Carleson measures and absolute continuity of elliptic/parabolic measure

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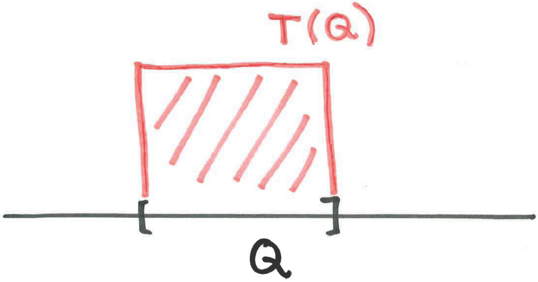
Background and definitions

$Q \subset \mathbb{R}^n$ is a cube with side length $l(Q)$ set $T(Q) = \{(x,t) \in \mathbb{R}^{n+1}_+ : x \in I, 0 < t < l(Q)\}$, a cube sitting above its boundary face Q:



Definition

The measure $d\mu$ is a Carleson measure in the upper half space $\mathbb{R}^{n+1}_{+}$ if there exists a constant C such for all cubes $Q \subset \mathbb{R}^n$, $\mu(T(Q)) < C|Q|$, where $|Q|$ denotes the Lebesgue measure of the cube Q.

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L. Carleson, An interpolation problem for bounded analytic functions, *Amer. J. Math.* **80** (1958), 921-930.

Carleson measures and potential theory

If $\Delta u(x,t) = 0$ in $\mathbb{R}^{n+1}_+$, $u(x,0) = f(x)$ with $f \in \text{BMO}$, then

$$t|\nabla u(x,t)|^2 dxdt$$

is a Carleson measure.

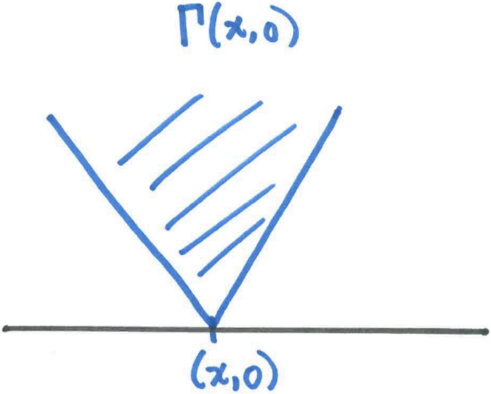
That is, there exists a constant C such that for all cubes Q,

$$\int_{T(Q)} t|\nabla u(x,t)|^2 dxdt \le C|Q|$$

C. Fefferman, Characterizations of bounded mean oscillation, *Bulletin of the American Mathematical Society*, 77, (1971), no. 4, 587–588.

Square functions

Let $\Gamma(x,0) = \{(y,t) : |x-y| < ct\}$ denote the cone at $(x,0)$ and define



Square functions

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$$S(u)(x) = \left\{ \int_{\Gamma(x)} t^{1-n} |\nabla u(y,t)|^2 dydt \right\}^{1/2}$$

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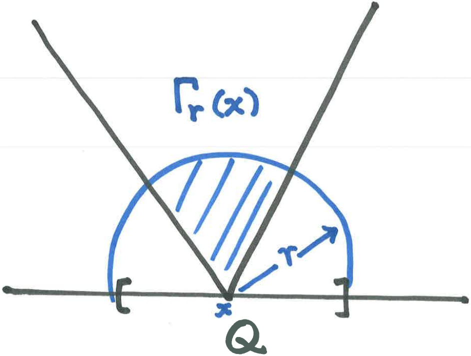
If $\Delta u(x,t) = 0$ in $\mathbb{R}^{n+1}_+$, $u(x,0) = f(x)$ with $f \in L^2$, then

$$\int_{\mathbb{R}^n} S^2(u)(x)dx = \int_{\mathbb{R}^n} t\nabla u(y,t)|^2 dydt = c \int_{\mathbb{R}^n} f^2(x)dx$$

Square functions

And if $f \in$ BMO, then a localized version holds for the truncated square function:

$$S_r(u)(x) = \left\{ \int_{\Gamma_r(x)} t^{1-n} |\nabla u(y,t)|^2 dydt \right\}^{1/2}$$



The Dirichlet Problem

The Dirichlet problem with data in L^p, $p > 1$:

$$\Delta u = 0 \in \mathbb{R}^{n+1}_+$$

$$u(x,0) = f(x) \in L^p(\mathbb{R}^n)$$

in the sense of nontangential convergence.

$$\|u^*\|_p \leq C\|f\|_p$$

where $u^* = \sup\{|u(y,t) : (y,t) \in \Gamma(x,0)\}$.

And for all p,

$$\|u^*\|_p \approx \|S(u)\|_p$$

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$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle := \sum_{i,j=1}^{n+1} A_{ij}(x)\xi_j\xi_i, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \lambda^{-1}, \quad (1)$$

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If $Lu = 0$, $u(x,0) = f(x)$, then

$$u(X) = \int_{\mathbb{R}^n} f(x)d\omega^X(x)$$

Properties of solutions

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- D_p is solvable for some p if and only if the weights $d\omega^X$ belong to A_∞

Definitions of A_∞

Definition

A measure ω defined on $\mathbb{R}^n$ belongs to the weight class $A_\infty(dx)$ if any of the following equivalent conditions hold:

(i) For every $\varepsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that for any cube $Q \subset \mathbb{R}^n$ and $E \subset Q$ with

$$\frac{\omega(E)}{\omega(Q)} < \delta \text{ then } \frac{|E|}{|Q|} < \varepsilon. \tag{2}$$

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(iii) there exists a $p > 1$ such that ω belongs to $A_p(dx)$ or to RH_p.

$$L := -\operatorname{div} A(x,t)\nabla, \quad (x,t) \in \mathbb{R}^{n+1}$$

What conditions on A guarantee that the elliptic measure and Lebesgue (or surface) measure are mutually absolutely continuous, thus ensuring well-posedness and unique solvability of a Dirichlet problem for L?

What criteria for A_∞, expressed in terms of properties of the solution, can be verified for large classes of operators in the absence of L^2 estimates?

Sharp, optimal answers to these questions are often found by means of, or in terms of, Carleson measures.

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- L^2 identity of Rellich type (Jerison-Kenig); Carleson measures in perturbation theory
- In the absence of an L^2 theory, new machinery needed to prove mutual absolute continuity, and A_∞: ε-approximation of bounded solutions

Definition of ε-approximation of bounded solutions

Definition

Let $u \in L^\infty(\mathbb{R}^{n+1}_+)$, with $\|u\|_\infty \leq 1$. Given $\varepsilon > 0$, we say that u is ε-approximable if for every cube $Q_0 \subset \mathbb{R}^n$, there is a $\varphi = \varphi_{Q_0} \in W^{1,1}(T_{Q_0})$ such that

$$\|u - \varphi\|_{L^\infty(T_{Q_0})} < \varepsilon \,, \tag{4}$$

and

$$\sup_{Q \subset Q_0} \frac{1}{|Q|} \iint_{T(Q)} |\nabla \varphi(x,t)| \, dxdt \leq C_\epsilon \,, \tag{5}$$

where C_ε depends also upon dimension and ellipticity, but not on Q_0.

Connection between A_∞ and ε-approximability

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- How to verify ε-approximability? KKPT introduced another characterization, namely:
- A_∞ is equivalent to the existence of a p such that $\|u^*\|_p \approx \|S(u)\|_p$ (on all Lipschitz subdomains).
- Applications to several classes of non-symmetric elliptic operators, classes for which the conclusion A_∞ is sharp.

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- Integrate over E, to obtain $|E|/|Q| < Ck^{-2}$, verifying A_∞.

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Theorem

Let $L := -\operatorname{div} A(x)\nabla$, be an elliptic divergence form operator, not necessarily symmetric, with bounded measurable coefficients, defined in $\mathbb{R}^{n+1}_+$. Then $\omega \in A_\infty$ if and only if, for every solution u to $Lu = 0$ with boundary data $f \in BMO$, one has the Carleson measure estimate:

$$\sup_Q \frac{1}{|Q|} \iint_{T(Q)} t|\nabla u(x,t)|^2 \, dxdt \leq C||f||^2_{BMO}, \tag{6}$$

- Dindos-Kenig-P., 2011

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- Use the Jones-Journé construction to find a function f satisfying $f = 1$ on E and $f = 0$ outside $2Q$ but with $\|f\|_{BMO}$ small.
- Specifically,

$$f = \max\{\delta \log(M\chi_E) + 1, \, 0\}$$

An optimal Carleson measure characterization of A_∞

Theorem

Let $L := -\operatorname{div} A(x)\nabla$, *be an elliptic divergence form operator, not necessarily symmetric, with bounded measurable coefficients, defined in* $\mathbb{R}^{n+1}_+$. *Then* $\omega \in A_\infty$ *if and only if, for every solution* u *to* $Lu = 0$ *with boundary data* $f \leq 1$, *one has the Carleson measure estimate:*

$$\sup_Q \frac{1}{|Q|} \iint_{T(Q)} t|\nabla u(x,t)|^2 \, dxdt \leq C. \qquad (7)$$

Kenig-Kirchheim-P.-Toro, 2014

Definition

Let $\varepsilon_0 > 0$ be given and small. If $E \subset \Omega_0$, a good ε_0-cover for E of length k is a collection of nested open sets $\{\Omega_i\}_{i=1}^{k}$ with $E \subseteq \Omega_k \subseteq \Omega_{k-1} \subseteq \cdots \subseteq \Omega_1 \subseteq \Omega_0$ such that for $l = 1, \dots, k$,

(i) $\Omega_l \subseteq \bigcup_{i=1}^{\infty} S_i^{(l)}, \bigcup_{i=1}^{\infty} S_i^{(l)} \backslash \Omega_l \subset \partial Q_0$, where each $S_i^{(l)}$ is a dyadic cube in Re^n,

(ii) $\bigcup_{i=1}^{\infty} S_i^{(l)} \subset \bigcup_{i=1}^{\infty} S_i^{(l-1)}$ and

(iii) for all $1 \leq l \leq k$, $\omega(\Omega_l \bigcap S_i^{(l-1)}) \leq \varepsilon_0 \omega(S_i^{(l-1)})$.

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- The solution u to $Lu = 0$ with boundary data f will oscillate by a fixed amount in a large number of dyadic blocks.

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Theorem

For some $C, c > 0$*, and every* $x \in E$*, there are sequences* $\{x_m, t_m\}_{m=0}^{k}$ *with* $ct_{m-1} < t_m < Ct_{m-1}$ *for which* $|u(x_m, t_m) - u(x_{m-1}, t_{m-1})| > \epsilon$.

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Corollary

For every $x \in E$,

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Corollary

For every $x \in E$,

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Integrate this inequality over E with respect to Lebesgue measure gives the result that E has small Lebesgue measure.

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- Hofmann-Lewis, 1991: study of Dirichlet problem and absolute continuity of parabolic measure.
- Dindos - P.- Petermichl: proved the parabolic analog of KKrPT, namely A_∞ of parabolic measure wrt surface measure if and only if bounded solutions satisfy a parabolic Carleson measure estimate.

Applications: establish absolute continuity of classes of elliptic/parabolic operators

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- Parabolic analogues follow from the Carleson measure characterization of bounded solutions (Dindos-Huang 2013, Dindos-P.-Petermichl 2014)