Quantizing G-actions on Poisson manifolds

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One of Bert Kostant's enduring contributions to mathematics: the theory of geometric quantization.

More explicitly: let (X^{2n}, w) be a symplectic manifold, let G be a Lie group and let $G \times X \to X$ be a Hamiltonian action of G on X

Then X is prequantizable if $[w] \in H^2(X, \mathbb{Z})$ in which case there exists a complex line $\mathbb{L} \to X$ and a connection on $\mathbb{L}$ with curvature form w.

The goal of geometric quantization: Associate with this data a Hilbert subspace $\mathcal{Q}(X)$ of the space of L^2 sections of $\mathbb{L}$ and a representation of G on $\mathcal{Q}(X)$ with "nice" properties.

For instance, for co-adjoint orbits, X, of G associate a subspace $\mathcal{Q}(X)$ of L^2 sections of $\mathbb{L}$ on which G acts irreducibly.

Our goal in today's lecture will be much more modest: To simplify we'll assume G us an m-torus and for X compact we'll take $\mathcal{Q}(X)$ to be the virtual vector space

$$-\operatorname{Ker} \partial_{\mathbb{L}} \oplus \operatorname{Im} \partial_{\mathbb{L}}$$

where $\partial_{\mathbb{L}}$ is the $\mathbb{L}$-twisted spin $\mathbb{C}$ Dirac operator.

What about X non-compact? The two main goals of today's lecture

1. Describe a method for quantizing non-compact prequantizable Hamiltonian G-manifolds based upon the "quantization commutes with reduction" principle.
2. Apply this method to an interesting class of examples: b-symplectic manifolds

Remark 1

What I am about to report on is joint work with Jonathan Weitsman and Eva Miranda.

Remark 2

According to David Vogan, Bert was probably not the inventor of $[Q, R] = 0$*, but it was certainly inspired by his ideas.*

Item 1: $[Q, R] = 0$ and formal quantization

Let (X, w) be a pre-quantizable symplectic manifold and $G \times X \to X$ a Hamiltonian G-action with moment map $\phi : X \to \mathfrak{g}^*$. In lieu of assuming X compact we'll assume that ϕ is proper.

Let $\mathbb{Z}_G \subseteq \mathfrak{g}^*$ be the weight lattice of G and $\alpha \in \mathbb{Z}_G$ a regular value of ϕ. For the moment we'll also assume that G acts freely on $\phi^{-1}(\alpha)$ in which case the reduced spree $X_\alpha = \phi^{-1}(\alpha)/G$ is a prequantizable symplectic manifold and $[Q, R] = 0$ asserts that $Q(X)_\alpha = Q(X_\alpha)$ where $Q(X)_\alpha$ is the α-weight space of $Q(X)$.

What if α is a regular value of ϕ but G *doesn't* act freely on $\phi^{-1}(\alpha)$? Then this assertion is true modulo some complications owing to the fact that X_α is, in this case, a symplectic orbifold, not a symplectic manifold.

What is α is a singular value of ϕ?

The Meinrenken desingularization trick

Replace α by a nearby α' which is a regular value of ϕ then convert the reduced symplectic form on $X_{\alpha'}$ into a symplectic form on $X_{\alpha'}$ which *would be*, up to symplectomorphism, the reduced symplectic form on X_{α} if α were a regular value of ϕ.

Inspired by these comments we'll define the formal quantization of X to be the sum

$$\bigoplus_{\alpha} \mathcal{Q}(X_\alpha)$$

Item 2: b-symplectic manifolds

These objects are easiest to define as Poisson manifolds. Namely, let $X = X^{2n}$ be a compact oriented Poisson manifold and let π be its Poisson bi-vector field. Then π^n is a section of the real line bundle $\Lambda^{2n}(TX) \to X$.

The definition of b-Poisson

We will say that π has this property if π^n intersects the zero section of $\Lambda^{2n}(TX)$ transversally in a codimension one submanifold Z.

Some implications of this assumption

1. Lets denote this intersection by Z. Then $\mathcal{U} = X - Z$ inherits from π a symplectic structure whose symplectic form we'll denote by w.
2. The symplectic leaves of the restriction, $\pi \,|\, Z$ are $2n-2$ dimensional symplectic submanifolds of Z and these manifolds define a foliation of Z by symplectic manifolds.

Let Z_i be a connected component of Z. Then on a tubular neighborhood $Z_i \times (-\epsilon, \epsilon)$ the symplectic form w has the form

$$w = \alpha \wedge \frac{dt}{t} + w_0, \quad -\epsilon < t < \epsilon$$

where α is a closed one-form on Z_i and w_0 a closed two form on Z_i.

Moreover, α and w_0 have the following properties:

I. For every symplectic leaf, L of $Z_i, \iota_L^* \alpha = 0$, i.e. α defines the symplectic foliation of Z

II. For every symplectic leaf, L, of Z_i, $\iota_L^* w_0$ is the symplectic form on this symplectic leaf

Prequantizability

We'll say that the Poisson manifold (X, π) is prequantizable if

1. $w\,|\,\mathcal{U}$ is prequantizable: i.e. there exists a line bundle-connection pair (L, ∇) on $\mathcal{U}$ with

$$w = \text{curv}(\nabla)$$

2. On the tubular neighborhood $\mathcal{U}_i = Z_i \times (-\epsilon, \epsilon)$ the two components. α and w_0 of the symplectic form

$$w\,|\,\mathcal{U}_i = \alpha \wedge \frac{dt}{t} + w_0$$

are pre-quantizable: For α this means that there exists a fibration $\gamma : Z_i \to S^1$ with

$$\alpha = \gamma^* \frac{d\theta}{2\pi}$$

Remark

An implication of this condition is that the symplectic leaves of Z_i, i.e. the level sets of γ, are compact.

Item 4: Hamiltonian G actions

We will denote by ${}^bC^\infty(X)$ the space of functions φ on X having the properties

1. $\varphi \,|\, X - Z \in C^\infty(X - Z)$
2. On the tubular neighborhood, $Z_i \times (-\epsilon, \epsilon)$ defined in item 2, $\varphi = c_i \,\mathrm{Log}\, |t| + \psi$, with $c_i \in R$ and $\psi \in C^\infty(Z_i \times (-\epsilon, \epsilon))$.

Remark

Notice that if t is the defining function for $Z \times \{0\}$ in the neighborhood $Z \times (-\epsilon, \epsilon)$ and $t' = ft$, $f \in C^\infty(Z_i \times (-\epsilon, \epsilon))$, is another defining function, i.e. $f \neq 0$, then

$$c \operatorname{Log} |t'| = c \operatorname{Log} |t| + c \operatorname{Log} |f|$$

with $c \operatorname{Log} |f| \in C^\infty(Z_i \times (-\epsilon, \epsilon))$

Hence this definition is an intrinsic definition: i.e. doesn't depend on the choice of defining function.

Now let $G \times X \to X$ be a Poisson action of G on X and let w be the b-symplectic form that we defined in item 3.

Definition

This action is Hamiltonian if for every $v \in \mathfrak{g}$

$$\iota(v_X)w = d\varphi_v, \quad \varphi_v \in {}^bC^\infty(X)$$

Thus $\varphi_v = u_i(v)\,\mathrm{Log}\,|t| + \psi_v$ where ψ_v is in $C^\infty(Z_i \times (-\epsilon, \epsilon))$ and u_i is an element of $\mathfrak{g}^*$.

Definition

u_i is the modular weight of the hypersurface Z_i.

Note that since

$$\iota(v_X)w = \iota(v_X)\alpha\frac{dt}{t} + \iota(v_X)w_0$$

$$u_i(v) = \iota(v_X)\alpha = \alpha(v_X) \tag{1}$$

Thus since the symplectic leaves of Z_i are defined by the condition, $\iota_L^* \alpha = 0$, we conclude that $u_i(v) = 0$ implies that v_X is everywhere tangent to the symplectic leaves of Z.

In particular if K is the subtorus of G having

$$k = \{v \in \mathfrak{g},\ u_i(v) = 0\}$$

as Lie subalgebra then the action of K on Z_i preserves the symplectic leaves of Z_i.

An important property of modular weights

Theorem

If X is connected and $u_i \neq 0$ for some Z_i then $u_i \neq 0$ for all Z_i's.

We'll henceforth assume that this is the case.

Another assumption we will make:
The map $\gamma : Z_i \to S^1$ defined by the prequantizability hypothesis

$$\alpha = \gamma^* \frac{d\theta}{2\pi}$$

intertwines the action of G on Z with the action of $S^1 = G/K$ on itself.

Therefore if S^1 is a complementary circle to K in G, S^1 acts freely on Z_i and interchanges the leaves of the symplectic foliation of Z_i.

In fact one can assume a bit more. Let $v \in \mathfrak{g}$ be the generator of this complementary circle and let $f = \iota(v_X)\alpha$. Then by the Moser trick one can prove

Lemma

The b-symplectic forms w and $w' = w - \alpha \wedge df$ are G-symplectomorphic.

Hence replacing w by w' we get as a corollary the following result:

Theorem

If v is the generator of the S^1 action above

$$\iota(v)w = d\,\mathrm{Log}\,|t|$$

and

$$\iota(v)w_0 = 0$$

In particular if u_i is the modular weight associated with Z_i, $u_i(v) = 1$.

Moreover the condition

$$\iota(v_X)w_0 = 0$$

implies the following: Lets denote by Z_i/S^1 the symplectic reduction by S^1 of $Z_i \times (-\epsilon, \epsilon)$ at some moment level $\lambda = \operatorname{Log}|t|,\ |t| \neq 0$.

Then the reduced symplectic form on Z_i/S^1 has the defining property

$$\pi^* w_{\text{red}} = w_0$$

where π is the projection

$$Z_i \to Z_i/S^1$$

Next let k_i be the Lie subalgebra of $\mathfrak{g}$ defined by

$$v \in k_i \Leftrightarrow u_i(v) = 0$$

Since u_i is a weight of $\mathfrak{g}$, (i.e., the *modular* weight) k_i is the Lie algebra of a subtorus K_i of G.

Moreover, $u_i(v) = 0$ implies $\iota(v_Z)\alpha = 0$ which in turn implies

$$\iota(v_Z)\omega = \iota(v_Z)\omega_0$$

Thus the projection, π, intertwines the actions of K on Z with a reduced symplectic action of K on Z/S^1.

Theorem 1

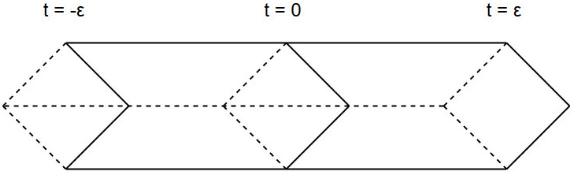
The moment map associated with the action of G on $Z_i \times (-\epsilon, \epsilon)$ is the map

$$\phi(z,t) = -\operatorname{Log}|t| u_i + \phi_K \circ \pi(z)$$

where ϕ_K is the moment map associated with the K-action on the symplectic quotient Z_i/S^1 and π is the projection of Z_i onto Z_i/S^1.

Corollary

The moment image of $Z_i \times (-\epsilon, \epsilon)$ *is the polytope in the figure below*



The vertical slice at $t = 0$ *being the moment polytope,* Δ_K*, for the action of* K *on* Z_i/S^1.

Note that as the moment image of $Z_i \times (-\epsilon, \epsilon)$ this vertical slice lies on the hyperplane $\operatorname{Log}|t| = -\infty$ and its edges are the lines

$$p + tu_i, \; -\infty \leq \operatorname{Log}|t| < \log \epsilon$$

the p's being the vertices of Δ_K.

In particular the weights figuring in the formal quantization of $Z_i \times (-\epsilon, \epsilon)$ all lie along the lines

$$\beta + \operatorname{Log}|t| u, \quad -\infty < \log|t| < \log\epsilon$$

where β is a weight of K lying in the moment polytope.

We'll now describe our second main result

Theorem 2

The formal geometric quantization of $\mathcal{U} = X - Z$ is a finite dimensional virtual vector space.

Remark 1

The proof of this will be based on the following well-known fact.

Theorem 3

The index of the $\mathbb{L}$-twisted spin $\mathbb{C}$ Dirac operator changes by a factor of (-1) if one changes the orientation fo the underlying manifold.

Proof of Theorem 2.

If we delete the moment images of the regions $Z_i \times (-\epsilon, \epsilon)$ from the moment image of $\mathcal{U}$ the remaining image is compact. Hence the contributions to the formal quantization of $\mathcal{U}$ from weights in this region is a finite dimensional vector space. $\square$

What about contributions coming from the moment image on slide 40 of $Z_i \times (-\epsilon, \epsilon)$?
We'll prove these contributions add up to $\{0\}$. In fact we'll prove a much stronger result.
Let S^1 be the circle group in the factorization

$$G = S^1 \times K$$

We will prove

For every weight, $-n,\ e^{-n} < \epsilon$ of the weight lattice, $\mathbb{Z}$, of S^1 let $X_{i,n}$ be the symplectic reduction of $Z_i \times (-\epsilon, \epsilon)$ with respect to the Hamiltonian action of S^1. Then

$$\mathcal{Q}(X_{i,n}) = \{0\}$$

Proof.

This reduction consists of two copies of the symplectic manifold, Z_i/S^1, hence if we quantize them by assigning to them the orientations defined by their symplectic volumes we get two copies of $\mathcal{Q}(Z_i/S^1)$.
But these orientations correspond, upstairs on $Z_i \times (-\epsilon, 0)$ and $Z_i \times (0, \epsilon)$, to the orientations defined by the symplectic volume form associated with the b-symplectic form

$$\alpha \wedge \frac{dt}{t} + w_0$$

If instead we fix a global volume form v on X and assign to these two spaces the orientations defined by v then we get the same orientation as before on one of these two copies of Z_i/S^1 and the opposite orientation on the other.
Hence by theorem 3 the quantization of this symplectic reduction is zero. $\square$