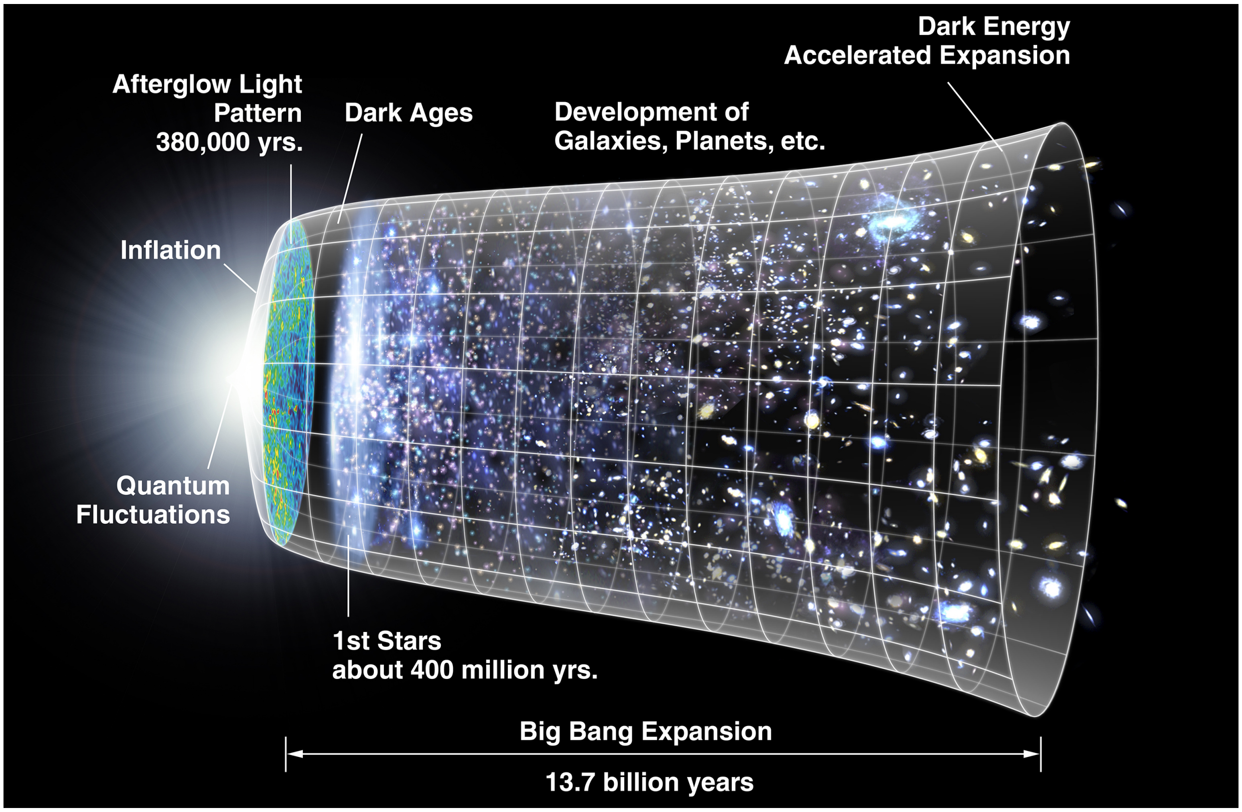
If $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra then let

$$\pi_{\mathfrak{h}} = \prod_{\alpha > 0} \alpha$$

for some choice of positive roots relative to $\mathfrak{h}$.

$$\mathcal{I} = \{f \in P(V) | \pi_{\mathfrak{h}} | f_{|\mathfrak{h}}\}.$$

Then $M \subset \mathcal{I}$.



$$Ric - \frac{1}{2}Rg + \Lambda g = \frac{8\pi G}{c^4}T$$

Λ the cosmological constant.

g a Lorentzian (- - - +) metric on a 4 manifold M.

Associated to g is the Hodge $*$–operator mapping $\wedge^k T(M)^* \longrightarrow \wedge^{4-k} T(M)^*$.

Maxwell's equations $\omega \in \wedge^2 T(M)^*$:

$$d\omega = d * \omega = 0.$$

These equations are conformally invariant.

Examples:

$M = \mathbb{R}^4, g = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$.

$H = 2 \times 2$ Hermitian matrices

$$X = \begin{bmatrix} x_4 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_4 - x_3 \end{bmatrix}$$

with the metric $(X, X) = \det(X)$.

$M = U(2) \cong S^1 \times S^3$. $T_I(M) = iH$ and the left invariant metric $\langle X, X \rangle = -\det(X)$.

$$F : H \to M,$$

$$F(X) = \frac{I + iX}{I - iX},$$

$$(F^* \langle ..., ... \rangle)_X = 4(1 + 2 \sum_{i=1}^{4} x_i^2 + (X, X)^2)(..., ...)_X.$$

Two incarrnations of $U(2, 2)$:

$$J_1 = \begin{bmatrix} 0 & iI \\ -iI & 0 \end{bmatrix}, J_2 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

$$G_i = \{g \in GL(4, \mathbb{C}) | gJ_i g^* = J_i\}.$$

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix}$$

$$\sigma(g) = LgL^* = LgL^{-1}, \sigma : G_1 \to G_2.$$

If

$$\sigma(g) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$gZ = (AZ + B)(CZ + D)^{-1}$$

for $ZZ^* < I$.

This realizes $U(2)$ as the Shilov boundary of $ZZ^* < I$ and $U(2,2)$ as a group of conformal transformations of M.

$$Maxw \subset \Omega^2(M)_{\mathbb{C}}$$

the space of all smooth solutions to Maxwell's equations with the C^∞ topology. Then this representation of $U(2,2)$ is an admissible, Smooth, Fréchet module of moderate growth. It breaks up into four irreducible submodules. $Maxw^{\pm}_{0,2}, Maxw^{\pm}_{2,0}$.

Note

$$H^2(U(2), \mathbb{C}) = 0,$$

$$H^3(U(2), \mathbb{C}) = \mathbb{C}, H_3(U(2), \mathbb{Z}) = \mathbb{Z}SU(2).$$

This implies that if $\eta \in \Omega^3(\mathbb{C})$ and $d\eta = 0$ then of $g \in U(2,2)$ then

$$\int_{SU(2)} \eta = \int_{gSU(2)} \eta = \int_{SU(2)} g^* \eta.$$

If $\alpha, \beta \in Maxw$ then $\alpha = d\mu$. Set

$$\langle \alpha, \beta \rangle = \int_{SU(2)} \mu \wedge \overline{\beta}$$

The formula is independent of the choices and

$$\left\langle Maxw^{\varepsilon}_{p,q}, Maxw^{\nu}_{rs} \right\rangle = 0$$

unless $\varepsilon = \nu$ and $(p,q) = (r,s)$. Furthermore,

$$\varepsilon \left\langle ..., ... \right\rangle_{|Maxw^{\varepsilon}_{p,q}} > 0.$$

If V is the underlying $(\mathfrak{g}, K)$–module of one of the components of $Maxw$ then

$$H^2(\mathfrak{g}, K, V) = \mathbb{C}.$$

It also has a realization in the continuation of the holomorphic discrete series or the antiholomophic discrete series.

$SU(2,2)$ is also the quaternionic real form of $SL(4,\mathbb{C})$ and so these representations can be realized as first cohomology of holomorphic or antiholomorphic line bundles on the open orbit in $\mathbb{P}^3$ that has no non-constant functions. This corresponds to the twister transform.