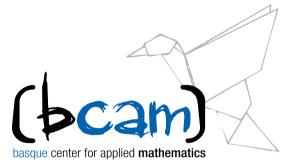
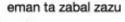
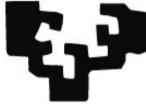
## Dirac Equations with Singular Potentials Shell interactions for Dirac operators: point spectrum and confinement

## Luis VEGA, BCAM-UPV/EHU







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### Free Dirac operator in $\mathbb{R}^3$

<u>**Definition.**</u>- $H : \mathcal{C}^{\infty}_{c}(\mathbb{R}^{3})^{4} \to \mathcal{C}^{\infty}_{c}(\mathbb{R}^{3})^{4}$  free Dirac operator in  $\mathbb{R}^{3}$ ,

$$H = -i\alpha \cdot \nabla + m\beta = \begin{pmatrix} m & 0 & -i\partial_3 & -\partial_2 - i\partial_1 \\ 0 & m & \partial_2 - i\partial_1 & i\partial_3 \\ -i\partial_3 & -\partial_2 - i\partial_1 & -m & 0 \\ \partial_2 - i\partial_1 & i\partial_3 & 0 & -m \end{pmatrix}$$

- 1st order symmetric differential operator.
- Local version of  $\sqrt{-\Delta + m^2}$ :  $H^2 = (-\Delta + m^2)I_4$ .
- Dirac (1928)

## Coupling with a singular potential

#### First Question.-

 $\Omega \subset \mathbb{R}^3$  bounded regular domain,  $\Sigma = \partial \Omega, \quad \sigma \text{ surface measure on } \Sigma,$   $V \text{ potential } L^2(\sigma)^4 \text{-valued.}$ To find  $D \subset L^2(\mathbb{R}^3)^4$  such that H + V defined on D is self-adjoint.

#### Motivation.-

- Quantum Physics requires self-adjointness.
- $\frac{\lambda}{|x|}$  critical (scaling) for H,  $|\lambda| < 1$  (Dolbeault, Esteban, Sere, '00; Hardy Inequality, Uncertainty Principle).
- $H + \lambda \delta_{|x|=1}$  (and other critical V's on  $S^2$ ) (Dittrich, Exner & Seba '89; Spherical Harmonics. Albeverio, Gesztesy, Hoegh-Krohn & Holden '88 -'05).
- Previous results on  $-\Delta + \lambda \delta_{\Sigma}$  for Lipschitz surfaces  $\Sigma$ . Subcritical/Critical.

### **Initial Approach**

#### First Question.-

To find  $D \subset L^2(\mathbb{R}^3)^4$  such that H + V defined on D is self-adjoint.

#### Our Approach.-

Take  $\varphi \in D$ , V potential  $L^2(\sigma)^4$ -valued  $\implies V(\varphi) = -g$  for some  $g \in L^2(\sigma)^4$ .  $(H+V)(\varphi) \in L^2(\mathbb{R}^3)^4 \implies (H+V)(\varphi) = G$  for some  $G \in L^2(\mathbb{R}^3)^4$ .

 $H(\varphi) = G + g$  in the sense of distributions.

Therefore  $\varphi = \phi * (G + g)$  and

$$(H+V)(\varphi) = G, \qquad V(\varphi) = -g,$$

where  $\phi$  is the fundamental solution of  $H = -i\alpha \cdot \nabla + m\beta$ ,

$$\phi(x) = \frac{e^{-m|x|}}{4\pi|x|} \left( m\beta + (1+m|x|)\,i\alpha \cdot \frac{x}{|x|^2} \right) \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\}.$$

### Self-adjointness of H + V

 $\frac{\mathbf{Property.}-\text{ If }G\in L^2(\mathbb{R}^3)^4,\text{ then }\phi\ast G\in W^{1,2}(\mathbb{R}^3)^4\text{ and }}{(\phi\ast G)|_{\Sigma}\in L^2(\sigma)^4.}$ 

Theorem (Self-adjointness). – Given  $\Lambda : L^2(\sigma)^4 \to L^2(\sigma)^4$ bounded, self-adjoint and with closed range, define

$$D = \left\{ \phi * (G+g) : (\phi * G)|_{\Sigma} = \Lambda(g) \right\} \subset L^2(\mathbb{R}^3)^4.$$

If  $V(\phi * (G + g)) = -g$ , then H + V defined on D is essentially self-adjoint.

- Under more assumptions on  $\Lambda$ , H + V is self-adjoint. Posilicano '08-'09.
- Other differential operators and measures are considered.
- Other relations between  $(\phi * G)|_{\Sigma}$  and g are considered.

#### **Resolvent of** H

**<u>Resolvent</u>**.- Given  $a \in (-m, m)$ , let  $\phi^a$  be the fundamental solution of  $H - a = -i\alpha \cdot \nabla + m\beta - a$ ,

$$\phi^{a}(x) = \frac{e^{-\sqrt{m^{2} - a^{2}}|x|}}{4\pi|x|} \left(a + m\beta + \left(1 + \sqrt{m^{2} - a^{2}}|x|\right) i\alpha \cdot \frac{x}{|x|^{2}}\right).$$

Our Setting.  $-\Omega_+ \subset \mathbb{R}^3$  bounded regular domain,  $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega_+}$ ,  $\Sigma = \partial \Omega_{\pm}, \sigma$  surface measure on  $\Sigma$ , N normal vector on  $\Sigma$  w.r.t.  $\Omega_+$ .

**Properties.**– If  $g \in L^2(\sigma)^4$ , then  $(H-a)(\phi^a * g) = 0$  in  $\Sigma^c$ . For  $x \in \Sigma$ , set

$$C^a_{\pm}g(x) = \lim_{\Omega_{\pm} \ni y \xrightarrow{nt} x} (\phi^a * g)(y), \quad C^a_{\sigma}g(x) = p.v. \ (\phi^a * g)(x).$$

Then,

•  $C^a_{\pm} = \mp \frac{i}{2} (\alpha \cdot N) + C^a_{\sigma}$  (Plemelj-Sokhotski jump formulae),

• 
$$\left(C^a_\sigma(\alpha \cdot N)\right)^2 = -\frac{1}{4}.$$

#### **Point spectrum and confinement for** H + V

Our Setting.- Set  $D = \{\varphi = \phi * (G + g) : (\phi * G)|_{\Sigma} = \Lambda(g)\}$ and  $H + V : D \subset L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4$  defined by  $V(\varphi) = -g$  and  $(H + V)(\varphi) = G$  for  $\varphi \in D$ .

 $\frac{\text{Our Theorem (Point Spectrum)}}{\text{Ker}(H+V-a) \neq \emptyset \text{ iff Ker}(\Lambda + C_{\sigma} - C_{\sigma}^{a}) \neq \emptyset}. \quad \in \quad (-m, m),$ 

### Point spectrum and confinement for H + V

**Definition.**– V generates confinement w.r.t. H and  $\Sigma$  iff supp  $\left(e^{-it(H+V)}(f)\right) \subset \Omega_{\pm}$  for all  $f \in L^2(\Omega_{\pm})^4$  and all  $t \in \mathbb{R}$ . This is equivalent to require that  $\chi_{\Omega_{\pm}}\varphi \in D$  for all  $\varphi \in D$ .

**Theorem (Confinement)**.– Assume that H + V is self-adjoint on D. Then, V generates confinement w.r.t. H and  $\Sigma$  if  $\{C_{\sigma}(\alpha \cdot N), \Lambda(\alpha \cdot N)\} = -(\Lambda(\alpha \cdot N))^2.$ 

## Some applications. Electrostatic shell potentials

**<u>Theorem</u>**.- Let  $\lambda \in \mathbb{R} \setminus \{0\}$  and  $a \in (-m, m)$ . Take  $\Lambda = -(1/\lambda + C_{\sigma}), D = \{\varphi = \phi * (G + g) : (\phi * G)|_{\Sigma} = \Lambda(g)\},$ and  $V_{\lambda}(\varphi) = \frac{\lambda}{2} (\varphi_{+} + \varphi_{-}) \quad (\varphi_{\pm} \text{ n.t. boundary values of } \varphi \text{ on } \Sigma).$ 

- $H + V_{\lambda}$  defined on D is self-adjoint for all  $\lambda \neq \pm 2$ .
- Ker  $(H + V_{\lambda} a) \neq \emptyset$  iff Ker  $(1/\lambda + C_{\sigma}^{a}) \neq \emptyset$ .
- $H + V_{\lambda}$  and  $H + V_{-4/\lambda}$  have the same eigenvalues in (-m, m).
- If  $|\lambda| \notin [1/\|C^a_{\sigma}\|, 4\|C^a_{\sigma}\|]$ , then Ker $(H + V_{\lambda} a) = \emptyset$ .
- If  $|\lambda| \notin [1/C, 4C]$ , where  $C = \sup_{a \in (-m,m)} ||C_{\sigma}^{a}|| < \infty$ , then  $H + V_{\lambda}$  has no eigenvalues in (-m, m).

<u>**Theorem.**</u> – Let  $H + V_{\lambda}$  be as above. If  $\Omega_{-}$  is connected, then  $H + V_{\lambda}$  has no eigenvalues in  $\mathbb{R} \setminus [-m, m]$ .

### Some applications.

### **Electrostatic plus Lorentz scalar shell potentials**

**Theorem.**- Let 
$$\lambda_e, \lambda_s \in \mathbb{R}$$
 be such that  $\lambda_e^2 - \lambda_s^2 \neq 0, 4$ . Take  

$$\Lambda = \frac{\lambda_s \beta - \lambda_e}{\lambda_e^2 - \lambda_s^2} - C_{\sigma},$$

$$D = \{\varphi = \phi * (G + g) : (\phi * G)|_{\Sigma} = \Lambda(g)\}, \text{ and}$$

$$V_{es}(\varphi) = \frac{1}{2}(\lambda_e + \lambda_s \beta)(\varphi_+ + \varphi_-) \quad (\varphi_{\pm} \text{ n.t. boundary values of } \varphi).$$

- $H + V_{es}$  defined on D is self-adjoint.
- $V_{es}$  generates confinement w.r.t H and  $\Sigma$  iff  $\lambda_e^2 \lambda_s^2 = -4$ .

### Some applications.

**Electrostatic plus Lorentz scalar shell potentials** 

- That  $V_{es}$  generates confinement means that the particles modelized by the evolution  $\partial_t + i(H + V_{es})$  never cross  $\Sigma$ over time, i.e.,  $\Sigma$  becomes impenetrable.
- The impenetrability condition  $\lambda_e^2 \lambda_s^2 = -4$  was known for  $\Sigma = \{x \in \mathbb{R}^3 : |x| = R\}, R > 0$  (Dittrich-Exner-Seba).

#### Uncertainty Principle on the sphere $S^2$

We focus on  $H + V_{\lambda}$  for  $\Sigma = S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ 

**<u>Definition</u>**.– Let  $\tilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  be the family of Pauli matrices. Given  $a \in (-m, m)$ , define

$$k^{a}(x) = \frac{e^{-\sqrt{m^{2} - a^{2}}|x|}}{4\pi |x|} I_{2} \text{ and}$$
$$w^{a}(x) = \frac{e^{-\sqrt{m^{2} - a^{2}}|x|}}{4\pi |x|^{3}} \left(1 + \sqrt{m^{2} - a^{2}}|x|\right) i \widetilde{\sigma} \cdot x.$$

For  $f \in L^2(\sigma)^2$  and  $x \in S^2$ , set

 $K^{a}f(x) = (k^{a} * f)(x)$  and  $W^{a}(f) = p.v.(w^{a} * f)(x).$ 

## Uncertainty Principle on the sphere $S^2$

- $K^a$  and  $W^a$  are bounded operators in  $L^2(\sigma)^2$ .
- $K^a$  is a positive operator.

### Uncertainty Principle on the sphere $S^2$

**<u>Theorem</u>**.– Let  $\lambda > 0$  and  $a \in (-m, m)$ . The operator

 $1/\lambda + (m+a)K^a$ 

is invertible in  $L^2(\sigma)^2$ . Furthermore, for any  $f \in L^2(\sigma)^2$  and any  $\delta > 0$ ,

$$\begin{split} \int_{S^2} |f|^2 \, d\sigma &\leq \frac{1}{2M\delta} \int_{S^2} \left( 1/\lambda + (m+a)K^a \right)^{-1} \left( W^a(f) \right) \cdot \overline{W^a(f)} \, d\sigma \\ &+ \frac{\delta}{2M} \int_{S^2} \left( 1/\lambda + (m+a)K^a \right) \left( \left( \widetilde{\sigma} \cdot N \right) f \right) \cdot \overline{\left( \widetilde{\sigma} \cdot N \right) f} \, d\sigma, \end{split}$$

$$(1)$$

where M is a constant depending only on m and a. Moreover,  $M \ge \frac{1}{2} e^{-\sqrt{m^2 - a^2}} \sqrt{2 - e^{-2\sqrt{m^2 - a^2}}}.$ 

For suitable  $\delta$ 's, the inequality (1) is sharp and the equality can be attained.

## Uncertainty Principle on the sphere $S^2$ . Consequences

**Definition (2-dimensional Riesz transform).** – Given a finite Borel measure  $\nu$  in  $\mathbb{R}^3$ ,  $h \in L^2(\nu)$  and  $x \in \mathbb{R}^3$ , one defines the 2-dimensional Riesz transform of h as

$$R_{\nu}(h)(x) = \lim_{\epsilon \searrow 0} \int_{|x-y| > \epsilon} \frac{x-y}{|x-y|^3} h(y) \, d\nu(y),$$

whenever the limit makes sense.

## Uncertainty Principle on the sphere $S^2$ . Consequences

<u>Corollary</u>.-  $2\pi \|h\|_{L^2(\sigma)} \leq \|R_{\sigma}(h)\|_{L^2(\sigma)^3}$  for all real-valued  $h \in L^2(\sigma)$ , and the inequality is sharp. Hofmann, Marmolejo-Olea, Mitrea, Pérez-Esteva, & Michael Taylor '09

- For suitable elections of  $\lambda$ , a, and  $\delta$ , the minimizers of (??) give rise to eigenfunctions of  $H + V_{\lambda}$  with eigenvalue a.
- The set of  $\lambda$ 's for which  $H + V_{\lambda}$  has a non-trivial eigenfunction contains an interval.

# THANK YOU FOR YOUR ATTENTION