Dirac Equations with Singular Potentials
Shell interactions for Dirac operators: point spectrum and confinement

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Free Dirac operator in $\mathbb{R}^3$

Definition.– $H : \mathcal{C}_c^\infty(\mathbb{R}^3)^4 \to \mathcal{C}_c^\infty(\mathbb{R}^3)^4$ free Dirac operator in $\mathbb{R}^3$,

$$H = -i\alpha\cdot\nabla + m\beta = \begin{pmatrix} m & 0 & -i\partial_3 & -\partial_2 - i\partial_1 \\ 0 & m & \partial_2 - i\partial_1 & i\partial_3 \\ -i\partial_3 & -\partial_2 - i\partial_1 & -m & 0 \\ \partial_2 - i\partial_1 & i\partial_3 & 0 & -m \end{pmatrix}.$$

Remarks.–

- 1st order symmetric differential operator.
- Local version of $\sqrt{-\Delta + m^2}$: $\quad H^2 = (-\Delta + m^2) I_4$.
- Dirac (1928)

Coupling with a singular potential

First Question.–

$\Omega \subset \mathbb{R}^3$ bounded regular domain,
$\Sigma = \partial\Omega$, σ surface measure on Σ,
V potential $L^2(\sigma)^4$-valued.

To find $D \subset L^2(\mathbb{R}^3)^4$ such that $H + V$ defined on D is self-adjoint.

Motivation.–

- *Quantum Physics* requires self-adjointness.
- $\frac{\lambda}{|x|}$ critical (scaling) for H, $|\lambda| < 1$ (Dolbeault, Esteban, Sere, '00; Hardy Inequality, Uncertainty Principle).
- $H + \lambda\delta_{|x|=1}$ (and other critical V's on S^2) (Dittrich, Exner & Seba '89; Spherical Harmonics. Albeverio, Gesztesy, Hoegh-Krohn & Holden '88 -'05).
- Previous results on $-\Delta + \lambda\delta_\Sigma$ for Lipschitz surfaces Σ. Sub-critical/Critical.

Initial Approach

First Question.–

To find $D \subset L^2(\mathbb{R}^3)^4$ such that $H + V$ defined on D is self-adjoint.

Our Approach.–

Take $\varphi \in D$,
V potential $L^2(\sigma)^4$-valued $\Longrightarrow V(\varphi) = -g$ for some $g \in L^2(\sigma)^4$.
$(H{+}V)(\varphi) \in L^2(\mathbb{R}^3)^4 \Longrightarrow (H{+}V)(\varphi) = G$ for some $G \in L^2(\mathbb{R}^3)^4$.

$H(\varphi) = G + g$ in the sense of distributions.

Therefore $\quad \varphi = \phi * (G + g) \quad$ and

$$(H + V)(\varphi) = G, \qquad V(\varphi) = -g,$$

where ϕ is the fundamental solution of $H = -i\alpha \cdot \nabla + m\beta$,

$$\phi(x) = \frac{e^{-m|x|}}{4\pi|x|}\left(m\beta + (1 + m|x|)\, i\alpha \cdot \frac{x}{|x|^2}\right) \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\}.$$

Self-adjointness of $H+V$

Property.– If $G \in L^2(\mathbb{R}^3)^4$, then $\phi * G \in W^{1,2}(\mathbb{R}^3)^4$ and $(\phi * G)|_\Sigma \in L^2(\sigma)^4$.

Theorem (Self-adjointness).– Given $\Lambda : L^2(\sigma)^4 \to L^2(\sigma)^4$ bounded, self-adjoint and with closed range, define

$$D = \big\{\phi * (G+g) : (\phi * G)|_\Sigma = \Lambda(g)\big\} \subset L^2(\mathbb{R}^3)^4.$$

If $V\big(\phi * (G+g)\big) = -g$, then $H+V$ defined on D is essentially self-adjoint.

Remarks.–

- Under more assumptions on Λ, $H+V$ is self-adjoint. Posilicano '08-'09.
- Other differential operators and measures are considered.
- Other relations between $(\phi * G)|_\Sigma$ and g are considered.

Resolvent of H

Resolvent.– Given $a \in (-m, m)$, let ϕ^a be the fundamental solution of $H - a = -i\alpha \cdot \nabla + m\beta - a$,

$$\phi^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\left(a + m\beta + \left(1 + \sqrt{m^2 - a^2}|x|\right) i\alpha \cdot \frac{x}{|x|^2}\right).$$

Our Setting.– $\Omega_+ \subset \mathbb{R}^3$ bounded regular domain, $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega_+}$, $\Sigma = \partial\Omega_\pm$, σ surface measure on Σ, N normal vector on Σ w.r.t. Ω_+.

Properties.– If $g \in L^2(\sigma)^4$, then $(H - a)(\phi^a * g) = 0$ in Σ^c. For $x \in \Sigma$, set

$$C^a_\pm g(x) = \lim_{\Omega_\pm \ni y \overset{nt}{\to} x} (\phi^a * g)(y), \quad C^a_\sigma g(x) = p.v.\,(\phi^a * g)(x).$$

Then,

- $C^a_\pm = \mp \frac{i}{2}(\alpha \cdot N) + C^a_\sigma$ (*Plemelj-Sokhotski jump formulae*),
- $\left(C^a_\sigma(\alpha \cdot N)\right)^2 = -\frac{1}{4}$.

Point spectrum and confinement for $H + V$

Our Setting.– Set $D = \big\{\varphi = \phi * (G + g) \,:\, (\phi * G)|_\Sigma = \Lambda(g)\big\}$ and $H + V : D \subset L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4$ defined by $V(\varphi) = -g$ and $(H + V)(\varphi) = G$ for $\varphi \in D$.

Our Theorem (Point Spectrum).– Given $a \in (-m, m)$, $\mathrm{Ker}\,(H + V - a) \neq \emptyset$ iff $\mathrm{Ker}\,(\Lambda + C_\sigma - C_\sigma^a) \neq \emptyset$.

Point spectrum and confinement for $H + V$

Definition.– V generates confinement w.r.t. H and Σ iff $\text{supp}\left(e^{-it(H+V)}(f)\right) \subset \Omega_\pm$ for all $f \in L^2(\Omega_\pm)^4$ and all $t \in \mathbb{R}$.
This is equivalent to require that $\chi_{\Omega_\pm}\varphi \in D$ for all $\varphi \in D$.

Theorem (Confinement).– Assume that $H + V$ is self-adjoint on D.
Then, V generates confinement w.r.t. H and Σ if

$$\{C_\sigma(\alpha \cdot N), \Lambda(\alpha \cdot N)\} = -(\Lambda(\alpha \cdot N))^2.$$

Some applications.

Electrostatic shell potentials

Theorem.– Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $a \in (-m, m)$.
Take $\Lambda = -(1/\lambda + C_\sigma)$, $D = \big\{\varphi = \phi * (G + g) : (\phi * G)|_\Sigma = \Lambda(g)\big\}$,
and $V_\lambda(\varphi) = \frac{\lambda}{2}(\varphi_+ + \varphi_-)$ ($\varphi_\pm$ n.t. boundary values of φ on Σ).

- $H + V_\lambda$ defined on D is self-adjoint for all $\lambda \neq \pm 2$.
- $\operatorname{Ker}(H + V_\lambda - a) \neq \emptyset$ iff $\operatorname{Ker}(1/\lambda + C_\sigma^a) \neq \emptyset$.
- $H + V_\lambda$ and $H + V_{-4/\lambda}$ have the same eigenvalues in $(-m, m)$.
- If $|\lambda| \notin [1/\|C_\sigma^a\|, 4\|C_\sigma^a\|]$, then $\operatorname{Ker}(H + V_\lambda - a) = \emptyset$.
- If $|\lambda| \notin [1/C, 4C]$, where $C = \sup_{a \in (-m,m)} \|C_\sigma^a\| < \infty$, then $H + V_\lambda$ has no eigenvalues in $(-m, m)$.

Theorem.– Let $H + V_\lambda$ be as above. If Ω_- is connected, then $H + V_\lambda$ has no eigenvalues in $\mathbb{R} \setminus [-m, m]$.

Some applications.
Electrostatic plus Lorentz scalar shell potentials

Theorem.– Let $\lambda_e, \lambda_s \in \mathbb{R}$ be such that $\lambda_e^2 - \lambda_s^2 \neq 0, 4$. Take
$\Lambda = \dfrac{\lambda_s \beta - \lambda_e}{\lambda_e^2 - \lambda_s^2} - C_\sigma$,
$D = \big\{ \varphi = \phi * (G + g) : (\phi * G)|_\Sigma = \Lambda(g) \big\}$, and
$V_{es}(\varphi) = \frac{1}{2}(\lambda_e + \lambda_s \beta)(\varphi_+ + \varphi_-)$ ($\varphi_\pm$ n.t. boundary values of φ).

- $H + V_{es}$ defined on D is self-adjoint.
- V_{es} generates confinement w.r.t H and Σ iff $\lambda_e^2 - \lambda_s^2 = -4$.

Some applications.
Electrostatic plus Lorentz scalar shell potentials

Remarks.–

- That V_{es} generates confinement means that the particles modelized by the evolution $\partial_t + i(H + V_{es})$ never cross Σ over time, i.e., Σ becomes impenetrable.

- The impenetrability condition $\lambda_e^2 - \lambda_s^2 = -4$ was known for $\Sigma = \{x \in \mathbb{R}^3 : |x| = R\}$, $R > 0$ (Dittrich-Exner-Seba).

Uncertainty Principle on the sphere S^2

We focus on $H + V_\lambda$ for $\Sigma = S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$

Definition.– Let $\widetilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ be the family of Pauli matrices. Given $a \in (-m, m)$, define

$$k^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|} I_2 \quad \text{and}$$

$$w^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|^3} \Big(1 + \sqrt{m^2 - a^2}|x|\Big)\, i\,\widetilde{\sigma} \cdot x.$$

For $f \in L^2(\sigma)^2$ and $x \in S^2$, set

$$K^a f(x) = (k^a * f)(x) \quad \text{and} \quad W^a(f) = p.v.(w^a * f)(x).$$

Uncertainty Principle on the sphere S^2

Remarks.–

- K^a and W^a are bounded operators in $L^2(\sigma)^2$.

- K^a is a positive operator.

Uncertainty Principle on the sphere S^2

Theorem.– Let $\lambda > 0$ and $a \in (-m, m)$. The operator

$$1/\lambda + (m+a)K^a$$

is invertible in $L^2(\sigma)^2$.

Furthermore, for any $f \in L^2(\sigma)^2$ and any $\delta > 0$,

$$\int_{S^2} |f|^2 \, d\sigma \leq \frac{1}{2M\delta} \int_{S^2} (1/\lambda + (m+a)K^a)^{-1} (W^a(f)) \cdot \overline{W^a(f)} \, d\sigma + \frac{\delta}{2M} \int_{S^2} (1/\lambda + (m+a)K^a) ((\widetilde{\sigma} \cdot N) f) \cdot \overline{(\widetilde{\sigma} \cdot N) f} \, d\sigma, \tag{1}$$

where M is a constant depending only on m and a.
Moreover, $M \geq \frac{1}{2} \, e^{-\sqrt{m^2-a^2}} \sqrt{2 - e^{-2\sqrt{m^2-a^2}}}$.

For suitable δ's, the inequality (1) is sharp and the equality can be attained.

Uncertainty Principle on the sphere S^2. Consequences

Definition (2-dimensional Riesz transform).– Given a finite Borel measure ν in $\mathbb{R}^3$, $h \in L^2(\nu)$ and $x \in \mathbb{R}^3$, one defines the 2-dimensional Riesz transform of h as

$$R_\nu(h)(x) = \lim_{\epsilon \searrow 0} \int_{|x-y|>\epsilon} \frac{x-y}{|x-y|^3}\, h(y)\, d\nu(y),$$

whenever the limit makes sense.

Uncertainty Principle on the sphere S^2. Consequences

Corollary.– $2\pi\|h\|_{L^2(\sigma)} \leq \|R_\sigma(h)\|_{L^2(\sigma)^3}$ for all real-valued $h \in L^2(\sigma)$, and the inequality is sharp.
Hofmann, Marmolejo-Olea, Mitrea, Pérez-Esteva, & Michael Taylor '09

- For suitable elections of λ, a, and δ, the minimizers of (**??**) give rise to eigenfunctions of $H + V_\lambda$ with eigenvalue a.
- The set of λ's for which $H + V_\lambda$ has a non-trivial eigenfunction contains an interval.

THANK YOU FOR YOUR ATTENTION