The derivation of 1D focusing NLS from 3D quantum many-body evolution

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September 19, 2014

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We will work in 3 spatial dimensions, with N particles.

$$\mathbf{r}_N = (r_1, \ldots, r_N) \in \mathbb{R}^{3N}$$

An N-body wave function $\psi_N(t, \mathbf{r}_N)$ describes a system of N bosons if it is symmetric:

$$\forall\, \sigma \in S_N, \qquad \psi_N(t, r_1, \ldots, r_N) = \psi_N(t, r_{\sigma(1)}, \ldots, r_{\sigma(N)})$$

We normalize so $\|\psi_N\|_{L^2(\mathbb{R}^{3N})} = 1$.

We will introduce a suitable N-particle Hamiltonian H_N and consider the linear N-body Schrödinger evolution

$$i\partial_t \psi_N = H_N \psi_N$$

Bose-Einstein condensation (BEC) means that the wave function is approximately for large N a tensor product

$$\psi_N(t, \mathbf{r}_N) \sim \prod_{j=1}^{N} \phi(t, r_j)$$

for some one-particle wave function $\phi(t) \in L^2(\mathbb{R}^3)$.

As $N \to \infty$, the function space $L^2(\mathbb{R}^{3N})$ is changing, so in what sense do we require convergence as $N \to \infty$?

A pure quantum state described by $\psi_N \in L^2(\mathbb{R}^{3N})$ is alternatively described by a rank 1 orthogonal projection $L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$.

$$\begin{array}{lcl} \psi_N \in L^2(\mathbb{R}^{3N}) & & \gamma_N \in \mathcal{L}(L^2(\mathbb{R}^{3N}); L^2(\mathbb{R}^{3N})) \\ \|\psi_N\|_{L^2(\mathbb{R}^{3N})} = 1 & \Leftrightarrow & \gamma_N = \text{orth proj onto } \psi_N \end{array}$$

Schrödinger equation converts to von-Neumann equation

$$i\partial_t \psi_N = H_N \psi_N \quad \Leftrightarrow \quad i\partial_t \gamma_N = [H_N, \gamma_N]$$

At the level of kernels:

$$\gamma_N(t, \mathbf{r}_N; \mathbf{r}'_N) = \psi_N(t, \mathbf{r}_N)\overline{\psi_N}(t, \mathbf{r}'_N)$$

The k-particle marginal density ($k \leq N$) is

$$\gamma_N^{(k)} = \text{trace of } \gamma_N \text{ over last } (N-k) \text{ coords}$$

$\gamma_N^{(k)} \in \mathcal{L}(L^2(\mathbb{R}^{3k}); L^2(\mathbb{R}^{3k}))$ is no longer necessarily a pure state, and could be a more general operator with $\operatorname{Tr} \gamma_N^{(k)} = 1$ representing a mixed state.

It is customary to decompose

$$\mathbf{r}_N = (\mathbf{r}_k, \mathbf{r}_{N-k})$$

At the level of kernels, the k-particle marginal is

$$\begin{aligned}
\gamma_N^{(k)}(t, \mathbf{r}_k; \mathbf{r}'_k) &= \int_{\mathbf{r_{N-k}}} \gamma_N(t, \mathbf{r}_k, \mathbf{r}_{N-k}; \mathbf{r}'_k, \mathbf{r}_{N-k})\, d\mathbf{r}_{N-k} \\
&= \int_{\mathbf{r_{N-k}}} \psi_N(t, \mathbf{r}_k, \mathbf{r}_{N-k})\overline{\psi_N(\mathbf{r}'_k, \mathbf{r}_{N-k})}\, d\mathbf{r}_{N-k}
\end{aligned}$$

We said BEC is, informally, for large N,

$$\psi_N(t, \mathbf{r}_N) \sim \prod_{j=1}^{N} \phi(t, r_j)$$

for some one particle wave function $\phi(t) \in L^2(\mathbb{R}^3)$

Converted to a statement about k-particle marginal densities:

$$\forall k \leq N, \quad \gamma_N^{(k)}(t, \mathbf{r}_k, \mathbf{r}'_k) \sim \prod_{j=1}^{k} \phi(t, r_j)\overline{\phi}(t, r'_j)$$

A precise definition of BEC:

$$\forall\ k, \quad \lim_{N\to\infty} \left\| \gamma_N^{(k)}(t, \mathbf{r}_k, \mathbf{r}'_k) - \prod_{j=1}^{k} \phi(t, r_j)\overline{\phi}(t, r'_j) \right\|_{\mathrm{Tr}} = 0$$

Equivalently, BEC means that

$$\forall\, k\,, \qquad \gamma_\infty^{(k)} \stackrel{\text{def}}{=} \lim_{N\to\infty} \gamma_N^{(k)} \text{ is a pure state}$$

where limit is taken in trace norm.

The problem is to prove this holds and show that ϕ evolves according to the nonlinear Schrödinger equation (NLS), which we call the mean-field limit equation.

Next, we consider the form of the Hamiltonian H_N. We need to decompose

$$r_j = (x_j, z_j)\,, \qquad x_j \in \mathbb{R}^2\,,\ z_j \in \mathbb{R}$$

$x \in \mathbb{R}^2$ is the transverse direction and $z \in \mathbb{R}$ is the longitudinal direction.

$$H_N = \sum_{j=1}^{N}(-\Delta_{r_j} + \omega^2|x_j|^2 + \omega_z^2 z_j^2) + \sum_{1\le i<j\le N} \frac{1}{a^{3\beta-1}} V\left(\frac{r_i - r_j}{a^\beta}\right)$$

$V : \mathbb{R}^3 \to \mathbb{R}$ is the interatomic interaction potential, whose long range $|r_i - r_j| \gg a^\beta$ effect as observed by low energy (energy $O(1)$) particles is expressed by the scattering length.

$0 < \beta \le 1$

For $0 < \beta \leq 1$, and $a \ll 1$.

$$b = \mathrm{scat}\left(\frac{1}{a^{3\beta-1}} V\left(\frac{r}{a^\beta}\right)\right) \sim \begin{cases} \frac{a}{8\pi}\int_{\mathbb{R}^3} V & \text{if } 0 < \beta < 1 \\ a\,\mathrm{scat}(V) & \text{if } \beta = 1 \end{cases}$$

The scattering length can be positive or negative.

The 3D to 3D problem means keep $\omega = 1$, $\omega_z = 1$ fixed, send $N \to \infty$ with

$$a = \frac{1}{N}$$

Each particle x_i interacts with N other particles x_j, $i \neq j$ and the strength of each interaction is $\sim N^{-1}$.

3D NLS becomes the mean-field limit equation

$$i\partial_t \phi + \Delta_r \phi - b|\phi|^2\phi = 0, \qquad b = \begin{cases} \int_{\mathbb{R}^3} V & \text{if } 0 < \beta < 1 \\ 8\pi\,\mathrm{scat}(V) & \text{if } \beta = 1 \end{cases}$$

Results available for $b \geq 0$ (repulsive interaction, defocusing NLS)

The 3D to 1D problem means keep $\omega_z = 1$, but send $\omega \to \infty$, $N \to \infty$ simultaneously.

In the time-independent case, five different regimes have been considered

Chapter 8 of Lieb, Seiringer, Solovej, Yngvason, The mathematics of the Bose gas and its condensation.

We consider their "region 2"

$$a = \frac{1}{N\omega}$$

in

$$H_{N,\omega} = \sum_{j=1}^{N} (-\Delta_{r_j} + \omega^2 |x_j|^2 + \omega_z^2 z_j^2) + \sum_{1 \le i < j \le N} \frac{1}{a^{3\beta-1}} V\left(\frac{r_i - r_j}{a^\beta}\right)$$

In this case, the mean-field limit becomes, for large N, ω,

$$(*) \qquad \gamma^{(k)}_{N,\omega}(t, \mathbf{r}_k; \mathbf{r}'_k) \approx \prod_{j=1}^{k} \phi(t, r_j)\overline{\phi(t, r_j)}$$

where

$$\phi(t, r) = \sqrt{\omega} h(\sqrt{\omega}\, x)\varphi(t, z)\,, \qquad h(x) = \pi^{-1/2} e^{-|x|^2/2}$$

1D NLS becomes the mean-field limit equation:

$$i\partial_t \varphi + \partial_z^2 \varphi - b|\varphi|^2 \varphi = 0\,, \qquad b = \begin{cases} \int_{\mathbb{R}^3} V \int_{\mathbb{R}^2} |h|^4 & 0 < \beta < 1 \\ 8\pi \mathrm{scat}(V) \int_{\mathbb{R}^2} |h|^4 & \beta = 1 \end{cases}$$

By (*), we mean precisely $\forall\ t, \quad \forall\ k$

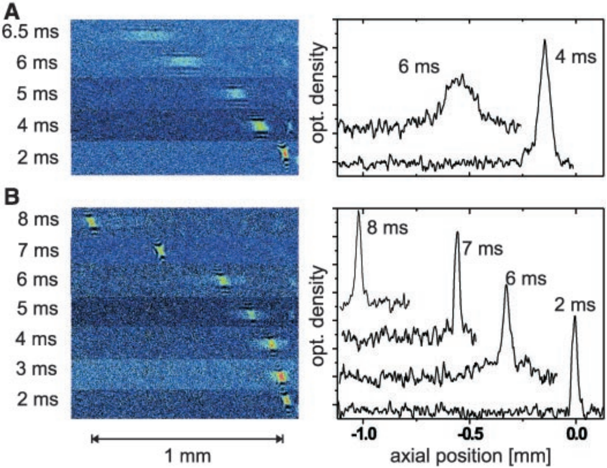
$$\lim_{(N,\omega)\to\infty} \left\| \frac{1}{\omega^k} \gamma^{(k)}_{N,\omega} \left(t, \frac{\mathbf{x}_k}{\sqrt{\omega}}, \mathbf{z_k}; \frac{\mathbf{x}'_k}{\sqrt{\omega}}, \mathbf{z}'_\mathbf{k} \right) - \prod_{j=1}^{k} h(x_j)\varphi(t, z_j)\overline{h(x'_j)\varphi(t, z'_j)} \right\|_{\mathrm{Tr}} = 0$$

The main new features of our work

- We are able to handle the 3D to 1D dimensional reduction in the BBGKY framework, where an $\infty - \infty$ cancelation is needed that does not occur in the 3D to 3D case.
- We are able to handle $b \leq 0$, attractive interactions, leading the focusing NLS. This is the only context in which focusing NLS has been derived as a mean-field limit.
- We claim that our assumptions correspond to the setting of successful physics experiments

Salomon et. al. (ENS) Formation of bright matter wave solitons, Science (2002), experiments in ^{7}Li condensates

Fig. 3. Absorption images at variable delays after switching off the vertical trapping beam. Propagation of an ideal BEC gas (**A**) and of a soliton (**B**) in the horizontal 1D waveguide in the presence of an expulsive potential. Propagation without dispersion over 1.1 mm is a clear signature of a soliton. Corresponding axial profiles are integrated over the vertical direction.



Strecker et. al. (Rice) Formation and propagation of matter wave soliton trains, Nature (2002), experiments in ^{7}Li condensates

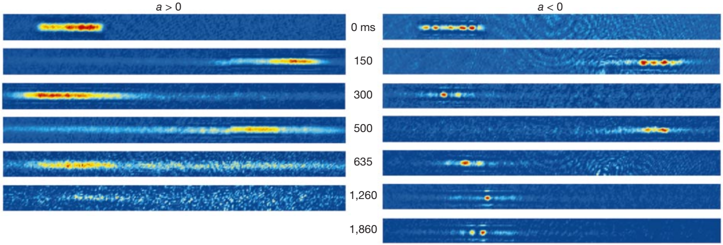


Figure 3 Comparison of the propagation of repulsive condensates with atomic solitons. The images are obtained using destructive absorption imaging, with a probe laser detuned 27 MHz from resonance. The magnetic field is reduced to the desired value before switching off the end caps (see text). The times given are the intervals between turning off the end caps and probing (the end caps are on for the $t = 0$ images). The axial dimension of each image frame corresponds to 1.28 mm at the plane of the atoms. The amplitude of oscillation is $\sim$370 μm and the period is 310 ms. The $a > 0$ data correspond to 630 G, for which $a \approx 10a_0$, and the initial condensate number is $\sim 3 \times 10^5$. The $a < 0$ data correspond to 547 G, for which $a \approx -3a_0$. The largest soliton signals correspond to $\sim$5,000 atoms per soliton, although significant image distortion limits the precision of number measurement. The spatial resolution of $\sim$10 μm is significantly greater than the expected transverse dimension $l_r \approx 1.5$ μm.

The typical experiment (taking $\omega_z = 0$ for convenience)

Step A. Confine bosons, initially repelling, inside a trap, with Hamiltonian

$$H_{N,\omega_0,0} = \sum_{j=1}^{N}(-\Delta_{r_j} + \omega_0^2|x_j|^2) + \sum_{1\leq i<j\leq N}(N\omega_0)^{3\beta-1}V_0((N\omega_0)^{\beta}(r_i - r_j))$$

where

$$V_0(r) \geq 0 \qquad \text{a repulsive pair interaction}$$

Reduce the temperature so that the bosons settle into the ground state $\psi_{N,\omega_0,0}$ for $H_{N,\omega_0,0}$.

Mathematical Problem 1. Show that the ground state $\psi_{N,\omega_0,0}$ exhibits BEC as $N, \omega_0 \to \infty$.

Recall this means $\forall\ k$

$$\lim_{N,\omega_0\to\infty} \left\| \frac{1}{\omega_0^k} \gamma^{(k)}_{N,\omega_0,0}\left(\frac{\mathbf{x_j}}{\sqrt{\omega_0}}, \mathbf{z_j}; \frac{\mathbf{x'_j}}{\sqrt{\omega_0}}, \mathbf{z'_j}\right) - \prod_{j=1}^{k} h(x_j)\varphi_0(z_j)\overline{h(x'_j)\varphi_0(z'_j)} \right\|_{\mathrm{Tr}} = 0$$

This has been addressed by other authors, and the field is summarized in the book by Lieb, Seiringer, Solovej, and Yngvason, The mathematics of the Bose Gas and Its Condensation (2005).

Recall the Hamiltonian from Step A:

$$H_{N,\omega_0,0} = \sum_{j=1}^{N}(-\Delta_{r_j} + \omega_0^2|x_j|^2) \qquad V_0(r) \geq 0$$
$$+ \sum_{1\leq i<j\leq N} (N\omega_0)^{3\beta-1} V_0((N\omega_0)^{\beta}(r_i - r_j))$$

Step B. Stengthen the trap (increase ω_0 to ω), which turns the interaction from repulsive $V_0 \geq 0$ to attractive $V \leq 0$, by a mechanism called Feshbach resonance. Assume that this is done quickly enough so that the wave function $\psi_{N,0}$ remains unchanged, but the Hamiltonian is now

$$H_{N,\omega} = \sum_{j=1}^{N}(-\Delta_{r_j} + \omega^2|x_j|^2) \qquad V(r) \leq 0$$
$$+ \sum_{1\leq i<j\leq N} (N\omega)^{3\beta-1} V((N\omega)^{\beta}(r_i - r_j))$$

Step C. Taking time $t = 0$, describe the subsequent evolution

$$i\partial_t \psi_{N,\omega} = H_{N,\omega}\psi_{N,\omega} \qquad (*)$$

Since $H_{N,\omega} \neq H_{N,\omega_0,0}$, the wave function $\psi_{N,\omega_0,0}$ is no longer the ground state, but it is asymptotically factorized (BEC).

Mathematical Problem 2. Show if $\psi_{N,\omega_0,0}$ exhibits BEC, i.e. $\forall\ k$,

$$\lim_{N,\omega\to\infty}\left\|\frac{1}{\omega^k}\gamma^{(k)}_{N,\omega_0,0}\Big(\frac{\mathbf{x}_k}{\sqrt{\omega}},\mathbf{z}_k;\frac{\mathbf{x}'_k}{\sqrt{\omega}},\mathbf{z}'_k\Big)-\prod_{j=1}^{k}h(x_j)\varphi_0(z_j)\overline{h(x'_j)\varphi_0(z'_j)}\right\|_{\mathrm{Tr}}=0$$

then the solution $\psi_N(t)$ to (*) also exhibits BEC, i.e. $\forall\ t,\quad \forall\ k$,

$$\lim_{N,\omega\to\infty}\left\|\frac{1}{\omega^k}\gamma^{(k)}_{N,\omega}\Big(t,\frac{\mathbf{x}_k}{\sqrt{\omega}},\mathbf{z}_k;\frac{\mathbf{x}'_k}{\sqrt{\omega}},\mathbf{z}'_k\Big)-\prod_{j=1}^{k}h(x_j)\varphi(t,z_j)\overline{h(x'_j)\varphi(t,z'_j)}\right\|_{\mathrm{Tr}}=0$$

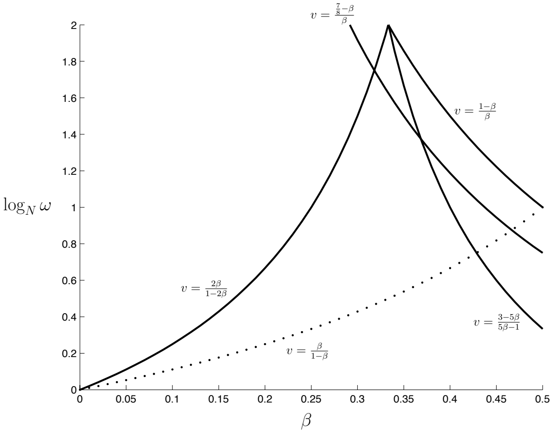
and moreover φ evolves according to the 1D focusing NLS

$$i\partial_t\varphi+\Delta\varphi-b|\varphi|^2\varphi=0$$

Theorem [X. Chen, Holmer, 2014]. Problem 2 is solved for attractive interatomic interactions $\int_{\mathbb{R}^3} V < 0$ leading to the 1D focusing NLS as a mean-field limiting equation when the $(N, \omega) \to \infty$ limit is taken under the constraints

$$C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}\,, \qquad 0 < \beta < \frac{3}{7}$$

for certain functions $v_1(\beta)$, $v_2(\beta)$.



$v_1(\beta)$ = dotted line

$v_2(\beta)$ = min of solid lines

For $V \geq 0$:

- 1D, Adami-Golse-Teta (2007)
- 3D, Elgart-Erdös-Schlein-Yau (2006-2010) Energy estimates, weak-* convergence of BBGKY to GP, uniqueness of GP by Feynman graph combinatorics
- 3D, Klainerman-Machedon (2008) uniqueness of GP via "board game", but under *a priori* space-time bound
- 1D,2D,3D, more on KM estimates and the needed *a priori* space-time bound (2008-2014): Kirkpatrick-Staffilani-Schlein, Gressman-Sohinger-Staffilani, T.Chen-Pavlovic, X.Chen, X.Chen-Holmer, Hong-Taliaferro-Xie, Tzirakis-T.Chen-Pavlovic
- 3D, T.Chen-Hainzl-Pavlovic-Seiringer (2013), new proof of uniqueness for GP using the quantum de Finetti theorem.
- 3D, Sohinger-Staffilani, Sohinger (2013-2014) randomized GP
- 3D, Fock space method (2010-2014), Grillakis-Machedon-Margetis, X.Chen, Benedikter-Oliveira-Schlein

The above results all assume $V \geq 0$ if the BBGKY$\to$GP derivation is considered. Also, the problems considered are 3D$\to$3D, 2D$\to$2D.

Our (X.Chen, Holmer) angle in the field has been to consider

- dimensional reduction in the mean-field limit
  3D$\to$2D defocusing, (2012)
- allow for attractive interactions and focusing NLS limit
  1D$\to$1D focusing (2013)

Our current paper 3D$\to$1D focusing (2014) combines the two.

Other different but related problems in which attractive interactions have been permitted is:

- Hartree problem ($\beta = 0$) with V of either sign, Erdös-Yau (2001), Michelangeli-Schlein (2010)
- 2D$\to$2D stationary problem, $\beta > 0$, $V < 0$ in region of stability, Lewin-Nam-Rougerie (2014)

Other different but related problems in which dimensional reduction has been considered:

- $(n+d)$D NLS $\to$ dD NLS, Abdullah-Méhats-Schmeiser-Weishaupl (2005). This corresponds to sending $N \to \infty$ first, then $\omega \to \infty$.
- Hani-Thomann (2014)

Cornell-Weiman (2000) did experiments with ^{85}Rb condensates without anisotropic confining. Once interaction tuned attractive the 3D condensate blows-up in a manner not described by NLS.

So the experiments by Strecker (2002) and Salomon (2002) employing strong anistropic confining are perhaps best mathematically modeled by sending $N, \omega \to \infty$ simultaneously. In the experiments, $N \sim 10^4$, $\omega \sim 10^3$.

Aspects of the proof. Let

$$h(x) = \pi^{-1/2} e^{-|x|^2/2}$$

It is the ground state:

$$(-\Delta_x + |x|^2)h = 2h$$

One key analytical component of the argument, quite different from earlier papers, is the energy estimates.

Before getting to that, let us view the overall picture:

Define the scaled density

$$\tilde{\gamma}^{(k)}_{N,\omega}(t,\mathbf{x}_k,\mathbf{z}_k;\mathbf{x}'_k,\mathbf{z}'_k) \overset{\text{def}}{=} \frac{1}{\omega^k}\gamma^{(k)}_{N,\omega}\left(t,\frac{\mathbf{x}_k}{\sqrt{\omega}},\mathbf{z}_k;\frac{\mathbf{x}'_k}{\sqrt{\omega}},\mathbf{z}'_k\right)$$

It satisfies the BBGKY hierarchy

$$\begin{aligned} i\partial_t\tilde{\gamma}^{(k)}_{N,\omega} &= \omega\sum_{j=1}^{k}[-\Delta_{x_j}+|x_j|^2-2,\tilde{\gamma}^{(k)}_{N,\omega}]+\sum_{j=1}^{k}[-\partial^2_{z_j},\tilde{\gamma}^{(k)}_{N,\omega}] \\ &+\frac{1}{N}\sum_{1\le i<j\le k}[V_{N,\omega}(r_i-r_j),\tilde{\gamma}^{(k)}_{N,\omega}] \\ &+\frac{N-k}{N}\sum_{j=1}^{k}\mathrm{Tr}_{k+1}[V_{N,\omega}(r_i-r_{k+1}),\tilde{\gamma}^{(k+1)}_{N,\omega}] \end{aligned}$$

where $V_{N,\omega}(x,z)=\frac{(N\omega)^{3\beta}}{\omega}V(\frac{(N\omega)^{\beta}x}{\sqrt{\omega}},(N\omega)^{\beta}z)$.

We seek to show that

$$(*) \qquad \tilde{\gamma}^{(k)}_{N,\omega}(t,\mathbf{x}_k,\mathbf{z}_k;\mathbf{x}'_k,\mathbf{z}'_k) \underset{(N,\omega)\to\infty}{\longrightarrow} \left(\prod_{j=1}^{k} h(x_j)h(x'_j)\right) \tilde{\gamma}_{\infty,z}(t,\mathbf{z}_k;\mathbf{z}'_k)$$

If we assume that almost of the limiting form of RHS, then

$$\omega \sum_{j=1}^{k} [-\Delta_{x_j} + |x_j|^2 - 2, \tilde{\gamma}^{(k)}_{N,\omega}]$$

is of $\infty \cdot 0$ limit form.

Our energy estimates show that

$$P_{\text{above ground}}\left(\tilde{\gamma}^{(k)}_{N,\omega}(t,\mathbf{x}_k,\mathbf{z}_k;\mathbf{x}'_k,\mathbf{z}'_k)\right) \to 0$$

sufficiently fast and implies

$$\omega \sum_{j=1}^{k} [-\Delta_{x_j} + |x_j|^2 - 2, \tilde{\gamma}^{(k)}_{N,\omega}] \to 0$$

3D BBGKY

$$i\partial_t \tilde{\gamma}^{(k)}_{N,\omega} = \underbrace{\omega \sum_{j=1}^{k} [-\Delta_{x_j} + |x_j|^2 - 2, \tilde{\gamma}^{(k)}_{N,\omega}]}_{\to 0} + \sum_{j=1}^{k} [-\partial^2_{z_j}, \tilde{\gamma}^{(k)}_{N,\omega}]$$

$$\underbrace{+ \frac{1}{N} \sum_{1 \le i < j \le k} [V_{N,\omega}(r_i - r_j), \tilde{\gamma}^{(k)}_{N,\omega}]}_{\to 0}$$

$$+ \frac{N-k}{N} \sum_{j=1}^{k} \mathrm{Tr}_{k+1} [V_{N,\omega}(r_i - r_{k+1}), \tilde{\gamma}^{(k+1)}_{N,\omega}]$$

collapses, as $(N, \omega) \to \infty$ to 1D GP

$$i\partial_t \tilde{\gamma}^{(k)}_{\infty,z} = \sum_{j=1}^{k} [-\partial^2_{z_j}, \tilde{\gamma}^{(k)}_{\infty,z}] + b \sum_{j=1}^{k} \mathrm{Tr}_{z_{k+1}} [\delta(z_j - z_{k+1}), \tilde{\gamma}^{(k+1)}_{\infty,z}]$$

$$\text{with} \quad b = \int_{\mathbb{R}^3} V \int_{\mathbb{R}^2} |h|^4$$

1D GP

$$i\partial_t \tilde{\gamma}^{(k)}_{\infty,z} = \sum_{j=1}^{k} [-\partial^2_{z_j}, \tilde{\gamma}^{(k)}_{\infty,z}] + b \sum_{j=1}^{k} \mathrm{Tr}_{z_{k+1}} [\delta(z_j - z_{k+1}), \tilde{\gamma}^{(k+1)}_{\infty,z}]$$

Take φ solving 1D NLS

$$i\partial_t \varphi + \partial_z^2 \varphi - b|\varphi|^2 \varphi = 0$$

Set

$$\tilde{\gamma}^{(k)}_{\infty,z}(t, \mathbf{z}_k; \mathbf{z}'_k) = \prod_{j=1}^{k} \varphi(t, z_j) \overline{\varphi(t, z'_j)}$$

Then $\tilde{\gamma}^{(k)}_{\infty,z}$ solves the 1D GP.

Let

$$\mathcal{K}_k = \text{ compact ops } L^2(\mathbb{R}^{3k}) \to L^2(\mathbb{R}^{3k})$$
$$\mathcal{L}^1_k = \text{ trace class ops } L^2(\mathbb{R}^{3k}) \to L^2(\mathbb{R}^{3k})$$

Then

$$(\mathcal{K}_k)^* = \mathcal{L}^1_k$$

Consider $(\mathcal{L}^1_k, \text{wk}*)$, the space $\mathcal{L}^1_k$ with the weak-star topology.

Since $\mathcal{K}_k$ is separable, $(\mathcal{L}^1_k, \text{wk}*)$ is metrizable.

Consider k-particle marginals

$$\gamma^{(k)}_{N,\omega} \in C([0,T]; (\mathcal{L}^1_k, \text{wk}*))$$

with the compact-open topology (topology of uniform in time convergence). Arzela-Ascoli characterizes compactness.

Boundedness and equicontinuity follow from the energy estimates, giving Step (A) below.

(A) Prove that for each k, the set $\{\tilde{\gamma}_{N,\omega}^{(k)}\}$ is compact in $C([0,T];(\mathcal{L}_k^1, \text{wk*}))$. Relies on energy estimates

(B) Prove that every limit point has the reduced form

$$\left(\prod_{j=1}^{k} h(x_j)h(x_j')\right) \tilde{\gamma}_{\infty,z}^{(k)}(t, \mathbf{z}_k; \mathbf{z}_k')$$

and $\tilde{\gamma}_{\infty,z}^{(k)}$ solves 1D GP. Relies on energy estimates

(C) Prove that, in the space in which all limit points lie, there is a unique solution to the GP hierarchy. A compact sequence with a unique limit point converges to that limit point. We use a 1D Klainerman-Machedon estimate that we previously proved.

(D) Upgrade convergence from wk* to strong by appealing to Grümm's convergence theorem.

Energy estimates. In the energy estimates, we must confront the focusing nonlinearity and the diverging ω.

Let

$$S_j \stackrel{\text{def}}{=} (1 - \Delta_{r_j} + \omega^2|x_j|^2 - 2\omega)^{1/2}$$

Since $-\Delta_{x_j} + \omega^2|x_j|^2 - 2\omega \geq 0$, we have $S_j \geq 0$. Notice that

$$S_j^2\left(\sqrt{\omega}h(\sqrt{\omega}x_j)\varphi(t,z_j)\right) = \sqrt{\omega}h(\sqrt{\omega}x_j)\,(1-\partial_{z_j}^2)\varphi(t,z_j)$$

so no diverging factor in ω is produced if solution is in x-ground state.

The energy estimate is: $\exists C > 0$ such that $\forall k$

$$\left\|\left(\prod_{j=1}^{k} S_j\right)\psi_{N,\omega}\right\|_{L^2(\mathbb{R}^{3N})}^2 \leq C^k\langle\psi_{N,\omega},(\alpha + N^{-1}H_{N,\omega} - 2\omega)^k\psi_{N,\omega}\rangle$$

for

$$C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}$$

and C_1, C_2, α depend on V.

The usefulness being that the RHS is constant in time.

To handle this, we use a decomposition of the energy different from earlier works:

$$\alpha + N^{-1} H_{N,\omega} - 2\omega = N^{-1} \sum_{1 \leq i < j \leq N} H_{ij}$$

where H_{ij} represents a two body interaction in r_i and r_j.

The most common terms are of the form

$$(*) \qquad (\alpha + N^{-1} H_{N,\omega} - 2\omega)^k \sim H_{i_1 j_1} H_{i_2 j_2} \cdots H_{i_k j_k}$$

where all i, j are distinct.

Need to use the spectral cluster estimate for the Hermite operator (Koch-Tataru (2005)) to

- extract positive lower bounds on each H_{ij}. [requires $\omega \leq N^{v_1(\beta)}$]
- bound cross terms $H_{12} H_{23}$ (which occur less frequently than those in (*)) [requires $N^{v_2(\beta)} \leq \omega$]

Another important ingredient is the Klainerman-Machedon board game argument and collapsing estimate. In our case, this takes the following form:

$$\|\theta(t) R_{\epsilon}^{(1)} U^{(1)}(-t) B_{1,2} U^{(2)}(t) R_{-\epsilon}^{(2)} \phi^{(2)}\|_{L_t^2 L^2_{\mathbf{x_1 x_1}'}} \leq C_{\epsilon,\theta} \|\phi^{(2)}\|_{L^2_{\mathbf{x}_2 \mathbf{x}_2'}}$$

where

$$R_{\epsilon}^{(1)} = \langle \partial_{x_1} \rangle^{\epsilon} \langle \partial_{x_1'} \rangle^{\epsilon}$$

$$R_{-\epsilon}^{(2)} = \langle \partial_{x_1} \rangle^{-\epsilon} \langle \partial_{x_1'} \rangle^{-\epsilon} \langle \partial_{x_2} \rangle^{-\epsilon} \langle \partial_{x_2'} \rangle^{-\epsilon}$$

$$U^{(1)} = e^{it\partial_{x_1}^2} e^{-it\partial_{x_1'}^2}$$

$$U^{(2)} = e^{it\partial_{x_1}^2} e^{-it\partial_{x_1'}^2} e^{it\partial_{x_2}^2} e^{-it\partial_{x_2'}^2}$$

and the collapsing operator

$$B_{1,2}\alpha^{(2)}(x_1, x_1') = \alpha^{(2)}(x_1, x_1, x_1', x_1)$$

Estimate fails if $\theta(t) \equiv 1$