On Nematic Liquid Crystal Flows

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Outline

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Mean orientation of molecule's optical axis,
$d : \Omega \to \mathbb{S}^2$, $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$,
minimizes Oseen-Frank's bulk energy functional.

Oseen-Frank density: Assume $W(d, \nabla d)$ quadratic in ∇d,

$$W(-d, -\nabla d) = W(Qd, Q\nabla d Q^t) = W(d, \nabla d), \ Q \in O(3) \Rightarrow$$

$$2W(d,\nabla d) = k_1(\mathrm{div} d)^2 + k_2(d \cdot \mathrm{curl} d + \tau)^2 + k_3|d \times \mathrm{curl} d|^2 \\ +(k_2 + k_4)[\mathrm{tr}(\nabla d)^2 - (\mathrm{div} d)^2]$$

$k_1, k_2, k_3 > 0$: splay, twist, and bending constants, $k_2 \geq |k_4|$,
$\tau \in \mathbb{R}$. ($\tau \neq 0$: cholesterics; $\tau = 0$: nematics).

Observation (Oseen, Ericksen): $\int[\mathrm{tr}(\nabla d)^2 - (\mathrm{div} d)^2]$ depends on $d|_{\partial\Omega}$ (or null-Lagrange).

Equilibrium configuration: Given $g : \partial\Omega \to \mathbb{S}^2$, there exists a d minimizing the energy functional $\mathcal{W}(d) = \int_\Omega W(d, \nabla d)$, i.e.,

$$\mathcal{W}(d) = \min\left\{\mathcal{W}(e) : \ e \in W^{1,2}(\Omega, \mathbb{S}^2), \ e = g \text{ on } \partial\Omega\right\}.$$

The Euler-Lagrange equation (not necessarily elliptic):

$$\frac{\delta \mathcal{W}}{\delta d} := \frac{d}{dt}\Big|_{t=0}\mathcal{W}(d_t) = 0, \quad d_t = \frac{d + t\phi}{|d + t\phi|}, \quad \phi \in C_0^\infty(\Omega, \mathbb{R}^3).$$

Simple case: $k_1 = k_2 = k_3 = 1, k_4 = 0 \Rightarrow \mathcal{W}(d) = \frac{1}{2}\int_\Omega |\nabla d|^2$

$$\frac{\delta \mathcal{W}}{\delta d} = \Delta d + |\nabla d|^2 d = 0 \text{ (harmonic map to } \mathbb{S}^2).$$

Regularity of minimizers

1. (Hardt-Lin-Kindelerhrer, 86's) $d \in C^\infty(\Omega \setminus \Sigma, \mathbb{S}^2)$, with

$$\boxed{\dim_H(\Sigma) < 1}$$

It covers the point defects observed in experiments and numerical stimulations:

$$d(x) = \frac{x}{|x|} : \mathbb{R}^3 \to \mathbb{S}^2 \quad \text{(hedgehog)}$$

2. (Open question) What's the optimal size estimate of the singular set Σ? Is Σ a finite set?

Outline

1. Brief review of the Ericksen-Leslie model
   - Static case: Oseen-Frank model
   - Dynamic case: Ericksen-Leslie model
2. Simplified EL system in dimensions two
3. Simplified EL system in higher dimensions
4. General Ericksen-Leslie system

Hydrodynamics (Ericksen and Leslie, 1958-1968): Assume liquid is homogeneous (density $\rho = 1$), $u : \Omega \to \mathbb{R}^n$ is the fluid velocity, based on:
i) Conservation of linear momentum,
ii) Conservation of angular momentum,
iii) Incompressibility of the fluid.
Ericksen-Leslie (EL) system takes the form:

$$u_t + u \cdot \nabla u = -\nabla P + \nabla \cdot \sigma \qquad (1)$$

$$\nabla \cdot u = 0 \qquad (2)$$

$$d \times \left(\frac{\delta W}{\delta d} - \gamma_1 N - \gamma_2 A d\right) = 0 \qquad (3)$$

σ is modeled by the phenomenological relation: $\sigma = \sigma^L + \sigma^E$.

i) σ^E – the elastic (Ericksen) stress: $\sigma^E = -\frac{\partial W}{\partial(\nabla d)} \cdot (\nabla d)^t$.

ii) σ^L – the viscous (Leslie) stress (α_i's - Leslie constants):

$$\sigma^L(u,d) = \begin{cases} \alpha_1 (d \otimes d : A) d \otimes d + \alpha_2 d \otimes N + \alpha_3 N \otimes d \\ +\alpha_4 A + \alpha_5 d \otimes (A \cdot d) + \alpha_6 (A \cdot d) \otimes d, \end{cases}$$

$$A = \frac{\nabla u + (\nabla u)^t}{2}, N = d_t + u \cdot \nabla d + \omega \cdot d, \ \omega = \frac{\nabla u - (\nabla u)^t}{2}.$$

iii) $\frac{\delta W}{\delta d}$ – first order variation of W.

$$\gamma_1 = \alpha_3 - \alpha_2, \ \gamma_2 = \alpha_6 - \alpha_5 \tag{4}$$

$$\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5 \ \text{ (Parodi's condition)} \tag{5}$$

One constant approx. of W:

$$W(d,\nabla d)=\frac{1}{2}|\nabla d|^2 \Rightarrow$$

$$\frac{\delta W}{\delta d}=(\Delta d+|\nabla d|^2 d),\ \frac{\partial W}{\partial(\nabla d)}=\nabla d,$$

$$\sigma^E=-\frac{\partial W}{\partial(\nabla d)}\cdot(\nabla d)^t=-\nabla d\odot\nabla d=-(\nabla_i d\cdot\nabla_j d)_{1\le i,j\le n}.$$

(EL) system can be written as

$$\begin{cases}\partial_t u+u\cdot\nabla u+\nabla P=-\nabla\cdot(\nabla d\odot\nabla d)+\nabla\cdot(\sigma^L(u,d))\\ \nabla\cdot u=0\\ N+\frac{\gamma_2}{\gamma_1}Ad=\frac{1}{|\gamma_1|}(\Delta d+|\nabla d|^2 d)+\frac{\gamma_2}{\gamma_1}(d^t Ad)d.\end{cases} \tag{6}$$

Consider $\Omega = \mathbb{R}^n$. Set $\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}^n} (|u|^2 + |\nabla d|^2)$.

Energy law under Parodi's condition (Lin–Liu; W. Wang–P. Zhang– Z. Zhang):

$$\frac{d}{dt}\mathcal{E}(t) + \int_{\mathbb{R}^n} \left[\alpha_4 |\nabla u|^2 + \frac{2}{|\gamma_1|}|\Delta d + |\nabla d|^2 d|^2\right] \quad (7)$$

$$= -2\int_{\mathbb{R}^n} \left[(\alpha_1 - \frac{\gamma_2^2}{\gamma_1})|A : d \otimes d|^2 + (\alpha_5 + \alpha_6 + \frac{\gamma_2^2}{\gamma_1})|A \cdot d|^2\right]$$

$$\leq 0,$$

provided $\alpha_1, \cdots, \alpha_6$ satisfy

$$\gamma_1 < 0,\ \alpha_1 - \frac{\gamma_2^2}{\gamma_1} \geq 0,\ \alpha_4 > 0,\ \alpha_5 + \alpha_6 \geq -\frac{\gamma_2^2}{\gamma_1}. \quad (8)$$

Basic Questions:

(A) Establish, in dimensions $n = 2, 3$, the global existence of Leray-Hopf type weak solutions to the Ericksen-Leslie system (EL (6)) under general initial-boundary conditions.

(B) (Partial) regularity and uniqueness issues of suitable weak solutions of (EL (6)).

(C) Global or local well-posedness of the (EL (6)) for rough initial data belonging to the largest possible function spaces.

Notation. Leray-Hopf's weak solutions refer to any weak solution in the energy space that satisfies a weak form of the above energy dissipation inequality (7).

Further Simplified EL system (L–, 1989's):

i) Neglect stretching, rigid rotation, and interacting Leslie: $\sigma^L(u,d) = \alpha_4 A \quad N = d_t + u \cdot \nabla d$.

ii) $\gamma_1 = -1,\ \gamma_2 = 0$.

(EL(6)) becomes

$$u_t + u\cdot\nabla u - \alpha_4 \triangle u + \nabla P = -\nabla\cdot(\nabla d \odot \nabla d) \quad (9)$$

$$\nabla \cdot u = 0 \quad (10)$$

$$d_t + u\cdot\nabla d = \underbrace{\Delta d + |\nabla d|^2 d}. \quad (11)$$

Remark. (9)-(11) is strongly coupling between Navier-Stokes equation (NSE) and harmonic heat flow:

i) d constant $\Rightarrow$ NSE.

ii) $u = 0 \Rightarrow$ harmonic heat flow to $\mathbb{S}^2$:

$$d_t = \triangle d + |\nabla d|^2 d, \text{ and } \underbrace{\Delta d \cdot \nabla d = 0} \text{ (stationarity)}.$$

Classical solutions (u, d) to (9)-(11) & (13)-(14) enjoy

Energy Dissipation Inequality:

$$\frac{d}{dt}\underbrace{\int_\Omega (|u|^2 + |\nabla d|^2)} \leq -2\int_\Omega (\mu|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) \quad (12)$$

Proof. Multiplying (9) by u and (11) by $(\Delta d + |\nabla d|^2 d)$ and integrating by parts a couple of times. □

Initial-boundary condition:

$$(u,d)\big|_{t=0} = (u_0,d_0) \in \mathbf{H} \times H^1(\Omega,\mathbb{S}^2) \tag{13}$$

$$(u,d)\big|_{\partial\Omega\times\mathbb{R}_+} = (0,d_0) \in C^{2,\beta}(\partial\Omega,\mathbb{S}^2) \tag{14}$$

$$H^1(\Omega,\mathbb{S}^2) = \left\{ d \in H^1(\Omega,\mathbb{R}^3) \;\middle|\; d(x) \in \mathbb{S}^2 \text{ a.e. } x \in \Omega \right\}.$$

$$\mathbf{H} = \overline{C_0^\infty(\Omega,\mathbb{R}^n) \bigcap \{v : \nabla\cdot v = 0\}} \text{ in } L^2(\Omega,\mathbb{R}^n).$$

$$\mathbf{J} = \overline{C_0^\infty(\Omega,\mathbb{R}^n) \bigcap \{v : \nabla\cdot v = 0\}} \text{ in } H_0^1(\Omega,\mathbb{R}^n).$$

Scaling invariance: For $\lambda > 0$, if (u, P, d) solves (9)-(11), so does

$$\boxed{(\mathrm{u}_\lambda, P_\lambda, d_\lambda)(x, t) = \left(\lambda u, \lambda^2 P, d\right)\left(\lambda x, \lambda^2 t\right)}$$

Dimensions:

$$\boxed{\begin{array}{c} \mathrm{u} \sim -1,\ P \sim -2,\ d \sim 0,\ x \sim 1, \\ \mathrm{t} \sim 2,\ P_r(= B_r^n \times [-r^2, 0]) \sim n + 2 \end{array}}$$

"Naive" approach:

i) Approximate initial data (u_0, d_0) by smooth $(u_0^\epsilon, d_0^\epsilon)$.

ii) Establish both lower bound of $T_\epsilon > 0$ and apriori estimates for short time smooth solutions $(u^\epsilon, d^\epsilon) : \Omega \times [0, T_\epsilon] \to \mathbb{R}^n \times \mathbb{S}^2$ to (9)-(11), in terms of the local energy profiles of $(u_0^\epsilon, d_0^\epsilon)$.

iii) Good news: This approach works for the critical dimension $n = 2$!

1. $\exists\ (u_0^k, d_0^k) \in C^\infty(\Omega, \mathbb{R}^2 \times \mathbb{S}^2)$, $\|(u_0^k - u_0, d_0^k - d_0)\|_{L^2 \times H^1} \to 0$.

$$\sup_k E_{r_0}(u_0^k, d_0^k) = \sup_k \max_{x \in \overline{\Omega}} \int_{B_{r_0}(x) \cap \Omega} (|u_0^k|^2 + |\nabla d_0^k|^2) \le \epsilon_0^2.$$

2. Local energy inequality and global energy inequality $\Rightarrow$
$\exists\ T_k \ge c_0 r_0^2$, $(u^k, d^k) \in C^\infty(\Omega \times [0, T_k], \mathbb{R}^2 \times \mathbb{S}^2)$ satisfy

$$\begin{cases} \sup_{0 \le t \le c_0 r_0^2} E_{r_0}(u^k(t), d^k(t)) \le 2\epsilon_0^2, \\ \int_0^{c_0 r_0^2} \int_\Omega (|\nabla u^k|^2 + |\Delta d^k + |\nabla d^k|^2 d^k|^2) \lesssim 1. \end{cases}$$

3. Ladyzhenskaya "$\leq$" & Calderon-Zgymund L^2-est. on P^k $\Rightarrow$

$$\max_{x\in\overline{\Omega}} \int_{B_{r_0}(x)\times[0,c_0r_0^2]} (|u^k|^4+|\nabla u^k|^2+|\nabla d^k|^4+|\nabla^2 d^k|^2+|P^k|^2) \lesssim \epsilon_0^2.$$

Ladyzhenskaya inequality ($n=2$):

$$\boxed{\int_\Omega |f|^4 \lesssim \left(\sup_{x\in\Omega}\int_{\Omega\cap B_r(x)}|f|^2\right)\int_\Omega |\nabla f|^2 + r^{-2}\left(\sup_{x\in\Omega}\int_{\Omega\cap B_r(x)}|f|^2\right)^2}$$

Equation of P:

$$\boxed{\Delta P = -\mathrm{div}^2\left(u\otimes u + \nabla d\odot\nabla d\right)}$$

Lemma 1 (ϵ_0-regularity lemma: $n = 2$). There exists $\epsilon_0 > 0$ such that if $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(P_1, \mathbb{R}^2)$, $d \in L_t^2 H_x^2(P_1, \mathbb{S}^2)$, and $P \in L^2(P_1)$ solves (9)-(11), satisfying

$$\begin{aligned} \Phi(u, d, P, r) \quad := \quad & \|u\|_{L^4(P_r)} + \|\nabla u\|_{L^2(P_r)} + \|\nabla d\|_{L^4(P_r)} \\ & + \|\nabla^2 d\|_{L^2(P_r)} + \|P\|_{L^2(P_r)} \leq \epsilon_0, \end{aligned} \tag{15}$$

for some $r > 0$, then $(u, d) \in C^\infty(P_{\frac{r}{2}})$, and

$$\|(u, \nabla d)\|_{C^k(P_{\frac{r}{2}})} \leq C(k)\epsilon_0 r^{-k}, \ \forall\, k \geq 0. \tag{16}$$

Theorem 2 (regularity, with J. Lin, C.Y.Wang).

If $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega_T, \mathbb{R}^2)$ and $d \in L^2([0,T], H^2(\Omega, \mathbb{S}^2))$ solves (9)-(11), then $(u, d) \in C^\infty(\Omega \times (0, T])$.
If $(u, d)|_{\partial\Omega} = (0, d_0)$ for $d_0 \in H^1(\Omega, \mathbb{S}^2) \cap C^{2,\beta}(\partial\Omega, \mathbb{S}^2)$ with $\beta \in (0, 1)$, then $(u, d) \in C_\beta^{2,1}(\overline{\Omega} \times (0, T])$.

Theorem 3 (existence, with J. Y. Lin, C.Y.Wang).

For $u_0 \in \mathbf{H}, d_0 \in H^1(\Omega, \mathbb{S}^2) \cap C^{2,\beta}(\partial\Omega, \mathbb{S}^2)$, there exists $u \in L^\infty([0,\infty), \mathbf{H}) \cap L^2([0,\infty), \mathbf{J})$, $d \in L^\infty([0,\infty), H^1(\Omega, \mathbb{S}^2))$ solve (9)-(14) such that

(i) $\exists L \in \mathbb{N}$ depending on (u_0, d_0) and $0 < T_1 < \cdots < T_L$ s.t.

$$(u,d) \in C^\infty(\Omega \times (\mathbb{R}_+ \setminus \{T_i\}_{i=1}^L)) \bigcap C_\beta^{2,1}(\overline{\Omega} \times (\mathbb{R}_+ \setminus \{T_i\}_{i=1}^L)).$$

(ii) Each T_i can be characterized by

$$\lim_{t \uparrow T_i} \max_{x \in \overline{\Omega}} \int_{\Omega \cap B_r(x)} (|u|^2 + |\nabla d|^2)(y,t)dy \geq 8\pi, \quad \forall r > 0. \quad (17)$$

(iii) $\exists\, x_k^i \to x_0^i \in \overline{\Omega}$, $t_k^i \uparrow T_i, r_k^i \downarrow 0$, constant $\neq \omega_i \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ harmonic map of finite energy such that

$$(u_k^i, d_k^i) \to (0, \omega_i) \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^2 \times [-\infty, 0]),$$

$$\begin{cases} u_k^i(x,t) = r_k^i u(x_k^i + r_k^i x, t_k^i + (r_k^i)^2 t), \\ d_k^i(x,t) = d(x_k^i + r_k^i x, t_k^i + (r_k^i)^2 t). \end{cases}$$

(iv) $\exists\, t_k \uparrow +\infty$ and harmonic $d_\infty \in C^\infty(\Omega, \mathbb{S}^2) \cap C^{2,\beta}_{d_0}(\overline{\Omega}, \mathbb{S}^2)$ s.t.

$$u(\cdot, t_k) \to 0 \text{ in } H^1(\Omega),\ d(\cdot, t_k) \rightharpoonup d_\infty \text{ in } H^1(\Omega),$$

$$|\nabla d(\cdot, t_k)|^2\, dx \rightharpoonup |\nabla d_\infty|^2\, dx + \sum_{i=1}^{l} 8\pi m_i \delta_{x_i} \tag{18}$$

for some $l \in \mathbb{N}$, $\{x_i\}_{i=1}^l \subset \overline{\Omega}$, and $\{m_i\}_{i=1}^l \subset \mathbb{N}$.

(v) If either $\int_\Omega (|u_0|^2 + |\nabla d_0|^2) \leq 8\pi$ or $d_0^3 \geq 0$, then

$$(u, d) \in C^\infty(\Omega \times (0, +\infty)) \cap C^{2,1}_\beta(\overline{\Omega} \times (0, +\infty)),$$

$\exists\ t_k \uparrow +\infty$ and a harmonic $d_\infty \in C^\infty(\Omega, \mathbb{S}^2) \cap C^{2,\beta}_{d_0}(\overline{\Omega}, \mathbb{S}^2)$ s.t.

$$(u(\cdot, t_k), d(\cdot, t_k)) \to (0, d_\infty) \text{ in } C^2(\overline{\Omega}).$$

Theorem 4 (uniqueness, with C.Y.Wang).

For $n = 2$ and $0 < T < +\infty$, suppose that for $i = 1, 2$, $(u_i, d_i) \in L^\infty_t L^2_x \cap L^2_t H^1_x(\mathbb{R}^2_T, \mathbb{R}^2) \times (L^\infty_t H^1_x \cap L^2_t H^2_x)(\mathbb{R}^2_T, \mathbb{S}^2)$ solves (9)-(11), under the same initial-boundary condition (13)-(14). Then $\boxed{(u_1, d_1) \equiv (u_2, d_2)}$.

"Tangible" approach (Ginzburg-Landau approximation)
For $\epsilon > 0$, consider the Ginzburg-Landau functional:

$$E_\epsilon(d) = \int_\Omega e_\epsilon(d) := \int_\Omega (\frac{1}{2}|\nabla d|^2 + \frac{1}{4\epsilon^2}(1-|d|^2)^2), \quad d : \Omega \to \mathbb{R}^3.$$

The system (9)-(11) becomes

$$\begin{cases} u_t + u \cdot \nabla u - \mu \Delta u + \nabla P = -\nabla \cdot (\nabla d \odot \nabla d) \\ \nabla \cdot u = 0 \\ d_t + u \cdot \nabla d = \Delta d + \frac{1}{\epsilon^2}(1-|d|^2)d. \end{cases} \tag{19}$$

Remark 1. L–& C. Liu (95-01's) showed, under (13)-(14),
(i) Global well-posedness of (19) for $n = 2$, for $n = 3$ if $\mu \gg 1$;
(ii) Partial regularity for suitable weak solutions of (19) for $n = 3$, similar to Caffarelli-Kohn-Nirenberg on NSE.

Creation of defect measures as $\epsilon \downarrow 0$:

Let (u^ϵ, d^ϵ) be a sequence of solutions to (19) such that

$$\boxed{u^\epsilon \rightharpoonup u \text{ in } L^2;\ d^\epsilon \rightharpoonup d \text{ in } H^1.}$$

$$\Downarrow$$

$$\boxed{\nabla \cdot u = 0 \ \ \& \ \ d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d.}$$

But u **may not** solve (9), since

$$(\nabla d^\epsilon \odot \nabla d^\epsilon)\, dxdt \rightharpoonup (\nabla d \odot \nabla d)\, dxdt + \nu \ \text{ as } \ \epsilon \downarrow 0,$$

for some $n \times n$ matrix ν of signed Radon measures in $\Omega \times (0, +\infty)$ (defect measure).

Question 3: How large is the size of support of ν? Under what conditions is $\mathrm{supp}(\nu)$ empty or $dvi\nu$ is given by a gradient?

Theorem 5 (with C.Y.Wang, preprint). For $n = 2, 3$, there exists δ_0 such that if

$$\boxed{|\mathrm{d}_\epsilon| \le 1,\ d_\epsilon^3 \ge -1 + \delta,\ \forall \epsilon > 0}$$

then $\nu \equiv 0$ and $(u_\epsilon, d_\epsilon) \to (u, d)$ strongly in $L^2 \times H^1$.

Theorem 6 (with C.Y.Wang, preprint).

For $n = 3$, if $(u_0, d_0) \in \mathbf{H} \times H^1(\Omega, \mathbb{S}^2)$, with $d_0^3 \ge 0$, then there is a Leray-Hopf type weak solution $(u, d) : \Omega \times [0, +\infty) \to \mathbb{R}^3 \times \mathbb{S}^2$ of the initial and boundary value problem of the simplifed Ericksen-Leslie system (9)-(11).

For $\delta \in (0,2)$ and $L_1, L_2 > 0$, define

$$\mathrm{d}_\epsilon \in H^1(\Omega,\mathbb{R}^3) \in \mathbf{X}(\delta, L_1, L_2) \Leftrightarrow \begin{cases} \Delta d_\epsilon + \frac{(1-|d_\epsilon|^2)}{\epsilon^2} d_\epsilon = f_\epsilon \\ |d_\epsilon| \le 1, d_\epsilon^3 \ge -1+\delta \\ \|f_\epsilon\|_{L^2(\Omega)} \le L_1 \\ E_\epsilon(d_\epsilon) = \int_\Omega e_\epsilon(d_\epsilon) \le L_2. \end{cases}$$

Lemma 2. $\mathbf{X}(\delta, L_1, L_2)$ is precompact in $H^1_{\text{loc}}(\Omega,\mathbb{R}^3)$.

Sketch of proof: Assume $d_\epsilon \rightharpoonup d$ in H^1, and

$$e_\epsilon(d_\epsilon)\,dx \rightharpoonup \mu := \frac{1}{2}|\nabla d|^2\,dx + \nu$$

for some Radon measure $\nu \ge 0$.

$$\boxed{\nu \equiv 0}$$

Step 1 (almost monotonicity). $d_\epsilon \in \mathbf{X}(\delta, L_1, L_2)$ satisfies

$$\Phi_\epsilon(R) \geq \Phi_\epsilon(r) + \int_{B_R \setminus B_r} |x|^{-1} \left|\frac{\partial d_\epsilon}{\partial |x|}\right|^2, \ \forall\, 0 < r \leq R, \tag{20}$$

where

$$\Phi_\epsilon(r) := \frac{1}{r} \int_{B_r} \left(e_\epsilon(d_\epsilon) - \langle x \cdot \nabla d_\epsilon, f_\epsilon \rangle\right) + \frac{1}{2} \int_{B_r} |x||f_\epsilon|^2.$$

Step 2 (δ_0-compactness). $\exists\ \delta_0 > 0, \alpha_0 \in (0,1), C_0 > 0$ such that for $d_\epsilon \in \mathbf{X}(\delta, L_1, L_2)$, it holds

$$\Phi_\epsilon(r_0) \le \delta_0 \Rightarrow d_\epsilon \to d \text{ in } H^1(B_{\frac{r_0}{2}}) \text{ and } \nu = 0 \text{ in } B_{\frac{r_0}{2}}.$$

Proof. $\omega_\epsilon = \frac{d_\epsilon}{|d_\epsilon|}$ solves

$$\operatorname{div}(|d_\epsilon|^2 \nabla\omega_\epsilon \times \omega_\epsilon) = \operatorname{div}(\nabla d_\epsilon \times d_\epsilon) = \tau_\epsilon \times d_\epsilon.$$

$$\Downarrow$$

$$\begin{cases} \left[\omega_\epsilon\right]_{C^{\alpha_0}(B_{\frac{r_0}{2}})} \le C_0 \\ \left\|\nabla\omega_\epsilon\right\|_{L^p(B_{\frac{r_0}{2}})} \lesssim \left[\left\|\nabla\omega_\epsilon\right\|_{L^2(B_{r_0})} + \left\|\tau_\epsilon\right\|_{L^2(B_{r_0})}\right], \end{cases}$$

for $2 < p < 3$.

Step 3 (almost monotonicity for μ)

$$\Theta^1(\mu, r) := \frac{1}{r}\mu(B_r) \leq \Theta^1(\mu, R) + C_0(R - r), \ \forall\, 0 < r \leq R. \quad (21)$$

$\Rightarrow \Theta^1(\mu) = \lim_{r \downarrow 0} \Theta^1(\mu, r)$ exists and is USC.

Step 4 (concentration set)

$$\boxed{\Sigma = \left\{ x \in \Omega : \Theta^1(\mu) \geq \delta_0 \right\}}$$

is 1-rectifiable, closed subset, with $H^1(\Sigma) < +\infty$.

$$\text{supp}(\nu) \subset \Sigma; \text{ and } \delta_0 \leq \Theta^1(\nu, x) \leq C_0, \quad H^1 \text{ a.e. } x \in \Sigma.$$

Step 5 (stratification and blow-up). Pick generic $x_0 \in \Sigma$:

$$\lim_{r_i \downarrow 0} r_i^{-1} \int_{B_{r_i}(x_0)} |\nabla d|^2 = 0; \ \Theta^1(\nu, \cdot) \text{ is } H^1 \text{ approx. continu. at } x_0,$$

and Σ has a tangent line at x_0.
Set $d_i(x) = d_{\epsilon_i}(x_0 + r_i x)$, $\nu_i(A) = r_i^{-1}\nu(x_0 + r_i A)$. Then

$$\boxed{\nu_i \rightharpoonup \nu_0 := \theta_0 H^1 LY, \ e_{\epsilon_i}(d_i)\, dx \rightharpoonup \nu_0 = \theta_0 H^1 LY,}$$

for some $\theta_0 > 0$ and line $Y = \{(0, 0, x_3) : x_3 \in \mathbb{R}\}$.

$$(20) \Rightarrow \int_{B_2} |\frac{\partial d_i}{\partial x_3}|^2 \to 0$$

$\Rightarrow \exists\, \omega(x) = \omega(x_1, x_2) : \mathbb{R}^3 \to \mathbb{S}^2_{-1+\delta}$ nontrivial, smooth harmonic map. This is impossible.

Sketch of Proof of Theorem 6:

Lemma 1. Assume $u_\epsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+, L^2)$ and $d_\epsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H^1)$ solve the transported GL:

$$\begin{cases} \partial_t d_\epsilon + u_\epsilon \cdot \nabla d_\epsilon = \Delta d_\epsilon + \frac{1}{\epsilon^2}(1 - |d_\epsilon|^2) d_\epsilon & \text{in } \ \Omega \times \mathbb{R}_+, \\ \nabla \cdot u_\epsilon = 0 & \text{in } \ \Omega \times \mathbb{R}_+, \\ d_\epsilon = g_\epsilon & \text{on } \partial_p(\Omega \times \mathbb{R}_+). \end{cases} \tag{22}$$

If $g_\epsilon \in H^1(\Omega, \mathbb{R}^3)$ satisfies

$$|g_\epsilon(x)| \le 1 \text{ and } g^3_\epsilon(x) \ge 0, \text{ a.e. } x \in \Omega.$$

Then

$$|d_\epsilon(x,t)| \le 1 \text{ and } d^3_\epsilon(x,t) \ge 0, \text{ a.e. } (x,t) \in \Omega \times \mathbb{R}_+.$$

Global energy inequality:

$$\sup_{\epsilon>0}\sup_{0<t<+\infty}\int_\Omega \left[|u_\epsilon|^2+|\nabla d_\epsilon|^2+\frac{1}{2\epsilon^2}(1-|d_\epsilon|^2)^2\right]$$
$$+2\int_0^\infty\int_\Omega\left[|\nabla u_\epsilon|^2+\left|\Delta d_\epsilon+\frac{1}{\epsilon^2}(1-|d_\epsilon|^2)d_\epsilon\right|^2\right]$$
$$\leq \int_\Omega\left(|u_0|^2+|\nabla d_0|^2\right):=C_0 \qquad (23)$$

$\Rightarrow \exists\, p>3$ such that for $0<T<+\infty$,

$$\sup_{\epsilon>0}\left[\|\partial_t u_\epsilon\|_{L^{\frac54}([0,T],L^{\frac54})+L^2([0,T],H^{-1})+L^2([0,T],W^{-1,p})}\right.$$
$$\left.+\|\partial_t d_\epsilon\|_{L^2([0,T],L^2)+L^{\frac54}([0,T],L^{\frac54})}\right]<+\infty. \qquad (24)$$

Aubin-Lions' Lemma $\Rightarrow \exists\, u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega \times \mathbb{R}_+, \mathbb{R}^3)$ and $d \in L_t^\infty H_x^1(\Omega \times \mathbb{R}_+, \mathbb{S}^2)$ such that

$$\begin{cases} (u_\epsilon, d_\epsilon) \to (u, d) \text{ in } L^2_{\mathrm{loc}}(\Omega \times \mathbb{R}_+), \\ (\nabla u_\epsilon, \nabla d_\epsilon) \rightharpoonup (\nabla u, \nabla d) \text{ in } L_t^2 L_x^2(\Omega \times \mathbb{R}_+). \end{cases}$$

Fatou's lemma $\Rightarrow$

$$\int_0^\infty \liminf_{\epsilon \to 0} \int_\Omega \left| \Delta d_\epsilon + \frac{1}{\epsilon^2}(1 - |d_\epsilon|^2) d_\epsilon \right|^2 \leq C_0. \tag{25}$$

For $\Lambda >> 1$, define

$$G_\Lambda^T := \left\{ t \in [0, T] : \liminf_{\epsilon \to 0} \int_\Omega \left| \Delta d_\epsilon + \frac{1}{\epsilon^2}(1 - |d_\epsilon|^2) d_\epsilon \right|^2 (t) \leq \Lambda \right\},$$

$$B_\Lambda^T = [0, T] \setminus G_\Lambda^T \Rightarrow \left| B_\Lambda^T \right| \leq \frac{C_0}{\Lambda}.$$

Claim 1. For any $t \in G_\Lambda^T$, it holds

$$d_\epsilon(t) \to d(t) \text{ in } H^1_{\rm loc}(\Omega), \ e_\epsilon(d_\epsilon(t))\,dx \rightharpoonup \frac{1}{2}|\nabla d(t)|^2\,dx. \qquad (26)$$

Proof. For $t \in G_\Lambda^T$, since

$$\{d_\epsilon(t)\} \subset \mathbf{X}(C_0, \Lambda, 1; \Omega),$$

by the precompactness theorem, (26) holds. □

Claim 2. $|d_\epsilon(x,t)| \geq \frac{1}{2}$, and $\omega_\epsilon := \frac{d_\epsilon}{|d_\epsilon|} \to d$ in $L^2(G_\Lambda^T, H^1_{\rm loc}(\Omega))$.

Claim 3. For $t \in G_\Lambda^T$, $\omega_\epsilon(t) \in W^{2,\frac{5}{4}}_{\rm loc}(\Omega, \mathbb{S}^2)$ and

$$\left\|\omega_\epsilon(t)\right\|_{W^{2,\frac{5}{4}}(K)} \leq C\big(\delta_0, K\big), \ \forall K \subset\subset \Omega. \qquad (27)$$

Claim 4. It holds

$$\left\|\nabla\omega_\epsilon - \nabla d\right\|_{L^2(K\times G_\Lambda^T)} \to 0, \ \forall K \subset\subset \Omega. \tag{28}$$

Proof. $W^{2,\frac{5}{4}}_{\mathrm{loc}}(\Omega) \subset H^1_{\mathrm{loc}}(\Omega)$ compact, $H^1_{\mathrm{loc}}(\Omega) \subset H^{-2}_{\mathrm{loc}}(\Omega)$.
Applying Aubin-Lions' Lemma yields Claim 4. □

Claim 5. It holds

$$\int_{K\times G_\Lambda^T} \left(\left|\nabla|d_\epsilon|\right|^2 + \frac{1}{\epsilon^2}(1-|d_\epsilon|^2)^2\right) \to 0, \ \forall K \subset\subset \Omega. \tag{29}$$

Combining (28) and (29) yields

$$\left\|\nabla d_\epsilon - \nabla d\right\|^2_{L^2\left(K\times G_\Lambda^T\right)} + \int_{K\times G_\Lambda^T} \frac{1}{\epsilon^2}(1-|d_\epsilon|^2)^2 \to 0. \tag{30}$$

Claim 6. It holds

$$\left\|\nabla d_\epsilon - \nabla d\right\|^2_{L^2\left(K\times B^T_\Lambda\right)} + \int_{K\times B^T_\Lambda} \frac{1}{\epsilon^2}(1-|d_\epsilon|^2)^2$$
$$\leq \left(2\sup_{t>0}\int_\Omega e_\epsilon(d_\epsilon)(t)\right)\left|B^T_\Lambda\right| \leq C\Lambda^{-1}. \qquad (31)$$

Putting all these estimates together yields we have

$$\lim_{\epsilon\to 0}\int_{K\times[0,T]} |\nabla d_\epsilon - \nabla d|^2 + \frac{1}{\epsilon^2}(1-|d_\epsilon|^2)^2\Big] \leq C\Lambda^{-1}. \qquad (32)$$

□

Well-posedness for rough initial data:

Definition 1. $f \in \mathrm{BMO}(\mathbb{R}^n) \Leftrightarrow f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies

$$\left[\, f \,\right]_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n, r>0} \inf_{c\in\mathbb{R}} \left\{ \frac{1}{r^n} \int_{B_r(x)} |f - c| \right\} < +\infty.$$

$f \in \mathrm{BMO}^{-1}(\mathbb{R}^n) \;\Leftrightarrow\; f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\exists\, (f_1, \cdots, f_n) \in \mathrm{BMO}(\mathbb{R}^n)$ such that

$$f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i},$$

$$\left[\, f \,\right]_{\mathrm{BMO}^{-1}(\mathbb{R}^n)} = \inf \left\{ \sum_{i=1}^{n} [f_i]_{\mathrm{BMO}(\mathbb{R}^n)} : \; f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \right\}.$$

Theorem 7(C.Y.Wang et al) $\exists\ \epsilon_0 > 0$ s.t. if $(u_0, d_0) \in \mathrm{BMO}^{-1}(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$ has

$$\boxed{\left[\,\mathrm{u}_0\,\right]_{\mathrm{BMO}^{-1}(\mathbb{R}^n)} + \left[\,d_0\,\right]_{\mathrm{BMO}(\mathbb{R}^n)} \leq \epsilon_0,} \tag{33}$$

$\exists\,!\ (u, d) \in C^\infty(\mathbb{R}^n \times (0, +\infty), \mathbb{R}^n \times \mathbb{S}^2)$ of (9)-(11), with $(u, d)\big|_{t=0} = (u_0, d_0)$.

Remark 2. (i) For $n \geq 2$, (33) holds, provided

$$\sup_{x\in\mathbb{R}^n, r>0} \left\{ r^{2-n} \int_{B_r(x)} (|u_0|^2 + |\nabla d_0|^2) \right\} \leq \epsilon_0^2.$$

(ii) Reduces to Koch-Tataru's classical theorem on NSE.
(iii) Answer an open question by Sverak on harmonic heat flow.

General Ericksen-Leslie system in 2D:

Consider the Ericksen-Leslie system in $\mathbb{R}^2$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = -\nabla \cdot (\nabla d \odot \nabla d) + \nabla \cdot (\sigma^L(u, d)) \\ \nabla \cdot u = 0 \\ N + \frac{\gamma_2}{\gamma_1} A d = \frac{1}{|\gamma_1|}(\Delta d + |\nabla d|^2 d) + \frac{\gamma_2}{\gamma_1}(d^t A d) d. \end{cases} \tag{34}$$

Assume

$$\gamma_1 = \alpha_3 - \alpha_2, \ \gamma_2 = \alpha_6 - \alpha_5 \tag{35}$$

$$\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5 \ \text{(Parodi's condition)} \tag{36}$$

and

$$\gamma_1 < 0, \ \alpha_1 - \frac{\gamma_2^2}{\gamma_1} \geq 0, \ \alpha_4 > 0, \ \alpha_5 + \alpha_6 \geq -\frac{\gamma_2^2}{\gamma_1}. \tag{37}$$

The ϵ_0-regularity lemma also hold for (34):

Theorem 8 (existence of Ericksen-Leslie, with J. Huang, C. Y. Wang).

For $n = 2$, assume (35), (36), and (37). For any $u_0 \in L^2(\mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$ and $d_0 \in H^1_{e_0}(\mathbb{R}^2, \mathbb{S}^2)$, there exists a global weak solution (u, d) to EL (34) along with the initial condition (u_0, d_0), such that

(i) $\exists\ T_0 = 0 < T_1 < \cdots < T_L < T_{L+1} = +\infty$ such that

$$(u, d) \in C^\infty(\mathbb{R}^2 \times ((0 + \infty) \setminus \{T_i\}_{i=0}^{L+1}), \mathbb{R}^2 \times \mathbb{S}^2).$$

(ii) $u \in L_t^\infty L^2 \cap L_t^2 H^1(\mathbb{R}^2 \times (0, +\infty))$, $d \in L_t^\infty H_{e_0}^1(\mathbb{R}^2)$, and

$$d \in \bigcap_{\delta > 0} L_t^2([T_1, T_{i+1} - \delta], H_{e_0}^2(\mathbb{R}^2)), \ \forall i = 1, \cdots, L+1.$$

(iii) Each T_i can be characterized by

$$\lim_{t \uparrow T_i} \max_{x \in \mathbb{R}^2} \int_{B_r(x)} (|u|^2 + |\nabla d|^2)(t) \geq 8\pi, \ \forall r > 0.$$

(iv) $\mathcal{E}(0) \leq 8\pi$ or $d_0^3 \geq 0 \Rightarrow (u, d) \in C^\infty(\mathbb{R}^2 \times (0, +\infty))$.