The Calderón problem with partial data

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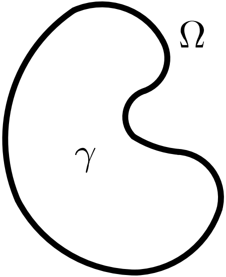
Calderón problem

Medical imaging, Electrical Impedance Tomography:

$$\begin{cases} \mathrm{div}(\gamma(x)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ bounded domain, $\gamma \in L^\infty(\Omega)$, and $\gamma \geq c > 0$.



Boundary measurements given by the *Dirichlet-to-Neumann (DN) map*

$$\Lambda_\gamma : f \mapsto \gamma \partial_\nu u|_{\partial\Omega}.$$

Inverse problem: given Λ_γ, determine γ.

Calderón problem

Model case of inverse boundary problems for elliptic equations (Schrödinger, Maxwell, elasticity).

Related to:

- optical tomography
- inverse scattering
- travel time tomography and boundary rigidity
- hybrid imaging methods
- invisibility

Calderón's approach

If $\mathrm{div}(\gamma \nabla u) = 0$ in Ω with $u|_{\partial\Omega} = f$, integrate by parts:

$$\int_{\partial\Omega} (\Lambda_\gamma f) f \, dS = \int_{\partial\Omega} \gamma (\partial_\nu u) u \, dS = \underbrace{\int_\Omega \gamma |\nabla u|^2 \, dx}_{=:Q_\gamma(f)} .$$

Thus $\Lambda_\gamma f$ determines $Q_\gamma(f)$[1]. Polarization:

$$\Lambda_\gamma \leftrightsquigarrow \int_\Omega \gamma \nabla u_1 \cdot \nabla u_2 \quad \forall\, u_j \in H^1(\Omega),\ \mathrm{div}(\gamma \nabla u_j) = 0.$$

Question: is the set $\{\nabla u_1 \cdot \nabla u_2\}$ complete in $L^1(\Omega)$?

[1] =the *power* needed to maintain boundary voltage f

Calderón's approach

Lemma (Calderón 1980)

The set $\{\nabla u_1 \cdot \nabla u_2\,;\ \Delta u_j = 0\}$ *is complete in* $L^1(\Omega)$.

Proof.

Let $u_j = e^{\rho_j \cdot x}$ where $\rho_j \in \mathbb{C}^n$ and $\rho_j \cdot \rho_j = 0$. Then $\Delta u_j = 0$. Given $\xi \in \mathbb{R}^n$, let $\eta \in \mathbb{R}^n$ satisfy $|\eta| = |\xi|$ and $\eta \cdot \xi = 0$. Take

$$\rho_1 = \eta + i\xi, \qquad \rho_2 = -\eta + i\xi.$$

Then $\nabla(e^{\rho_1 \cdot x}) \cdot \nabla(e^{\rho_2 \cdot x}) = c e^{(\eta + i\xi)\cdot x} e^{(-\eta + i\xi)\cdot x} = c e^{2ix\cdot\xi}$. $\square$

Exponentially growing solutions, or *complex geometrical optics solutions*, are a central tool in inverse boundary problems.[2]

[2] Earlier uses: Hadamard 1923, Faddeev 1966.

Calderón problem

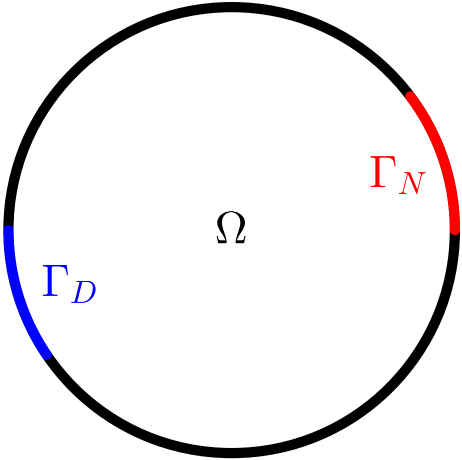
Uniqueness results ($\gamma \mapsto \Lambda_\gamma$ injective):

- Calderón (1980): linearized problem
- Sylvester-Uhlmann (1987): $n \geq 3$, $\gamma \in C^2$
- Nachman (1996): $n = 2$, $\gamma \in W^{2,p}$
- Astala-Päivärinta (2006): $n = 2$, $\gamma \in L^\infty$
- Haberman-Tataru (2013): $n \geq 3$, $\gamma \in C^1$

We are interested in the *partial data problem* where measurements are made only on subsets of the boundary.

Partial data problem

Prescribe voltages on Γ_D, measure currents on Γ_N:



Partial data problem

Particular case (*local data problem*): $\Gamma_D = \Gamma_N = \Gamma$.

Uniqueness known for arbitrary open $\Gamma \subset \partial\Omega$

- if $n = 2$
- for piecewise real-analytic conductivities if $n \geq 3$.

Open in general if $n \geq 3$. This talk will survey known results.

Partial data problem

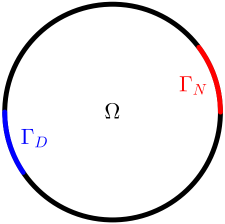
Substitution $u = \gamma^{-1/2} v$ reduces conductivity equation $\mathrm{div}(\gamma \nabla u) = 0$ to *Schrödinger equation* $(-\Delta + q)v = 0$.

Let $\Gamma_D, \Gamma_N \subset \partial\Omega$ be open and let $q \in L^\infty(\Omega)$. Define[3]

$$C_q^{\Gamma_D,\Gamma_N} = \{(u|_{\Gamma_D}, \partial_\nu u|_{\Gamma_N})\,;\, (-\Delta + q)u = 0 \text{ in } \Omega,\ u \in H_\Delta(\Omega),\\ \mathrm{supp}(u|_{\partial\Omega}) \subset \Gamma_D\}.$$

Prescribe Dirichlet data on Γ_D, measure Neumann data on Γ_N.

Inverse problem: given $C_q^{\Gamma_D,\Gamma_N}$, recover q.



[3] $H_\Delta(\Omega) = \{u \in L^2(\Omega)\,;\, \Delta u \in L^2(\Omega)\}$

Partial data problem

Four main approaches for uniqueness:

1. Carleman estimates (Kenig-Sjöstrand-Uhlmann 2007)
2. Reflection approach (Isakov 2007)
3. 2D case (Imanuvilov-Uhlmann-Yamamoto 2010)
4. Linearized case (Dos Santos-Kenig-Sjöstrand-Uhlmann 2009)

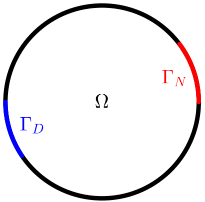
The first two approaches apply when $n \geq 3$. Other results:

- piecewise analytic conductivities (Kohn-Vogelius)
- other equations (Dos Santos et al, Chung, Chung-S-Tzou)
- stability (Heck-Wang, Caro-Dos Santos-Ruiz)
- numerics (Garde-Knudsen, Hamilton-Siltanen)

Strategy of proof

Integration by parts: if $\Gamma_D, \Gamma_N \subset \partial\Omega$ are open, then

$$C_{q_1}^{\Gamma_D,\Gamma_N} = C_{q_2}^{\Gamma_D,\Gamma_N} \implies \int_\Omega (q_1 - q_2) u_1 u_2 = 0$$



for any u_j with $(-\Delta + q_j)u_j = 0$ in Ω and

$$\mathrm{supp}(u_1|_{\partial\Omega}) \subset \Gamma_D, \quad \mathrm{supp}(u_2|_{\partial\Omega}) \subset \Gamma_N. \quad (*)$$

To show $q_1 = q_2$, enough that products of solutions

$$\{u_1 u_2 \,;\, (-\Delta + q_j)u_j = 0 \text{ in } \Omega, \quad u_j \text{ satisfy } (*)\}$$

are complete in $L^1(\Omega)$.

Strategy of proof

Use special *complex geometrical optics solutions*

$$u \approx e^{\pm\tau\varphi} a, \quad (-\Delta + q)u = 0, \quad \mathrm{supp}(u|_{\partial\Omega}) \subset \Gamma_{D,N}.$$

Here $\tau > 0$ is a large parameter. Want to show that

$$\{\lim_{\tau\to\infty} u_1 u_2\} \text{ dense in } L^1(\Omega).$$

- take $u_1 \approx e^{\tau\varphi} a_1$, $u_2 \approx e^{-\tau\varphi} a_2$ to kill exponential growth of $\lim_{\tau\to\infty} u_1 u_2$
- correct approximate solution $u_0 = e^{\pm\tau\varphi} a$ to exact solution $u = e^{\pm\tau\varphi}(a + r)$ by solving $(-\Delta + q)e^{\pm\tau\varphi}(a + r) = 0$
- possible if φ is a *limiting Carleman weight*

Strategy of proof

Condition for a *limiting Carleman weight* φ, $\nabla\varphi \neq 0$:

$$\|v\|_{L^2(\Omega)} \leq \frac{C}{\tau}\|e^{\pm\tau\varphi}\Delta e^{\mp\tau\varphi}v\|_{L^2(\Omega)}, \quad v \in C_c^\infty(\Omega), \; \tau \gg 1.$$

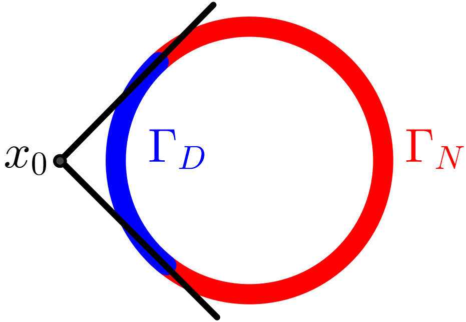
Results from Dos Santos-Kenig-S-Uhlmann (2009):

- conformally invariant condition
- if $n \geq 3$, only six basic forms for φ:

$$x_1, \quad \log|x|, \quad \frac{x_1}{|x|^2}, \quad \arctan\frac{x_2}{x_1}, \quad \ldots.$$

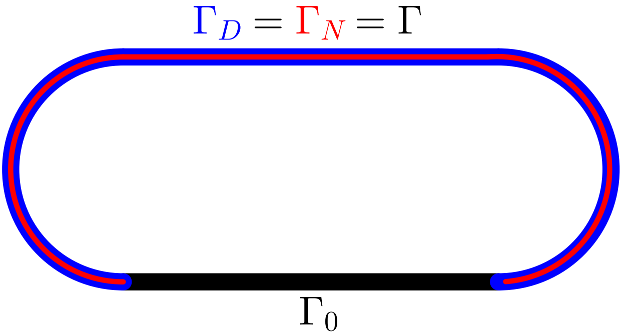
- if $n = 2$, any harmonic function is OK

1. Carleman estimate approach (KSU 2007)



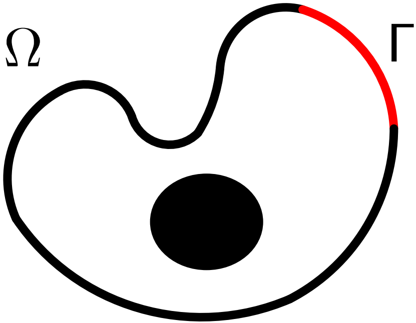
- Γ_D and Γ_N roughly complementary, need to overlap
- Γ_D can be very small, but then Γ_N has to be very large
- proof uses weights $\varphi(x) = \log|x - x_0|$ and Carleman estimates with boundary terms

2. Reflection approach (Isakov 2007)



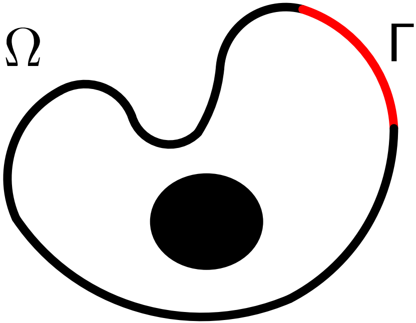
- local data: $\Gamma_D = \Gamma_N = \Gamma$, no measurements needed on Γ_0
- the *inaccessible part of the boundary*, Γ_0, has strict restrictions (part of a hyperplane or part of a sphere)
- proof uses weights $\varphi(x) = x_1$ and reflection about Γ_0

3. 2D case (IUY 2010)



- $\Omega \subset \mathbb{R}^2$ and $\Gamma_D = \Gamma_N = \Gamma$ is *any open set* in $\partial\Omega$
- any harmonic function is a limiting Carleman weight
- solutions $u = e^{\tau\Phi}(a + r)$, Φ is a *Morse holomorphic function* with prescribed critical point
- coefficients recovered via *stationary phase*

4. Linearized case (DKSU 2009)



- $\Omega \subset \mathbb{R}^n$ and $\Gamma_D = \Gamma_N = \Gamma$ is *any open set* in $\partial\Omega$
- if $\int_\Omega f u_1 u_2 = 0$ for all *harmonic* u_j with $\mathrm{supp}(u_j|_{\partial\Omega}) \subset \Gamma$, then $f = 0$ near Γ
- based on analytic microlocal analysis (FBI transform, Kashiwara's watermelon theorem)

New results (Kenig-S 2013)

Recall main approaches:

1. Carleman estimates
2. Reflection approach
3. 2D case
4. Linearized case

We unify approaches 1 and 2 and extend both. In particular, we relax the requirements on the inaccessible part in 2, and allow to use complementary (sometimes disjoint) sets as in 1.

The methods work for $n \geq 3$, also on certain Riemannian manifolds, and sometimes reduce the question to integral geometry problems of independent interest.

New results

The first results are local results: given measurements on $\Gamma \subset \partial\Omega$, coefficients are determined in a neighborhood of Γ.

Proof reduces to an integral geometry problem (*Helgason support theorem*): recover a function locally from its integrals over *lines*, *great circles*, or *hyperbolic geodesics* in a certain neighborhood.

Instead of being completely flat or spherical, the inaccessible part Γ_0 can be *conformally flat only in one direction*, e.g.

- *cylindrical* set (leads to integrals over lines)
- *conical* set (integrals over great circle segments)
- *surface of revolution* (integrals over hyperbolic geodesics).

Cylindrical sets

Theorem (Kenig-S 2013)

Let $\Omega \subset \mathbb{R} \times \Omega_0$ where $\Omega_0 \subset \mathbb{R}^2$ is convex, let $\Gamma = \partial\Omega \setminus \Gamma_0$, and suppose that Γ_0 satisfies

$$\Gamma_0 \subset \mathbb{R} \times (\partial\Omega_0 \setminus E)$$

for some open set $E \subset \partial\Omega_0$. If $q_1, q_2 \in C(\overline{\Omega})$ and if

$$C_{q_1}^{\Gamma,\Gamma} = C_{q_2}^{\Gamma,\Gamma},$$

then $q_1 = q_2$ in $\overline{\Omega} \cap (\mathbb{R} \times \mathrm{ch}_{\mathbb{R}^2}(E))$.

Corresponds to $\varphi(x) = x_1$. Similar result obtained independently by Imanuvilov-Yamamoto (2013).

Conical sets

Theorem (Kenig-S 2013)

Let $\Omega \subset \{r\omega\,;\, r > 0, \omega \in M_0\}$ where $M_0 \subset S^2$ is convex, let $\Gamma = \partial\Omega \setminus \Gamma_0$, and suppose that Γ_0 satisfies

$$\Gamma_0 \subset \{r\omega\,;\, r > 0, \omega \in \partial M_0 \setminus E\}$$

for some open set $E \subset \partial M_0$. If $q_1, q_2 \in C(\overline{\Omega})$ and if

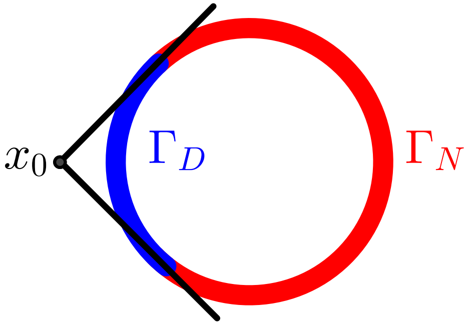
$$C_{q_1}^{\Gamma,\Gamma} = C_{q_2}^{\Gamma,\Gamma},$$

then $q_1 = q_2$ in $\overline{\Omega} \cap (\mathbb{R} \times \mathrm{ch}_{S^2}(E))$.

Corresponds to $\varphi(x) = \log|x|$. Convex hull in S^2 taken with respect to great circle segments.

Remarks

- convexity not required: if the inaccessible part is concave, recover the coefficient everywhere
- it is not required that $\Gamma_D = \Gamma_N$, can use somewhat complementary sets as in Kenig-Sjöstrand-Uhlmann
- sometimes Γ_D and Γ_N can be disjoint

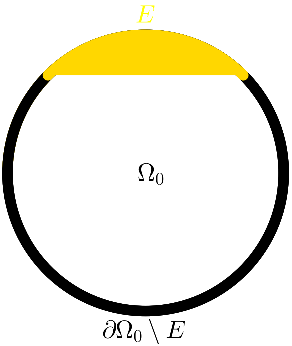


Beyond the convex hull

Let $\Omega \subset \mathbb{R} \times \Omega_0$ where $\Omega_0 \subset \mathbb{R}^2$ is convex, let $\Gamma = \partial\Omega \setminus \Gamma_0$, and suppose that Γ_0 satisfies

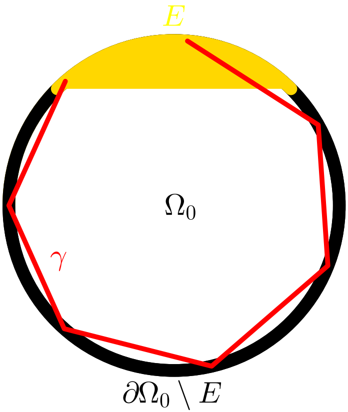
$$\Gamma_0 \subset \mathbb{R} \times (\partial\Omega_0 \setminus E)$$

for some open set $E \subset \partial\Omega_0$. From measurements on Γ, recover coefficient in $\overline{\Omega} \cap (\mathbb{R} \times \mathrm{ch}_{\mathbb{R}^2}(E))$. Can one go beyond the convex hull?



Beyond the convex hull

A continuous curve $\gamma : [0, L] \to \overline{\Omega}_0$ is a *broken ray* if it consists of straight line segments that are reflected according to geometrical optics (angle of incidence = angle of reflection) when they hit $\partial\Omega_0$.



Beyond the convex hull

Theorem (Kenig-S 2013)

Let $\Omega \subset \mathbb{R} \times \Omega_0$ where $\Omega_0 \subset \mathbb{R}^2$ is a bounded domain, let $\Gamma = \partial\Omega \setminus \Gamma_0$ where Γ_0 satisfies for some open $E \subset \partial\Omega_0$

$$\Gamma_0 \subset \mathbb{R} \times (\partial\Omega_0 \setminus E).$$

If $q_1, q_2 \in C(\overline{\Omega})$ and $C_{q_1}^{\Gamma,\Gamma} = C_{q_2}^{\Gamma,\Gamma}$, then *for any nontangential broken ray* $\gamma : [0, L] \to \overline{\Omega}_0$ *with endpoints on* E, and given any real number λ, one has

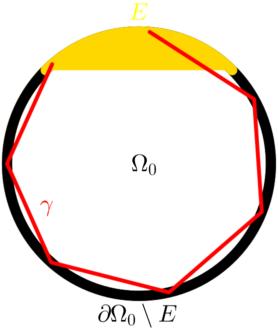
$$\int_0^L e^{-2\lambda t}(q_1 - q_2)\hat{\ }(2\lambda, \gamma(t))\,dt = 0.$$

Here $(\,\cdot\,)\hat{\ }$ is the Fourier transform in the x_1 variable, and $q_1 - q_2$ is extended by zero to $\mathbb{R}^3 \setminus \overline{\Omega}$.

Beyond the convex hull

Question

Let $\Omega_0 \subset \mathbb{R}^n$ strictly convex and $E \subset \partial\Omega_0$ open. Is a function $f \in C(\overline{\Omega}_0)$ determined by its integrals over broken rays starting and ending on E?



Partial results: Mukhometov (1980's), Eskin (2004), Hubenthal (2013), Ilmavirta (2013-4)

Components of proof

Need Carleman estimate with boundary terms:

$$
\begin{aligned}
&-\frac{1}{\tau}\int_{\partial\Omega}(\partial_\nu\varphi)e^{\pm 2\tau\varphi}|\partial_\nu v|^2\,dS + \|e^{\pm\tau\varphi}v\|^2_{L^2(\Omega)} \\
&\quad\le \frac{C}{\tau^2}\|e^{\pm\tau\varphi}(-\Delta+q)v\|^2_{L^2(\Omega)}, \quad v\in C^\infty(\overline{\Omega}),\ v|_{\partial\Omega}=0.
\end{aligned}
$$

Kenig-Sjöstrand-Uhlmann (2007) use convexified weights

$$\varphi_\varepsilon = \varphi + \frac{1}{\varepsilon\tau}\frac{\varphi^2}{2}, \quad \varepsilon > 0 \text{ small.}$$

Carleman estimate leads to solutions of $(-\Delta+q)u=0$ with

- good control on $\{x\in\partial\Omega\,;\,\partial_\nu\varphi(x)<0\}$
- *no control* on "flat" part $\{x\in\partial\Omega\,;\,\partial_\nu\varphi(x)=0\}$.

Components of proof

Need Carleman estimate with boundary terms:

$$
\begin{aligned}
&-\frac{1}{\tau}\int_{\partial\Omega}(\partial_\nu\varphi)e^{\pm 2\tau\varphi}|\partial_\nu v|^2\,dS + \|e^{\pm\tau\varphi}v\|^2_{L^2(\Omega)} \\
&\quad \leq \frac{C}{\tau^2}\|e^{\pm\tau\varphi}(-\Delta+q)v\|^2_{L^2(\Omega)}, \quad v\in C^\infty(\overline{\Omega}),\ v|_{\partial\Omega}=0.
\end{aligned}
$$

We use modified weights

$$
\varphi_\varepsilon = \varphi + \frac{1}{\varepsilon\tau}\frac{\varphi^2}{2} + \frac{1}{\tau}\kappa, \quad \varepsilon > 0 \text{ small}, \quad \partial_\nu\kappa|_{\partial\Omega} < 0.
$$

Carleman estimate leads to solutions of $(-\Delta+q)u=0$ with

- good control on $\{x\in\partial\Omega\,;\,\partial_\nu\varphi(x)<0\}$
- *weak control* on "flat" part $\{x\in\partial\Omega\,;\,\partial_\nu\varphi(x)=0\}$.

Components of proof

Some arguments can also be done by reflection, e.g. if Γ_0 is part of a graph

$$\Gamma_0 \subset \{(x_1, x_2, \eta(x_2))\,;\, x_1, x_2 \in \mathbb{R}\}$$

where η is a function $\mathbb{R} \to \mathbb{R}$. Flattening the boundary by $x_3 \mapsto x_3 - \eta(x_2)$ transforms the Euclidean Laplacian into

$$\Delta_g \approx \sum_{j,k=1}^{3} g^{jk} \partial_{x_j} \partial_{x_k}, \quad (g_{jk}(x)) = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x_2, x_3) \end{pmatrix}.$$

Reflecting across $x_3 = 0$ generates a *Lipschitz singularity* in the metric g_0. However, the singularity only appears in the lower right corner, and methods for the anisotropic Calderón problem (Kenig-S-Uhlmann 2011) still apply.

Components of proof

Suppose Ω is part of a cylinder $\mathbb{R} \times \Omega_0$ and

$$\Gamma_0 \subset \mathbb{R} \times (\partial\Omega_0 \setminus E)$$

where $\Omega_0 \subset \mathbb{R}^2$ and $E \subset \partial\Omega_0$. Use complex geometrical optics solutions as $\tau \to \infty$,

$$u(x_1, x') \approx e^{\pm\tau x_1} v_\tau(x')$$

where $v_\tau(x')$ is a *reflected Gaussian beam quasimode* in Ω_0, concentrating near a broken ray γ with endpoints on E:

$$\|(-\Delta - \tau^2)v_\tau\|_{L^2(\Omega_0)} = O(\tau^{-K}), \quad \|v_\tau\|_{L^2(\partial\Omega_0 \setminus E)} = O(\tau^{-K}),$$
$$|v_\tau|^2 \, dx' \rightharpoonup \delta_\gamma.$$

Cf. Dos Santos-Kurylev-Lassas-S (2013).

Summary

Calderón problem with local data for $n \geq 3$ still open, but

- possible to ignore measurements on sets that are part of *cylindrical sets*, *conical sets*, or *surfaces of revolution*
- *local uniqueness* results that determine coefficients near the measurement set
- *global uniqueness* under certain size or concavity conditions, or if the broken ray transform is invertible

Survey with Kenig: "Recent progress in the Calderón problem with partial data" (2014).

Open questions

Question (Local data for $n \geq 3$)

If $\Omega \subset \mathbb{R}^n$, $n \geq 3$, if Γ is any open subset of $\partial\Omega$, and if $q_1, q_2 \in L^\infty(\Omega)$, show that $C_{q_1}^{\Gamma,\Gamma} = C_{q_2}^{\Gamma,\Gamma}$ implies $q_1 = q_2$.

Question (Data on disjoint sets for $n = 2$)

If $\Omega \subset \mathbb{R}^2$, if Γ_D and Γ_N are disjoint open subsets of $\partial\Omega$, and if $q_1, q_2 \in L^\infty(\Omega)$, show that $C_{q_1}^{\Gamma_D,\Gamma_N} = C_{q_2}^{\Gamma_D,\Gamma_N}$ implies $q_1 = q_2$.