Supersymmetric gauge theory, representation schemes and random matrices

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joint work with Y. Berest, M. Müller-Lennert, S. Patotsky, A. Ramadoss and T. Willwacher

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Nekrasov partition functions

Derived representation schemes

Analytic properties of the partition function

Instanton partition function of $\mathcal{N} = 2$ Yang–Mills theory

The Nekrasov instanton partition function of $\mathcal{N} = 2$ supersymmetric Yang–Mills theory with gauge group $U(r)$ on $\mathbb{R}^4$ in the Ω-background with parameters ϵ_1, ϵ_2 is given as a sum over r-tuples $\vec{Y} = (Y_i)_{i=1}^r$ of Young diagrams of total size $|\vec{Y}|$:

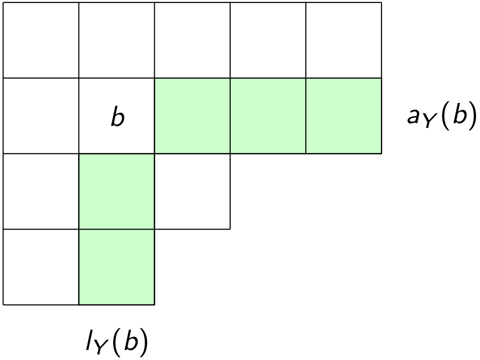
$$Z_{4D}(\epsilon_1, \epsilon_2, a, \mathfrak{q}, \lambda) = \sum_{\vec{Y}} \mathfrak{q}^{|\vec{Y}|} \prod_{\alpha,\beta=1}^{r} \prod_{b \in Y_\alpha} \frac{1}{E_{\alpha\beta}(b)(\epsilon_1 + \epsilon_2 - E_{\alpha\beta}(b))},$$

$$E_{\alpha\beta}(b) = a_\alpha - a_\beta - l_{Y_\beta}(b)\epsilon_1 + (a_{Y_\alpha}(b) + 1)\epsilon_2.$$

- $a = (a_1, \ldots, a_r) \in \mathfrak{u}(r)$ parametrizes boundary conditions of scalar fields
- ϵ_1, ϵ_2 equivariant parameters for the action of $U(1)^2$ on $\mathbb{R}^4 = \mathbb{C}^2$.
- $a_Y(b), l_Y(b)$ arm and leg length of the box $b \in Y$.

Arm and leg length

The arm and leg length of a box b in a Young diagram Y are the number of boxes to its right and below it, respectively.



Five-dimensional supersymmetric theory

Z_{4D} is the limit of the instanton partition function on $\mathbb{R}^4 \times S^1_\lambda$ as the radius λ of the circle tends to 0:

$$Z_{5D}(\epsilon_1, \epsilon_2, a, \mathfrak{q}, \lambda) = \sum_{\vec{Y}} \mathfrak{q}^{|\vec{Y}|} \prod_{\alpha,\beta=1}^{r} \prod_{b \in Y_\alpha} \frac{(\lambda/2)^2}{\sinh\left(\frac{\lambda}{2} E_{\alpha\beta}(b)\right) \sinh\left(\frac{\lambda}{2}(\epsilon_1 + \epsilon_2 - E_{\alpha\beta}(b))\right)},$$

$$E_{\alpha\beta}(b) = a_\alpha - a_\beta - l_{Y_\beta}(b)\epsilon_1 + (a_{Y_\alpha}(b) + 1)\epsilon_2.$$

Instanton count and equivariant cohomology

- Z_{4D} is the contribution of instantons to the partition function. Roughly,

$$Z_{4D} = \sum_{n=0}^{\infty} \mathfrak{q}^n \int_{M_{r,n}} 1$$

  is the generating function of counts of framed instantons of instanton number $n = 0, 1, 2, \ldots$.

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- The integral over moduli space $M_{r,n}$ makes sense if we consider 1 as an equivariant differential form for $T = U(1)^2 \times U(1)^r$ where $U(1)^2$ acts on $\mathbb{R}^4 = \mathbb{C}^2$ via $(z_1, z_2) \mapsto (e^{i\phi_1} z_1, e^{i\phi_2} z_2)$ and $U(1)^r$ is the Cartan torus of the group $U(r)$ of constant gauge transformations.

Instanton count and equivalent cohomology

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- The integral is then computed/defined by the localization formula: fixed points are labeled by r-tuples of partitions.

Instanton count in the 5D theory and equivariant K-theory

- In the five-dimensional case equivariant cohomology is replaced by equivariant K-theory:

$$Z_{5D} = \sum_n v^n p_*[\mathcal{O}_{M_{n,r}}] \in \bar{K}_T(\mathrm{pt})[[v]], \quad v = \mathfrak{q}\lambda^{2r} e^{-r\lambda(\epsilon_1+\epsilon_2)}.$$

is the generating function of the direct image by the map to a point of the class of the trivial line bundle.

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- T acts on $M_{r,n}$ and thus $H^i(M_{r,n}, \mathcal{O})$ is a representation of T. Let

$$\mathrm{ch}_T\, V \in K_T(\mathrm{pt}) = \mathbb{Z}[\hat{T}] = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 2}, u_1^{\pm 1}, \ldots, u_r^{\pm 1}]$$

  denote the character of a finite dimensional virtual representation V. Then

$$Z_{5D} = \sum_{n=0}^{\infty} v^n\, \mathrm{ch}_T \sum_{i=0}^{2rn} (-1)^i H^i(M_{r,n}, \mathcal{O}).$$

The cohomology groups $H^i(M_{r,n}, \mathcal{O})$ are infinite dimensional with finite dimensional weight spaces for the action of $U(1)^2$. Then Z_{5D} takes value in a completion of $K_T(\mathrm{pt})$. In fact

$$Z_{5D} \in \mathbb{Z}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}][[q_1, q_2, v]]$$

ADHM equations

- Atiyah, Drinfeld, Hitchin, Manin (ADHM) gave a description of the moduli space of framed instantons (torsion free sheaves on $\mathbb{CP}^2$ with trivialization at infinity) of instanton number n for the group $U(r)$ in terms of linear algebra data:

$$M_{r,n} = T^*(\mathrm{Mat}_{n,n}(\mathbb{C}) \times \mathrm{Mat}_{n,r}(\mathbb{C}))//GL_n$$

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- GIT means that we restrict to the four-tuples (X, I, Y, J) obeying the stability condition: there is no non-trivial proper subspace of $\mathbb{C}^n$ containing $I(\mathbb{C}^r)$ that is invariant under X and Y.

Gauge theory on S^4, AGT correspondence

- The square of the absolute value of the Nekrasov partition function $|Z_{4D}|^2$ (or $|Z_{5D}|^2$) appears in the integrand over the Coulomb parameters of the partition function of $\mathcal{N} = 2$ supersymmetric gauge theory on S^4 (or $S^4 \times S^1$) with ellipsoidal metric with half-axes ϵ_1, ϵ_2 (Pestun).

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- Partition functions for gauge theory with matter fields are obtained by replacing the trivial bundle by suitable vector bundles (or their Chern characters in the $4D$ case).
- By the AGT (Alday–Gaiotto–Tachikawa) correspondence, Nekrasov instanton partition functions are related to conformal blocks of Liouville or Toda theories, or their q-deformations for the 5D theory.

A special case of the AGT correspondence in 5D: Gaiotto states

- Deformed Virasoro algebra

$$[T_n, T_m] = -\sum_{l=1}^{\infty} r_l (T_{n-l} T_{m+l} - T_{m-l} T_{n+l}) - \frac{(1-q_1)(1-q_2)}{1-q_1 q_2} (q_1^n q_2^n - q_1^{-n} q_2^{-n}) \delta_{m+n,0}.$$

$$\sum_{l \geq 0} r_l x^l = \exp \sum_{n \geq 1} \frac{(1-q_1^n)(1-q_2^n)}{1+q_1^n q_2^n} \frac{x^n}{n}$$

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- The Verma module M_h is generated by $|h\rangle$ with relations $T_n|h\rangle = \delta_{n,0} h|h\rangle$, for $n \geq 0$. It has a grading $M_h = \oplus_{n=0}^{\infty} M_{h,n}$ by eigenspaces of T_0 to the eigenvalues $h+n$. It is orthogonal for the Shapovalov bilinear form on M_h so that $S(|h\rangle, |h\rangle) = 1$ and $S(T_n x, y) = S(x, T_{-n} y)$.

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- A Gaiotto state is a formal power series $|G\rangle = \sum_{n=0}^{\infty} \xi^n |G_n\rangle$ with coefficients $|G_n\rangle \in M_{h,n}$ such that

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- We will see that the norm squared, which is a priori a formal power series in ξ has in fact a finite radius of convergence.

Representation schemes

- Let S be an associative unital algebra over $\mathbb{C}$ and $S \to A$ an algebra over S (the basic example is $S = \mathbb{C}$). Let V be a finite dimensional S-module, and let $\rho\colon S \to \operatorname{End} V$ be the associated representation.

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- The (relative) representation scheme $R_V(S\backslash A)$ is the space of representations $A \to \operatorname{End}(V)$ restricting to ρ on S. It is an affine algebraic scheme. The character scheme is $R_V(S/A)/\operatorname{Aut}_S(V)$.

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- Example: $S = \mathbb{C}$, $V = \mathbb{C}^n$. $R_V(\mathbb{C}\backslash A)$ parametrizes the representations of A in $n \times n$ matrices. The character scheme parametrizes equivalence classes of n-dimensional representations.

Example

A path algebra of a quiver with vertex set I, S subalgebra generated by idempotents $(e_i)_{i \in I}$. $R_V(S \backslash A)$ parametrizes representations of the quiver on $(\operatorname{Im} \rho(e_i))_{i \in I}$.

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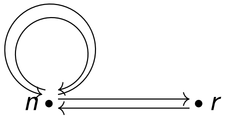
Example

$\mu^{-1}(0)$ is the representation scheme of the path algebra of



on $V = \mathbb{C}^r \oplus \mathbb{C}^n$ with ADHM relations $XY - YX + IJ = 0$ on the generators.

Derived representation schemes and representation homology

- The assignment $A \mapsto \mathcal{O}(R_V(S \backslash A))$ extends to a well-defined functor from the category DGA_S of differential graded (dg) S-algebras to the category $CDGA_{\mathbb{C}}$ of commutative dg algebra.

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- The derived representation scheme of A is obtained by replacing A be a weakly equivalent cofibrant object $QA \in DGA_S$ and applying the representation functor:

$$\mathrm{DRep}_V(S\backslash A) := \mathcal{O}(R_V(S\backslash QA)) \in \mathrm{CDGA}_{\mathbb{C}}$$

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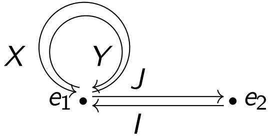
- The representation homology of A relative to V is the graded algebra

$$H_\bullet(S\backslash A, V) = H_\bullet(DRep_V(S\backslash A))$$

Derived representation scheme for ADHM relations

- The path algebra of the quiver



is generated by X, Y, I, J and the idempotents e_1, e_2.

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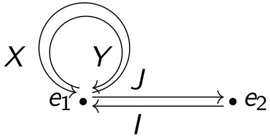

is generated by X, Y, I, J and the idempotents e_1, e_2.

- A cofibrant replacement QA of the path algebra with ADHM relation has an additional generator Θ of degree 1 whose differential enforces the relation on homology:

$$d\Theta = XY - YX - IJ, \quad dX = dY = dI = dJ = 0.$$

Derived representation scheme for ADHM relations

- $\mathrm{DRep}_V(S\backslash A)$ is the free graded commutative algebra generated by matrix entries

$$\mathbb{C}[x_{\alpha\beta}, y_{\alpha\beta}, i_{\alpha\mu}, j_{\mu\beta}, \theta_{\alpha\beta} | \alpha, \beta = 1, \dots, n, \mu = 1, \dots r]$$

with induced differential

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- The torus $(\mathbb{C}^{\times})^2$ acts by rescaling of X, Y. Also $GL_n \times GL_r$ acts on the derived representation scheme. In particular we have an action of $T = U(1)^2 \times U(1)^r$ on the GL_n-invariants of representation homology.

Observation (Y. Berest, G.F., A. Ramadoss, S. Patotsky, T. Willwacher)

The Nekrasov partition function on $\mathbb{R}^4 \times S^1$ coincides with the generating function of the weighted Euler characteristics of the representation homology of the ADHM quiver:

$$Z_{5D} = \sum_{n=0}^{\infty} v^n \sum_i (-1)^i \operatorname{ch}_T H_i(S \backslash A, V)^{GL_n}, \quad V = \mathbb{C}^n \oplus \mathbb{C}^r.$$

Integral formula for the partition function

The calculation of the weighted Euler characteristic of $H_i(S\backslash A, V)^{GL_n}$ leads to the integral formula for the partition function (due to Nekrasov, Shatashvili,...).

$$Z(v) = \sum_{n=0}^{\infty} v^n Z_n,$$

$$Z_n = \frac{1}{n!(2\pi i)^n} \left(\frac{1 - q_1 q_2}{(1-q_1)(1-q_2)} \right)^n$$

$$\oint_{|z_j|=1} \prod_{j=1}^{n} \prod_{\alpha=1}^{r} \frac{1}{(1 - u_\alpha / z_j)(1 - q_1 q_2 z_j / u_\alpha)} \prod_{j \neq k} \frac{(z_j - z_k)(z_j - q_1 q_2 z_k)}{(z_j - q_1 z_k)(z_j - q_2 z_k)} \prod_{j=1}^{n} \frac{dz_j}{z_j}.$$

more suitable to study analytic properties. The generators of the character group $\hat{T}$ are identified with the Ω-background and Coulomb parameters via

$$q_i = e^{-\lambda \epsilon_i}, \quad u_\alpha = e^{-\lambda a_\alpha}.$$

Analytic properties

Range of parameters of interest

- Gauge theory on $S^4 \times S^1$: $\epsilon_1, \epsilon_2 > 0$ and $a_i \in \sqrt{-1}\mathbb{R}$. Exponential variables $0 < q_1, q_2 < 1$, $|u_i| = 1$.
- AGT correspondence for Liouville theory. Central charge $c = 1 + 6(b + b^{-1})^2 \in (1, \infty)$, $\epsilon_1 = b$, $\epsilon_2 = b^{-1}$. Strongly coupled Liouville theory: $1 < c < 25$, $\epsilon_1 = \bar{\epsilon}_2 \in S^1$. Weakly coupled Liouville theory: $c > 25$, $\epsilon_1, \epsilon_2 > 0$.

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- Subtlety: The coefficents of the expansion of the partition functions have "small denominators" in these ranges: they are not defined for a dense set of parameters and they are arbitrarily small for the complement.

Analytic properties

Theorem (G.F., M. Müller-Lennert)

Let $|q_1|, |q_2| < 1$, $|u_\alpha| = 1$. Suppose that either $q_1 = \bar{q}_2$ or $q_1, q_2 \in \mathbb{R}_+$. Then the formal power series $Z_{5D}(v)$ has convergence radius (at least) 1 and depends analytically on the parameters.

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Corollary

The norm of the Gaiotto state for the deformed Virasoro algebra is analytic for $|\xi| < |q_1 q_2|^{1/2}$.

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The norm of the Gaiotto state for the deformed Virasoro algebra is analytic for $|\xi| < |q_1 q_2|^{1/2}$.

- The theorem amounts to an estimate of the asymptotic behaviour of the coefficients $Z_n = Z_n(q_1, q_2, \vec{u})$ of the formal power series $Z_{5D}(v)$: $\limsup_{n\to\infty} Z_n^{\frac{1}{n}} \leq 1$. This can be done with techniques from unitary random matrix theory.

Random matrices

- We write Z_n as an expectation value

$$Z_n = Z_n^0 E_n \left(\prod_{j=1}^{n} \prod_{\alpha=1}^{r} \frac{1}{(1 - u_\alpha / z_j)(1 - q_1 q_2 z_j / u_\alpha)} \right)$$

for a system of particles $z_1, \ldots, z_n$ on the unit circle with Boltzmann distribution

$$\frac{1}{Z_n^0} \exp\left(- \sum_{j \neq k} W(z_j / z_k) \right) \prod \frac{dz_i}{2\pi i z_i}$$

for some pair potential W which is repulsive at short distances.

Equilibrium measure

- The estimate of $E_n(\cdots)$ is standard. One proves that for large n the particle configurations approach a uniform distribution on the unit circle with high probability. The asymptotic behaviour of the integral is then calculated by evaluating the integrand on this distribution.

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- The normalization factor Z_n^0 (Z_n at $r = 0$) is the weighted Euler characteristic of the representation homology $H_\bullet(\mathbb{C}\langle x, y\rangle/(xy - tyx), \mathbb{C}^n)$ of quantum plane, that can be computed explicitly (Berest et al) with the result

$$\sum_{n=0}^{\infty} v^n Z_n^0 = \exp\left(\sum_{n=1}^{\infty} \frac{1 - q_1^n q_2^n}{(1 - q_1^n)(1 - q_2^n)} \frac{v^n}{n}\right).$$

The right-hand side converges for $|v| < 1$ so we get that $\lim_{n\to\infty} |Z_n^0|^{\frac{1}{n}} = 1$.

Open question

- The formal limit $\lambda \to 0$ of the estimated radius of convergence converges to the expected radius of convergence in the 4D theory. However the convergence is not uniform and we cannot deduce a result on the analyticity of Z_{4D}

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- From the point of view of random matrices the equilibrium measure in the 4D theory is no longer uniform as the two particle potential is attractive at intermediate distances. It would be interesting to describe this distribution.

Thank you for your attention