One-Phase Free Boundaries

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In honor of Carlos Kenig; September 2014

Joint work with Nikola Kamburov

One-Phase Free Boundary Problem

$$u \geq 0, \quad \text{continuous on} \quad D \subset \mathbf{R}^2$$

$$\Delta u = 0 \quad \text{on} \quad D^+ := \{u > 0\}$$ simply-connected

$$|\nabla u(x)| = 1, \; x \in F := D \cap \partial D^+$$

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$$\boxed{D^+ \cap D_{1/2} = \, ??}$$

Outline

- Double hairpin solutions
- Minimal surfaces!
- Removable singularities thm for Flat $\implies$ Lipschitz

Hauswirth, Hélein, Pacard 2011

$$S = \{\zeta = \xi + i\eta : |\eta| \leq \pi/2\}$$

$$\Omega := \varphi(S), \qquad \varphi(\zeta) = i(\zeta + \sinh\zeta).$$

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$$H_a(z) = aH(z/a); \qquad \Omega_a = a\Omega$$

Theorem. Up to translation and dilation, the only simply-connected global solutions in $\mathbf{R}^2$ are:

double hairpin (HHP) and 1-plane, x_1^+.

Traizet (2014), if $\partial\Omega =$ finitely many smooth strands.

Khavinson, Lundberg, Teodorescu (2013), if Ω is Smirnov; in particular assuming chord-arc condition.

Thm 1. There is $c > 0$ such that if $0 \in \partial D^+$ and D^+ is simply-connected, then either

$B_c(0) \cap \partial D^+$ has one strand of bounded curvature

or it resembles a piece of a double hairpin HHP

Thm 2 (Rigidity). $\forall \delta > 0$, $\exists c > 0$ such that if two strands of ∂D^+ are separated by $\varepsilon > 0$ near 0, then

$$\text{there is} \quad a \approx \varepsilon, \quad \psi : \Omega_a \cap B_c(0) \to D^+$$

Near isometry:

$$|\psi'(z) - 1| \leq \delta, \ z \in \Omega_a; \quad |\psi'(z)| = 1, \ z \in \partial\Omega_a$$

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Curvature bounds:

$$|\psi''(z)| \le \delta,\ z \in \Omega_a; \quad |\kappa(\psi(z)) - \kappa_a(z)| \le \delta;\ z \in \partial\Omega_a$$

Colding-Minicozzi: Embedded minimal annulus

$$M \subset B_1 \subset \mathbf{R}^3 \quad \text{with neck size } \varepsilon > 0,$$

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Traizet correspondence: free boundary solutions $\longleftrightarrow$ minimal surfaces with reflection symmetry

Where the two theorems overlap, ours is slightly stronger.

Colding-Minicozzi (Removable singularities)

$\forall \delta > 0$, $\exists C$, such that every minimal annulus

$$M \subset B_1 \backslash B_\varepsilon; \quad \partial M = \text{two loops in } \ \partial B_\varepsilon \cup \partial B_1,$$

satisfies

$$M \text{ is a } \delta\text{-Lipschitz graph in } \ B_{1/C} \backslash B_{C\varepsilon}$$

(Major estimate leading to classification of minimal disks.)

Thm 3. Suppose A^+ is simply-connected in

$$\text{annulus} \quad A = \{\varepsilon < |x| < 1\},$$

$F =$ two strands connecting ∂D_1 to ∂D_ε that don't get too close to each other.

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Significance: rules out spirals.

Thm 4. Blow-up limits. If u_k solves FBP with simply-connected $D_k^+ \subset D$, $0 \in \partial D_k^+$, $R_k \to \infty$, and

$$R_k u_k(x/R_k) \to U(x) \quad \text{unif. on compact} \subset \mathbf{R}^2$$

Then after a rigid motion either

$$U(x) = x_1^+; \quad U(x) = x_1^+ + (x_1 + b)^+, \ b \geq 0$$

or

$$U(x) = H_a(x), \quad a > 0$$

Proof of classification of blow-up limits U

- $|\nabla u_k| \leq C$
- nondegeneracy: $u_k(x) \geq c\,\mathrm{dist}(x, F)$
- All strands of F escape to ∞
- V+V = Viscosity + Variational solutions (Caffarelli + Georg Weiss)

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- $|\nabla U| \leq 1$.
- If $|\nabla U| \equiv 1$, then U is a 1-plane or 2-plane solution.
- If $|\nabla U| < 1$, then at all but one boundary point, the blow up is 1-plane x_1^+.
- If $|\nabla U| < 1$, then the zero set is convex at every point where it is smooth.

Small Lipschitz constant/Removable singularities

Goal: $A^+ = \{x \in D_{1/C} \setminus D_{C\varepsilon} : x_2 > f(x_1)\}; \quad |f'(x_1)| \le \delta.$

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Linear estimate for conjugate, $h \mapsto \tilde{h}$.

$$h(iy) = 0, \ \varepsilon \le |y| \le 1; \quad |h(w)| \le \delta, \ \varepsilon < |w| \le 1$$

$$\implies |\operatorname{osc} \tilde{h}(w)| \le C\delta, \quad C\varepsilon < |w| < 1/C.$$

Curvature bounds

$$\Phi_a = H_a + i\tilde{H}_a : \Omega_a \to \{\mathrm{Re}\, w > 0\}$$

$$\psi = \Phi^{-1} \circ \Phi_a$$

Both mappings are double coverings.

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$$\psi' = e^{f + i\tilde{f}} \implies f = 0 \text{ on } \partial\Omega_a$$

Need linear estimates for $|\nabla f|$. Valid uniformly in $a > 0$ because Ω_a has Green's function with slope 1.

TRAIZET CORRESPONDENCE

$$dX_1 + i dX_2 = \frac{1}{2} d\bar{z} - 2 \left(\frac{\partial u}{\partial z} \right)^2 dz$$

$$\boxed{z \mapsto (X_1, X_2, \pm u(z))}$$

The image is an immersed minimal surface with symmetry $x_3 \leftrightarrow -x_3$. Moreover,

$$|\nabla u| < 1 \iff \text{embedded}$$