The Euler–Maxwell system for electrons : global solutions in 2d

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Outline

- The "division problem"
- A quasilinear model : the Euler–Maxwell one-fluid model in 2D
- Existence of smooth global solutions
- Energy estimates using a quasilinear I-method
- Semilinear analysis : a Fourier integral operator
- Semilinear analysis : control of the Z-norm

The "division" problem. Consider a generic evolution problem of the type

$$\partial_t u + i\Lambda u = \mathcal{N}(u, D_x u)$$

where Λ is real and $\mathcal{N}$ is a quadratic nonlinearity. At first iteration

$$u(t) = e^{-it\Lambda}\phi.$$

At second iteration, assuming $\mathcal{N} = u^2$,

$$\widehat{u}(\xi, t) = e^{-it\Lambda(\xi)}\widehat{\phi}(\xi) + Ce^{-it\Lambda(\xi)} \int \widehat{\phi}(\xi - \eta)\widehat{\phi}(\eta) \frac{1 - e^{it[\Lambda(\xi)-\Lambda(\eta)-\Lambda(\xi-\eta)]}}{\Lambda(\xi) - \Lambda(\eta) - \Lambda(\xi - \eta)}\, d\eta.$$

One has to understand the contribution of the set of (time) resonances :

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In semilinear problems one can iterate using $X^{s,b}$ spaces (Bourgain, Kenig–Ponce–Vega, Klainerman–Machedon). The iteration method completely fails in quasilinear problems due to the unavoidable loss of derivative.

In quasilinear problems, the classical methods are energy and vector-field methods (Klainerman, Christodoulou). The long-term goal of our project is to implement Fourier analysis techniques in the study of quasilinear problems.

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- The solution has strictly less than $1/t$ pointwise decay ;
- There is a full set (codimension 1) of time resonances and no matching "null structure".

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The Euler–Maxwell two-fluid model

Two compressible ion and electron fluids interact with their own self-consistent electromagnetic field. The Euler-Maxwell system describes the dynamical evolution of the functions $n_e, n_i : \mathbb{R}^3 \to \mathbb{R}$, $v_e, v_i, E, B : \mathbb{R}^3 \to \mathbb{R}^3$, which evolve according to the quasi-linear coupled system,

$$
\begin{aligned}
&\partial_t n_e + \operatorname{div}(n_e v_e) = 0, \\
&n_e m_e \left[\partial_t v_e + v_e \cdot \nabla v_e\right] + \nabla p_e = -n_e e \left[E + \frac{v_e}{c} \times B\right], \\
&\partial_t n_i + \operatorname{div}(n_i v_i) = 0, \\
&n_i M_i \left[\partial_t v_i + v_i \cdot \nabla v_i\right] + \nabla p_i = Z n_i e \left[E + \frac{v_i}{c} \times B\right], \\
&\partial_t B + c\nabla \times E = 0, \\
&\partial_t E - c\nabla \times B = 4\pi e \left[n_e v_e - Z n_i v_i\right],
\end{aligned}
$$

together with the elliptic equations

$$\operatorname{div}(B) = 0, \quad \operatorname{div}(E) = 4\pi e(Z n_i - n_e)$$

and two equations of state expressing $p_e = p_e(n_e)$ and $p_i = p_i(n_i)$.

These equations describe a plasma composed of electrons and one species of ions. The electrons have charge $-e$, density n_e, mass m_e, velocity v_e, and pressure p_e, and the ions have charge Ze, density n_i, mass M_i, velocity v_i, and pressure p_i. In addition, c denotes the speed of light and E and B denote the electric and magnetic field. The two elliptic equations are propagated by the dynamic flow, provided that we assume that they are satisfied at the initial time.

At the linear level, there are ion-acoustic waves, Langmuir waves, as well as light waves. At the nonlinear level, the Euler-Maxwell system is the origin of many well-known dispersive PDE, such as KdV, KP, Zakharov, Zakharov-Kuznetsov, and NLS, which can be derived from via different scaling and asymptotic expansions. One can also derive the Euler–Poisson model, the cold-ion model, and quasi-neutral equations.

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Variants : 2D model Euler–Maxwell, relativistic versions, periodic solutions.

Main question : Are there any smooth nontrivial global solutions of the Euler–Maxwell system ? This is a system of nonlinear hyperbolic laws with no dissipation and no relaxation effects.

Constant solutions : $(n_e, v_e, n_i, v_i, E, B) = (n_0, 0, n_0/Z, 0, 0, 0)$.

Positive conserved energy for small perturbations of the constant solutions.

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Construction of global irrotational solutions of the full system (Guo–I–Pausader 2013) for data corresponding to small, smooth and localized perturbations of the constant solutions.

Earlier global existence results on simplified models : Guo (1998) for the Euler–Poisson electron model, Guo–Pausader for the Euler–Poisson ion model, Germain–Masmoudi for the Euler–Maxwell electron model (weak decay), I.–Pausader for the Euler–Maxwell electron model (robust decay).

Blow-up solutions for small irrotational initial data for the pure compressible Euler equations (John, Sideris).

In dimension 2 we consider the **Euler-Maxwell system for electrons**, namely

$$\begin{aligned}
&\partial_t n_e + \mathrm{div}(n_e v_e) = 0,\\
&n_e m_e \left(\partial_t v_e + v_e \cdot \nabla v_e\right) + \nabla p_e = -n_e e \left(E - \frac{b v_e^{\perp}}{c}\right),\\
&\partial_t b + c \cdot \mathrm{curl}(E) = 0,\\
&\partial_t E + c \nabla^{\perp} b = 4\pi e n_e v_e,
\end{aligned}$$

where $n_e, b : \mathbb{R}^2 \times I \to \mathbb{R}$ and $v_e, E : \mathbb{R}^2 \times I \to \mathbb{R}^2$ are C^2 functions, and e, m_e, c are strictly positive constants. We consider the quadratic pressure law $p_e = T n_e^2/2$ together with the compatibility and irrotationality equations

$$\mathrm{div}(E) = 4\pi e(n_0 - n_e), \quad \mathrm{curl}(v_e) = \frac{e}{m_e c} b.$$

We show that a constant neutral equilibrium,

$$(n, v, E, B) = (n_0, 0, 0, 0)$$

is asymptotically stable : small smooth perturbations leads to globally smooth solutions that return to equilibrium.

Main Theorem (Deng–I.–Pausader) : There exists a norm X and $\varepsilon > 0$ such that *irrotational* perturbations of size $O_X(\varepsilon)$ of a constant equilibrium lead to global solutions which remain globally smooth and scatter. Besides, the solution obeys some mild decay

$$\|(n - n_0, v, E, B)\|_{L^\infty} \lesssim (1 + |t|)^{-1+\delta}, \qquad \delta > 0.$$

Irrotationality and neutrality are conditions on the initial data :

$$\mathrm{div}(E) = 4\pi e(n_0 - n), \quad \mathrm{curl}(v) = \frac{e}{m_e c} b.$$

They are transported by the nonlinear flow.

After nondimensionalization the Euler–Maxwell system can be rewritten as a quasilinear dispersive system which only depends on the parameter

$$0 < d = \frac{Tn_0}{m_e c^2} = \frac{V_e^2}{c^2} \leq 1.$$

More precisely, after diagonalization, we get

$$(\partial_t + i\Lambda_e)\, U_e = \mathcal{N}_e,$$
$$(\partial_t + i\Lambda_b)\, U_b = \mathcal{N}_b,$$

where

$$\Lambda_e := \sqrt{1 - d\Delta}, \qquad \Lambda_b := \sqrt{1 - \Delta},$$

$$\mathcal{N}_e = (1/2)|\nabla|\big(u_1^2 + u_2^2\big) - i\Lambda_e\big(R_1(\rho u_1) + R_2(\rho u_2)\big),$$
$$\mathcal{N}_b = -i\big(R_1(\rho u_2) - R_2(\rho u_1)\big),$$

and

$$\rho = |\nabla|\Lambda_e^{-1}\Im U_e,$$
$$u_j = -R_j\Re(U_e) + \in_{jk} R_k\Lambda_b^{-1}\Re(U_b).$$

This is a quasilinear time-reversible system, with no dissipation and no relaxation effects.

Main theorem (quantitative). Assume that $d \in (0,1)$, let $N_0 := 10^{15}$, and assume that

$$\|(U_e^0, U_b^0)\|_{H^{N_0}} + \|(U_e^0, U_b^0)\|_Z = \varepsilon_0 \leq \overline{\varepsilon}.$$

Then there exists a unique global solution $(U_e, U_b) \in C([0,\infty) : H^{N_0})$ of the system with initial data $(U_e(0), U_b(0)) = (U_e^0, U_b^0)$. Moreover, for any $t \in [0,\infty)$

$$\|(U_e(t), U_b(t))\|_{H^{N_0}} + \sup_{|\alpha|\leq 4} (1+t)^{199/200} \|D_x^\alpha (U_e(t), U_b(t))\|_{L^\infty} \lesssim \varepsilon_0.$$

Main difficulties :

(1) Less than $1/t$ pointwise decay of the solutions. The optimal linear $1/t$ decay cannot be propagated by the nonlinear flow, even in simpler semilinear evolutions, as it was pointed out by Bernicot–Germain who found a logarithmic loss.

(2) Large set of time resonances. One can overcome sometimes the slow pointwise decay using the method of normal forms of Shatah. The main ingredient : absence of time resonances (or at least a suitable "null structure" of the quadratic part of the nonlinearity matching the set of time resonances). Our system has a full (codimension 1) set of time resonances, and no meaningful null structures.

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Main bootstrap proposition : Suppose (U_e, U_b) is a solution on some time interval $[0, T]$, $T \geq 1$, with initial data (U_e^0, U_b^0), and define $V_\sigma = e^{it\Lambda_\sigma} U_\sigma$, $\sigma \in \{e, b\}$. Assume that

$$\|(U_e^0, U_b^0)\|_{H^{N_0}} + \|(V_e^0, V_b^0)\|_Z \leq \epsilon_0 \ll 1$$

and

$$\|(U_e(t), U_b(t))\|_{H^{N_0}} + \|(V_e(t), V_b(t))\|_Z \leq \epsilon_1 \ll 1$$

for all $t \in [0, T]$. Then

$$\|(U_e(t), U_b(t))\|_{H^{N_0}} + \|(V_e(t), V_b(t))\|_Z \lesssim \varepsilon_0 + \epsilon_1^{3/2}$$

for any $t \in [0, T]$.

We use a "quasilinear I-method", in the spirit of the semilinear I-method of Colliander–Keel–Stafillani–Takaoka–Tao.

The energy estimate :

In our problem it is not hard to construct high order energy functionals E_N, which are of the same size as $\|(U_e, U_b)\|^2_{H^N}$ at least as long as solutions are "small", and which satisfy energy identities of the form

$$\partial_t E_N(t) = \text{ Semilinear cubic terms.}$$

The semilinear cubic terms are sums of nonsingular cubic paraproducts of the form

$$\text{Semilinear cubic terms } \approx \langle D\rangle^N U * \langle D\rangle^N U * \langle D\rangle^2 U,$$

which do not lose derivatives. Such basic energy estimates can be used to develop the local regularity theory of the equation. They can also be used, sometimes, as a step to proving global regularity, provided that one can also prove at least $1/t$ pointwise decay of solutions.

Shatah normal form method : "transform" the problem into a nonlinear problem with a cubic nonlinearity, in such a way that one could prove quartic energy estimates of the form

$$\partial_t E_N(t) = \text{ Semilinear quartic terms,}$$

for a suitable energy functional E_N.

Potential loss of derivatives in quasilinear problems : carefully constructed nonlinear changes of variables (as in Wu), or the "iterated energy method" of Germain–Masmoudi, or the "paradifferential normal form method" of Alazard–Delort, or the "modified energy method" of Hunter–Ifrim–Tataru.

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Regardless of the method used, the critical underlying ingredient needed to achieve a quartic energy identity is the effective absence of quadratic time resonances.

Construction of improved energy functionals : Our starting point is to construct improved high order energy functionals K_N, which are of the same size as $\|(U_e, U_b)\|^2_{H^N}$ as long as the solutions are "small" in suitable low-regularity norms, and which satisfy *improved energy identities* of the form

$$\partial_t K_N(t) = \text{Strongly semilinear cubic terms} \\ + \text{Semilinear quartic terms}.$$

In our case,

$$\text{Semilinear quartic terms} \approx \langle D\rangle^N U * \langle D\rangle^N U * \langle D\rangle^2 U * \langle D\rangle^2 U.$$

and

$$\text{Strongly semilinear cubic terms} \approx \langle D\rangle^N U * \langle D\rangle^{N-1} U * \langle D\rangle^3 U.$$

The main gain, compared with the standard semilinear cubic terms, is the gain of one derivative in one of the high order terms. This derivative transfer is the key algebraic ingredient in the problem.

The semilinear quartic terms can be estimated elliptically, using simple $L^2 * L^2 * L^\infty * L^\infty$ bounds on every time slice on the four components. On the other hand, to estimate the strongly semilinear cubic terms we decompose the (ξ, t) space and consider two cases : if

$$|\xi| \gtrsim |t|^\beta,$$

where ξ is the frequency of the high order derivatives and β is a suitable exponent, then we are in the *quasilinear case*. We estimate the resulting space-time integrals using again simple $L^2 * L^2 * L^\infty$ bounds.

On the other hand, if

$$|\xi| \lesssim |t|^\beta,$$

then we are in the *semilinear case*. In this case we use the Fourier transform method : our main ingredients are a critical L^2 bound and a bootstrap argument based on a suitable choice of a norm.

The Duhamel formula : The system can be rewritten as

$$(\partial_t + i\Lambda_\sigma)U_\sigma = \sum_{\mu,\nu\in\mathcal{P}} \mathcal{N}_{\sigma\mu\nu}(U_\mu, U_\nu)$$

for all $\sigma \in \{e, b\}$, where the nonlinearities are defined by

$$(\mathcal{F}\mathcal{N}_{\sigma\mu\nu}(f,g))(\xi) = \int_{\mathbb{R}^2} \mathfrak{m}_{\sigma\mu\nu}(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta.$$

The multipliers $\mathfrak{m}_{\sigma\mu\nu}$ satisfy suitable symbol-type estimates. We define V_σ by

$$V_\sigma(t) = e^{it\Lambda_\sigma}U_\sigma(t).$$

The Duhamel formula :

$$\widehat{V_\sigma}(t,\xi) = \widehat{V_\sigma}(0,\xi) + \sum_{\mu,\nu\in\mathcal{P}} \int_0^t e^{is\Phi_{\sigma\mu\nu}(\xi,\eta)}\mathfrak{m}_{\sigma\mu\nu}(\xi,\eta)\widehat{V_\mu}(s,\xi-\eta)\widehat{V_\nu}(s,\eta)\,d\eta ds.$$

The phase functions are

$$\Phi_{\sigma\mu\nu}(\xi,\eta) = \Lambda_\sigma(\xi) - \Lambda_\mu(\xi-\eta) - \Lambda_\nu(\eta),$$

where

$$\Lambda_{\pm e}(\xi) = \pm\sqrt{1+d|\xi|^2}, \qquad \Lambda_{\pm b}(\xi) = \pm\sqrt{1+|\xi|^2}.$$

Main L^2 lemma : Assume that T is given by

$$Tf(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} a(\xi,\eta)\varphi(2^\nu\Phi(\xi,\eta))f(\eta)d\eta,$$

with $2^m - 1 \le |s| \le 2^{m+1}$, $m \in \mathbb{Z}_+$, and

$$2^{(1+\delta_1)\nu} \approx 2^m, \qquad \delta_1 > 0.$$

The function a is supported in the ball $\xi,\eta \lesssim 1$ and satisfies

$$\sup_{\xi,\eta\in\mathbb{R}^2} \left|D^\alpha_{\xi,\eta}a(\xi,\eta)\right| \lesssim_\alpha 2^{|\alpha|m/2}.$$

Then

$$\left\|Tf\right\|_{L^2} \lesssim_{\delta_1} 2^{-1.005\nu}\|f\|_{L^2}.$$

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$$\left\| Tf \right\|_{L^2} \lesssim_{\delta_1} 2^{-1.005\nu} \|f\|_{L^2}.$$

Main idea of proof : Let

$$R := 2^{-\nu/16}$$

and decompose at scale $\approx R$,

$$T = \sum_{(i,j)\in\mathbb{Z}^2\times\mathbb{Z}^2} T_{ij},$$

$$T_{ij}f(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} a(\xi,\eta)\varphi(2^\nu\Phi(\xi,\eta))f(\eta)\chi_i(\xi)\chi_j(\eta)d\eta,$$

The of the operator T is based on the size of the smooth function $\Upsilon : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$,

$$\Upsilon(\xi,\eta) = \nabla^2_{\xi,\eta}\Phi(\xi,\eta)\left[\nabla^\perp_\xi\Phi(\xi,\eta), \nabla^\perp_\eta\Phi(\xi,\eta)\right].$$

Let

$$V^1 := \{(i,j) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : \Upsilon(v^i, v^j) < D_2 R\},$$
$$V^2 := \{(i,j) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : \Upsilon(v^i, v^j) \geq D_2 R\}.$$

Lemma 1 : With the definitions above, for any $(i,j) \in V^2$,

$$\| T_{ij} \|_{L^2 \to L^2} \lesssim 2^{-5\nu/4} R^{-3/2}.$$

The proof uses a TT^* argument and Schur's lemma.

Lemma 2 : With the definitions above,

$$\Big\| \sum_{(i,j)\in V^1} T_{ij} \Big\|_{L^2 \to L^2} \lesssim 2^{-\nu} R^{1/25}.$$

The proof uses Schur's lemma and the following set bounds : if

$$E = \{(\xi,\eta) : \max(|\xi|,|\eta|) \lesssim 1, |\Phi(\xi,\eta)| \le \epsilon\}$$

then

$$\sup_\xi \int_{\mathbb{R}^2} \mathbf{1}_E(\xi,\eta)\, d\eta + \sup_\eta \int_{\mathbb{R}^2} \mathbf{1}_E(\xi,\eta)\, d\xi \lesssim \epsilon \log(1/\epsilon).$$

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If $\epsilon \leq \epsilon' \leq 1/2$, and

$$E' = \{(\xi, \eta) : |\Phi(\xi, \eta)| \leq \epsilon, |\Upsilon(\xi, \eta)| \leq \epsilon'\},$$

then we can write $E' = E'_1 \cup E'_2$ such that

$$\sup_{\xi} \int_{\mathbb{R}^2} \mathbf{1}_{E'_1}(\xi, \eta)\, d\eta + \sup_{\eta} \int_{\mathbb{R}^2} \mathbf{1}_{E'_2}(\xi, \eta)\, d\xi \lesssim \epsilon \log(1/\varepsilon) \cdot (\epsilon')^{1/12}.$$

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This L^2 lemma and the improved energy identity allow us close the energy argument to control the increment of high order Sobolev norms.

The L^2 lemma can also be used to control the growth of the weighted norm

$$\|u\|_{Z_2} := \sup_{b\in[0,N_1]} \|\Omega^b u\|_{L^2}$$

where $\Omega = x_1\partial_2 - x_2\partial_1$ is the rotation vector-field. Note : combining the quasilinear I-method and the vector-field method could be challenging, leading to the notion of "compatible vector-field structures".

The main remaining issue is to prove $t^{-1+\delta}$ decay. We use the Duhamel formula

$$\widehat{V_\sigma}(t,\xi) = \widehat{V_\sigma}(0,\xi) + \sum_{\mu,\nu\in\mathcal{P}} \int_0^t e^{is\Phi_{\sigma\mu\nu}(\xi,\eta)} \mathfrak{m}_{\sigma\mu\nu}(\xi,\eta) \widehat{V_\mu}(s,\xi-\eta)\widehat{V_\nu}(s,\eta)\, d\eta ds.$$

We need a space Z_1 such that

$$T : Z_1 \cap Z_2 \cap H^{N_0} \times Z_1 \cap Z_2 \cap H^{N_0} \to Z_1.$$

and

$$\|e^{it\Lambda_\sigma} f\|_{L^\infty} \lesssim (1+t)^{-1+\delta} \|f\|_{Z_1}.$$

Critical points (spacetime resonances) :

$$\{(\xi,\eta) : \Phi_{\sigma\mu\nu}(\xi,\eta) = 0 \text{ and } \nabla_\eta \Phi_{\sigma\mu\nu}(\xi,\eta) = 0\}.$$

In our case

$$(\xi,\eta) = (r\omega, R\omega),$$

where $\omega \in \mathbb{S}^2$.

We need to study the contributions of sub-level sets

$$\{(\xi,\eta) : |\Phi_{\sigma\mu\nu}(\xi,\eta)| \le \delta_1 \text{ and } |\nabla_\eta \Phi_{\sigma\mu\nu}(\xi,\eta)| \le \delta_2\}.$$

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We define

$$Q_{jk}f := \varphi_j(x)P_k f(x).$$

For $\sigma \in \{e, b\}$ we define

$$Z_1^\sigma := \{f \in L^2(\mathbb{R}^3) : \|f\|_{Z_1^\sigma} := \sup_{(k,j)\in\mathcal{J}} 2^{10k_+}\|Q_{jk}f\|_{B_j^\sigma} < \infty\},$$

where

$$\|g\|_{B_j^\sigma} := 2^{(1-10\delta)j}\|A^\sigma_{\leq\overline{D}}g\|_{L^2} + 2^{(1-10\delta)j}\sup_{\overline{D}<n<j} 2^{-(1/2-9\delta)n}\|A^\sigma_n g\|_{L^2} + 2^{(1/2-\delta)j}\|A^\sigma_{\geq j}g\|_{L^2}.$$

The operators A^σ are projection operators relative to the location of the spheres of space-time resonances.

Perspective : the goal is to implement Fourier analysis methods in the study of global solutions of quasilinear equations and systems.

P. Germain, N. Masmoudi, and J. Shatah, Global solutions for the gravity water waves equation in dimension 3, Ann. of Math. (2) 175 (2012), 691–754.

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