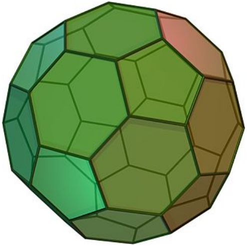
Defining equations for some nilpotent varieties

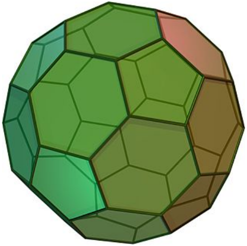
Eric Sommers (UMass Amherst)
Ben Johnson (Oklahoma State)

The Mathematical Legacy of Bertram Kostant
MIT
June 1, 2018

- Kostant's interest in the Buckyball



- Kostant's interest in the Buckyball



- He didn't like the Brooklyn Dodgers

Giants Top Dodgers and Win Flag on Thomson's Homer, 5-4; World Series to Open Today

Three-Run Blast In Ninth Climaxes New York Surge

Dramatic Wallop Off Branca Brings Giants First Pennant Since 1937; Brooklyn Loses Three-Run Lead

By DUTCH ROBBINS

Polo Grounds, New York, Oct. 3—Everybody said it couldn't be done but the New York Giants did it. They said it all during the month of August and they said it again today after 8½ innings of play here at the Polo Grounds before a crowd of 34,320. Then in the most dramatic finish that baseball will ever know, Bobby Thomson stepped up to the plate and hit a three-run homer that made the rags-to-riches Giants the National League champions of 1951. Thomson's smash, which will go down in history as one of the greatest clutch wallops of its kind, gave the Cinderella Kids of Manager Leo Durocher a 5-4 victory over the Brooklyn Dodgers in the deciding game of the three-game series

KEYSTONE ACTION—Going for a head-over-heels spill is Brooklyn shortstop Pee Wee Reese as Giants' Wes Westrum comes bowling into second base on a force play. The Giants won, 5-4, to win the National League pennant.

Reynolds Picked As Yankee Hurler In Series Opener

Giants Expected to Counter With Koslo, Although Jansen Given Outside Chance of Starting in Stadium Today

By VIC WALL

New York, Oct. 3—The New York Yankees, fresh as daisies, old hands at this World Series business, open defense of their world championship in their own park tomorrow afternoon with a delicious New York Giant team, the new Cinderella club of baseball, as opponents. But until Bobby Thomson crashed a dramatic home run to pull the Giants out of the gutter late this afternoon, the Yankees didn't know what club they would meet.

Anticlimax in Playoffs

In some respects this World Series that opens tomorrow in the vast Yankee Stadium is one of the most remarkable in history. In truth, an anticlimax to what has gone on in

Kostant's most highly cited paper in Math Reviews

Let $\mathcal{N}$ be nilpotent cone in $\mathfrak{g}$. Kostant showed

- $\mathcal{N}$ is a normal variety
- The defining ideal of $\mathcal{N}$ is generated by

  $$u_1, \ldots, u_\ell,$$

  a set of basic invariants in $S\mathfrak{g}^*$. Assume $\deg(u_\ell) = h$.
- As a module for G, write

  $$\mathbb{C}^\bullet[\mathcal{N}] \simeq \oplus p_\lambda V_\lambda$$

  where $p_\lambda = q^{m_1^\lambda} + \cdots + q^{m_{\ell_\lambda}^\lambda}$. Then

  $$\ell_\lambda = \dim V_\lambda^T.$$

  These exponents are called the *generalized exponents* of λ.

Key fact

Let e be a principal nilpotent and e, h, f basis of $\mathfrak{s}l_2$-triple.

Form slice

$$\mathfrak{v} := f + \mathfrak{g}^e.$$

The $\mathbb{C}^*$-action on $\mathfrak{v}$ coming from h and scaling so that that f is in degree 0.

Then $\mathfrak{g}^e$ is graded in degrees $2m_1 + 2, \ldots, 2m_\ell + 2$, where $m_1, \ldots, m_\ell$ are the usual exponents.

Restrict u_i to $\mathfrak{v}$. Then

Theorem (Kostant)

u_i has linear term when expressed in the graded basis of $\mathfrak{g}^e$.

Jacobian matrix of u_i's is rank ℓ everywhere on $\mathfrak{v}$, including at f.

Explicit list of basic invariants

- Can take a possible list and restrict to a Cartan subalgebra $\mathfrak{h}$. Look for a point where determinant of Jacobian is nonzero.

Explicit list of basic invariants

- Can take a possible list and restrict to a Cartan subalgebra $\mathfrak{h}$. Look for a point where determinant of Jacobian is nonzero.
- Take a possible single basic invariant and restrict to $\mathfrak{v}$ and see that it has a linear term.

Theorem

The invariants

$$tr((ad\,X)^2), \ldots tr((ad\,X)^{d_i}), \ldots tr((ad\,X)^{30})$$

is a list of basic invariants for E_8.

Can use smaller representations for other types if adjoint representation doesn't work (e.g., if there is an odd fundamental degree).

Example in MAGMA

Adjoint representation of the slice $\mathfrak{v}$ for F_4, using 4 variables: $m[1], m[2], m[3], m[4]$.

```
> M;
[0   m[1]   0   0   0   0   m[2]   -m[2]   0   0   0   0   -m[3]   -m[3]   0   0   0   0   0   0   0   0   0   0   m[4]   0   0
[22   0   m[1]   0   0   0   0   0   m[2]   0   2*m[2]   0   0   0   -m[3]   -m[3]   0   0   0   0   0   0   0   0   0   0   0
[0   42   0   m[1]   0   0   0   0   0   0   0   m[2]   0   2*m[2]   0   0   0   -m[3]   0   0   0   0   0   0   0   0   0
[0   0   60   0   2*m[1]   m[1]   0   0   0   0   0   0   0   0   m[2]   0   2*m[2]   0   0   0   -m[3]   m[3]   0   0   0   0
[0   0   0   30   0   0   m[1]   m[1]   0   0   0   0   0   0   0   0   0   0   0   -2*m[2]   0   0   0   0   0   -m[3]   m[3]
[0   0   0   16   0   0   0   m[1]   0   0   0   0   0   0   0   0   0   m[2]   m[2]   0   0   0   0   0   0   m[3]   -m[3]
[0   0   0   0   42   0   0   0   m[1]   m[1]   0   0   0   0   0   0   0   0   0   0   -m[2]   0   0   2*m[2]   0   0   0   0
[0   0   0   0   32   30   0   0   0   m[1]   -2*m[1]   0   0   0   0   0   0   0   0   0   0   m[2]   0   -2*m[2]   0   0   0   0
[0   0   0   0   0   0   22   0   0   0   0   m[1]   0   0   0   0   0   0   0   0   0   0   0   0   2*m[2]   0   0   -2*m[2]   0
[0   0   0   0   0   0   32   42   0   0   0   m[1]   m[1]   -2*m[1]   0   0   0   0   0   0   0   0   0   0   2*m[2]   -2*m[2]
[0   0   0   0   0   0   0   -16   0   0   0   0   0   m[1]   0   0   0   0   0   0   0   0   0   0   m[2]   0   -m[2]   0
[0   0   0   0   0   0   0   0   32   22   0   0   0   0   m[1]   -2*m[1]   0   0   0   0   0   0   0   0   0   0   0   0   -2*m[
[0   0   0   0   0   0   0   0   0   30   0   0   0   0   m[1]   0   -m[1]   0   0   0   0   0   0   0   0   0   0   0   0   0
[0   0   0   0   0   0   0   0   0   -16   42   0   0   0   0   m[1]   m[1]   0   0   0   0   0   0   0   0   0   0   0   0   -m[
[0   0   0   0   0   0   0   0   0   0   0   30   22   0   0   0   0   m[1]   -m[1]   0   0   0   0   0   0   0   0   0   0   0
[0   0   0   0   0   0   0   0   0   0   0   -16   0   22   0   0   0   0   m[1]   0   0   0   0   0   0   0   0   0   0   0   0   0
[0   0   0   0   0   0   0   0   0   0   0   0   -16   60   0   0   0   0   m[1]   -2*m[1]   0   0   0   0   0   0   0   0   0
[0   0   0   0   0   0   0   0   0   0   0   0   0   0   42   0   0   0   0   0   m[1]   -m[1]   0   0   0   0   0   0   0   0
[0   0   0   0   0   0   0   0   0   0   0   0   0   0   -16   60   22   0   0   0   0   m[1]   -2*m[1]   0   0   0   0   0   0
[0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   -30   0   0   0   0   0   m[1]   -m[1]   0   0   0   0   0   0
[0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   30   0   0   0   0   0   0   0   0   -m[1]   2*m[1]   0   0
[0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   -16   42   0   0   0   0   0   0   -2*m[1]   2*m[1]   -m[1]
```

```
> Trace(M^2);
5616*m[1]
> Trace(M^6);
506278656*m[1]^3 - 50523782400*m[2]
> Trace(M^8);
19991582208*m[1]^4 - 5492841984000*m[1]*m[2] - 173283618816000*m[3]
> Trace((M/6)^12);
1004977308700/59049*m[1]^6 - 9278201548000/729*m[1]^3*m[2] - 23228446472000/243*m[1]^2*m[3] + 1461034225000/9*m[2]^2 - 697250400000*m[4]
```

Linear term in each case.

Usual exponents

When $\lambda = \theta$ is the highest root, the generalized exponents are the usual exponents.

Two ways to see this:

- Generalized exponents come from grading of

  $$V_\lambda^{\mathfrak{g}^e} = \mathfrak{g}^{\mathfrak{g}^e} \subset \mathfrak{g}^e.$$

  Have equality since $\mathfrak{g}^e$ is abelian.

- Fix i. Then

  $$\left\{ \frac{\partial u_i}{\partial x_j} \right\}$$

  is a basis of a copy of the adjoint representation in $S\mathfrak{g}^*$.

  Non-zero on $\mathcal{N}$; in fact, the copies are linearly independent.

  So $\{d_i - 1\}$ are generalized exponents for $\lambda = \theta$.

Applications to intersections

- $\mathcal{N} \cap \mathfrak{h} = \{0\}$. Functions at 0 are

$$S\mathfrak{h}^*/(\text{positive invariants}) \simeq H^*(G/B).$$

  Starting point to look at $\overline{\mathcal{O}} \cap \mathfrak{h}$.
  We obtain cohomology of Springer fiber for dual orbit in type A.

  Kraft, De Concini-Procesi, Tanisaki, Carrell

- $\mathcal{N} \cap \mathcal{S}_{f'}$ for smaller nilpotent, where

$$S_{f'} = f' + \mathfrak{g}^{e'}$$

  Also can do this by replacing $\mathcal{N}$ by smaller nilpotent orbit $\mathcal{O}$:

$$\overline{\mathcal{O}} \cap S_{f'}.$$

  Brieskorn, Slodowy, Kraft-Procesi, Fu-Juteau-Levy-S.

Subregular slice

Let f' be in the subregular orbit. Take slice to f':

$$S_{f'} = f' + \mathfrak{g}^{e'}$$

This has dimension $\ell + 2$.

Subregular slice

Let f' be in the subregular orbit. Take slice to f':

$$S_{f'} = f' + \mathfrak{g}^{e'}$$

This has dimension $\ell + 2$.

Then $u_1, \ldots, u_{\ell-1}$ have (linearly independent) linear terms on $S_{f'}$.

While u_ℓ of highest degree, the Coxeter number, is exactly the defining equation in the remaining three dimensions of an ADE-singularity.

Carried out by Slodowy

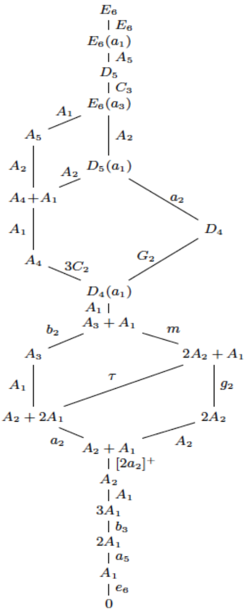
Example in E_6

```
> invs;
[
    1008*m[8],
    -11232000*m[5] + 1140480*m[7]*m[8],
    -161740800*m[4] + 11404800*m[6]*m[8] + 58934304*m[7]^2 + 9232128*m[8]^3,
    9754214400*m[2] - 54809395200*m[4]*m[8] - 31788288000*m[5]*m[7] + 12814848000*m[6]^2 + 3328819200*m[6]*m[8]^2 + 16539780096*m[7]^2*m[8] +
        1134185472*m[8]^4,
    263363788800*m[1] - 921773260800*m[4]*m[7] - 399748608000*m[5]*m[6] - 709619097600*m[5]*m[8]^2 - 58002739200*m[6]*m[7]*m[8] -
        90860507136*m[7]^3 + 69352464384*m[7]*m[8]^3,
    15643809054720000*m[1]*m[7] + 10429206036480000*m[2]*m[6] + 919315198771200*m[2]*m[8]^2 - 929147446886400*m[3]^2 +
        929147446886400*m[3]*m[4] + 321798301286400*m[4]^2 - 346851901440000*m[4]*m[6]*m[8] - 252240162324480*m[4]*m[7]^2 -
        252627305103360*m[4]*m[8]^3 + 114803619840000*m[5]^2*m[8] + 884620154880000*m[5]*m[6]*m[7] - 521149908787200*m[5]*m[7]*m[8]^2 -
        239446056960000*m[6]^3 + 123273662054400*m[6]^2*m[8]^2 + 316344700108800*m[6]*m[7]^2*m[8] + 127622814105600*m[6]*m[8]^4 +
        63774280824883*m[7]^4 + 71315439914188*m[7]^2*m[8]^3 + 20115886620672*m[8]^6
]

>
> GroebnerBasis(IAdjNil);
[
    m[1] - 1037/6400*m[7]^3,
    m[2] + 103/784*m[6]^2,
    m[3]^2 - 583/1600*m[3]*m[7]^2 + 6655/38416*m[6]^3 - 92323/320000*m[7]^4,
    m[4] - 583/1600*m[7]^2,
    m[5],
    m[8]
]
> Evaluate(x[3], [m[1], m[2], m[3]+583/3200*m[7]^2, m[4],m[5],m[6],m[7],m[8]]);
m[3]^2 + 6655/38416*m[6]^3 - 131769/409600*m[7]^4
>
```

The latter is the equation for the E_6 singularity in $\mathbb{C}^3$: $x^2 + y^3 + z^4 = 0$.

Singularities in E_6



Find equations for other orbits

- Weyman for $GL_n(\mathbb{C})$.

  $\mathcal{O} = \mathcal{O}_\lambda$

  $\lambda = (\lambda_1, \lambda_2, ...)$ partition of n

  Let $k_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i - i + 1$

  Equations come from subspace of $k_i \times k_i$-minors isomorphic to representation of highest weight $\varpi_i + \varpi_{n-i}$, plus the basic invariants.

Find equations for other orbits

- Weyman for $GL_n(\mathbb{C})$.

  $\mathcal{O} = \mathcal{O}_\lambda$

  $\lambda = (\lambda_1, \lambda_2, ...)$ partition of n

  Let $k_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i - i + 1$

  Equations come from subspace of $k_i \times k_i$-minors isomorphic to representation of highest weight $\varpi_i + \varpi_{n-i}$, plus the basic invariants.
- Hook $\lambda = (a, 1, ...1)$.

  Minimal generators: all $a \times a$ minors. Rank conditions plus basic invariants up to degree a.

Find equations for other orbits

- Weyman for $GL_n(\mathbb{C})$.

  $\mathcal{O} = \mathcal{O}_\lambda$

  $\lambda = (\lambda_1, \lambda_2, ...)$ partition of n

  Let $k_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i - i + 1$

  Equations come from subspace of $k_i \times k_i$-minors isomorphic to representation of highest weight $\varpi_i + \varpi_{n-i}$, plus the basic invariants.
- Hook $\lambda = (a, 1, ...1)$.

  Minimal generators: all $a \times a$ minors. Rank conditions plus basic invariants up to degree a.
- Almost rectangular: $\lambda = (a, a, ...a, b)$.

  Minimal generators: Just need a copy of the adjoint in degree a and basic invariants up to degree a.

  Take entries X^a where $X = (x_{ij})$ is a generic matrix for a copy of the adjoint rep.

Broer's result on the subregular orbit

- Let ϕ be the dominant short root.
- The highest generalized exponent for V_ϕ occurs in $\mathrm{ht}(\phi)$ degree, the dual Coxeter number. This is true for any representation V_λ.
- The ideal for the subregular nilpotent variety is given by a copy of

  $$V_\phi \text{ in this top degree}$$

  together with

  $$u_1, \ldots, u_{\ell-1}.$$

- These are minimal generators.

Main result

- Let Ω be a set of orthogonal, short, simple roots. Let $s = |\Omega|$.

  Let $\mathfrak{n}_\Omega$ be nilradical of the parabolic subalgebra attached to Ω.
- Let $\mathcal{O}_\Omega$ be the Richardson orbit in $\mathfrak{n}_\Omega$.

  These orbits were considered by Broer in Kostant 65th volume.

  For $s = 1$, we get the subregular orbit. For $s = 0$, we get principal nilpotent orbit.
- Let r be dimension of zero weight space of V_ϕ, which is the number of short simple roots, and order the generalized exponents for V_ϕ by $m_1^\phi \leq \cdots \leq m_r^\phi$.

Main result

- Let Ω be a set of orthogonal, short, simple roots. Let $s = |\Omega|$.

  Let $\mathfrak{n}_\Omega$ be nilradical of the parabolic subalgebra attached to Ω.
- Let $\mathcal{O}_\Omega$ be the Richardson orbit in $\mathfrak{n}_\Omega$.

  These orbits were considered by Broer in Kostant 65th volume.

  For $s = 1$, we get the subregular orbit. For $s = 0$, we get principal nilpotent orbit.
- Let r be dimension of zero weight space of V_ϕ, which is the number of short simple roots, and order the generalized exponents for V_ϕ by $m_1^\phi \leq \cdots \leq m_r^\phi$.

Theorem (Johnson, S-)

The ideal for $\overline{\mathcal{O}_\Omega}$ is minimally generated by:

- *a copy of V_ϕ in either degree m_{r-s+1}^ϕ or $m_{\lfloor \frac{r}{2} \rfloor}^\phi$.*
- *(sometimes) a copy of V_ϕ is degree m_{r-s+2}^ϕ.*
- *$r - s$ of the basic invariants*

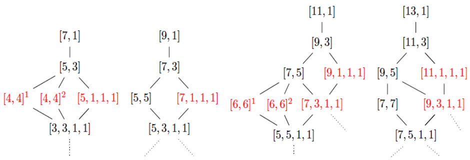
Main result

Theorem (Johnson, S-, arXiv:1706.04820)

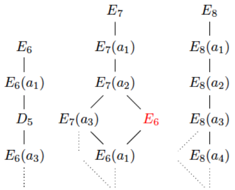
The ideal for $\overline{\mathcal{O}}_\Omega$ is minimally generated by:

- *a copy of V_ϕ in either degree m^ϕ_{r-s+1} or $m^\phi_{\lfloor \frac{r}{2} \rfloor}$.*
- *(sometimes) a copy of V_ϕ is degree m^ϕ_{r-s+2}.*
- *$r - s$ of the basic invariants*

FIGURE 1. The studied nilpotent varieties with second family in red



(a) In types D_4, D_5, D_6, and D_7



(b) In types E_6, E_7, and E_8

Flat basis

- Pick a basis $\{x_i\}$ of $\mathfrak{g}$ and a dual basis $\{y_i\}$ with respect to the Killing form $(\cdot,\cdot)$.
- Let p and q be two homogeneous invariants of degree $a+1$ and $b+1$, respectively. Then

  $$p \circ q := \sum_i \frac{\partial p}{\partial x_i} \frac{\partial q}{\partial y_i}$$

  is again an invariant.

  Homogeneous of degree $a+b$.
- Saito's flat basis, first considered in a paper by Saito, Yano, Sekiguchi: unique basis (up to scalars) with $u_i \circ u_j \in \mathbb{C}[u_1, \ldots, u_{\ell-1}] + cu_\ell$, where c is a constant.
- De Concini, Papi, Procesi:

  $u_i \circ u_j$ a generator of the invariants when $u_i \circ u_j$ is the degree of some u_k.
  A weaker statement is true in type D_{2k}.

Containment of ideals

- Consider the copy of the adjoint representation V_{u_i} determined by u_i by taking derivatives.
- Take its ideal (V_{u_i}) in $\mathbb{C}[\mathcal{N}]$.

Theorem (Johnson, S-)

The following are equivalent:

- *Containment:* $V_{u_j} \subset (V_{u_i})$
- *There exists an invariant* p *such that* $p \circ u_i = u_j$ *modulo expressions in lower degree invariants.*

Hence, by DPP result, containment question, outside of D_{2k}, is equivalent to $(m_j + 1) - m_i$ is an exponent. This helps us find minimal generators.

For example, in E_7, adjoint rep in degree 13 is not in the ideal generated by copy in degree 11, but it is in ideal generated by copy in degree 9, since 3 is not an exponent, but 5 is.

Cohomology on the resolution

Next we search for a set of generators. Let $P = P_\Omega$. Consider the Springer type map:

$$G \times^P \mathfrak{n}_\Omega \to \overline{\mathcal{O}}_\Omega$$

If this is a resolution,

$$\mathbb{C}[\mathcal{O}_\Omega] = \mathbb{C}[G \times^P \mathfrak{n}_\Omega] = H^0(G/P, S^\bullet \mathfrak{n}_\Omega^*)$$

For $\Omega = \emptyset$, $\mathcal{O}_\Omega$ is regular nilpotent orbit.

$$\mathbb{C}[\mathcal{N}] = \mathbb{C}[\mathcal{O}_{reg}] = \mathbb{C}[G \times^B \mathfrak{n}] = H^0(G/B, S^\bullet \mathfrak{n}^*)$$

Paper by R. Brylinski (Twisted Ideals paper):

- thinking about ideals in $\mathbb{C}[\mathcal{N}]$ coming from cohomology
- subregular ideal

Twisted ideals

Higher vanishing

$H^i(G/B, S^\bullet \mathfrak{n}^* \otimes \mathbb{C}_\mu) = 0$ for $i > 0$ when

- $\mu = 0$ (Borho-Kraft, Hesselink)
- μ dominant (Broer)
- μ slightly not dominant (Broer)

To compute the occurrences of V_λ in H^0, compute Euler characteristic $\sum(-1)^i H^i$, and thus replace $S^\bullet \mathfrak{u}^*$ by a sum of one-dimensional representations and then use Bott-Borel-Weil:

Bott-Borel-Weil

$H^i(G/B, \mathbb{C}_\lambda) = 0$ except if $w \cdot \lambda$ is dominant and $i = \ell(w)$, in which case it is $V_{w\cdot\lambda}$. Here, $w \in W$, the Weyl group, is unique.

Conclude: $H^0(G/B, S^\bullet \mathfrak{u}^* \otimes \mathbb{C}_\mu)$ is computable in terms of Lusztig's q-analog of Kostant weight multiplicity.

Twisted Ideals

- Hence, multiplicity of V_λ in $H^0(G/B, S^\bullet\mathfrak{u}^* \otimes \mathbb{C}_\mu)$ is the dimension of the μ-weight space in V_λ.
- The graded version is an affine Kazhdan-Lusztig polynomial (Lusztig).
- If $\mu = 0$, get a formula for generalized exponents and also get another way of seeing that their number is dimension of the zero weight space of V_λ.

Broer

For μ dominant:

$$H^0(G/B, S^\bullet\mathfrak{u}^* \otimes \mathbb{C}_\mu)$$

will identify with an ideal in

$$\mathbb{C}[\mathcal{N}]$$

and the unique copy of V_μ in lowest degree generates the ideal.

Sketch proof in A_3 with $s = 2$

Let $\Omega = \{\alpha_1, \alpha_3\}$. Consider parabolic and nilradical for subregular: $\mathfrak{n}_{\alpha_3}$. Take Koszul resolution:

$$0 \to S^{n-1}\mathfrak{n}^*_{\alpha_3} \otimes \mathbb{C}_{\alpha_1} \to S^n\mathfrak{n}^*_{\alpha_3} \to S^n\mathfrak{n}^*_{\alpha_1,\alpha_3} \to 0.$$

Take cohomology over G/B:

$$0 \to H^0(S^{n-1}\mathfrak{n}^*_{\alpha_3} \otimes \mathbb{C}_{\alpha_1}) \to H^0(S^n\mathfrak{n}^*_{\alpha_3})$$
$$\to H^0(S^n\mathfrak{n}^*_{\alpha_1,\alpha_3}) \to H^1(S^{n-1}\mathfrak{n}^*_{\alpha_3} \otimes \mathbb{C}_{\alpha_1}) \to \dots.$$

Key facts are that H^1 vanishes and
$H^0(S^{n-1}\mathfrak{n}^*_{\alpha_3} \otimes \mathbb{C}_{\alpha_1}) \simeq H^0(S^{n-2}\mathfrak{n}^*_{\alpha_2} \otimes \mathbb{C}_{\phi})$.
This is a generalization of Broer's result for $\mathfrak{n}$ when weights are slightly not dominant (see next slide).

Hence, the ideal of the orbit is cut out by a copy of V_ϕ in degree 2 in the closure of the subregular orbit. General case uses this kind of induction.

Type A_l cohomological theorem

Let $\mathfrak{n}_m$ be the nilradical for the maximal parabolic in type A_l with simple root α_m not in the Levi subalgebra.

Theorem (S-)

Let r be in the range $-|l+1-2m|-1 \leq r \leq 0$. Then there is a G-module isomorphism :

$$H^i(S^n\mathfrak{n}_m^* \otimes r\varpi_m) \simeq H^i(S^{n+rm}\mathfrak{n}_{l+1-m}^* \otimes -r\varpi_{l+1-m})$$

for all $i, n \geq 0$.

This is always an isomorphism for H^0 when $r < 0$.

Type A_2

$$H^i(S^n\mathfrak{n}_1^* \otimes -\varpi_1) \simeq H^i(S^{n-1}\mathfrak{n}_2^* \otimes \varpi_2)$$

Can use this A_2 result in any bigger Lie algebra. In A_3 it says $H^i(S^n\mathfrak{n}_3^* \otimes \alpha_1) \simeq H^i(S^{n-1}\mathfrak{n}_2^* \otimes (\alpha_1 + \alpha_2 + \alpha_3))$ since α_1 has inner product -1 with α_2 and 0 with α_3. This was the key fact on the previous slide.