On the anisotropic Calderón problem

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Outline

Prologue: solutions to the Schrödinger equation

The anisotropic Calderón problem

Carleman weights and warped products

The geodesic ray transform

Quasimodes and semiclassical measures

Solutions of the Schrödinger equation

Aim: Construct solutions of the Schrödinger equation on an open bounded set $\Omega \subset \mathbf{R}^n$

$$-\Delta u + qu = 0$$

with an L^∞ potential q on Ω.

One method: use complex geometrical optics solutions

Introduced by Sylvester and Uhlmann Construction presented in this introduction by Kenig, Sjöstrand and Uhlmann

Complex geometrical solutions

Pick x_0 not in the convex hull of $\bar{\Omega}$.
Coordinates in $\overline{\Omega}$

$$t = \log|x - x_0| \in \mathbf{R}, \quad \omega = \frac{x - x_0}{|x - x_0|} \in S^{n-1}_{>\alpha_0},$$

with respect to which the Laplace operator reads

$$\Delta = \mathrm{e}^{-\frac{n+2}{2}t}\big(\partial_t^2 + \widehat{\Delta}_{S^{n-1}}\big)\mathrm{e}^{\frac{n-2}{2}t}$$

where $\widehat{\Delta}_{S^{n-1}} = \Delta_{S^{n-1}} - (n-2)^2/4$.

Hemisphere caps

Definition

A compact manifold with boundary (M,g) is said to be simple if its boundary ∂M is stricly convex and if the exponential map $\exp_x : U_x \to M$ defined on its maximal set of definition is a diffeomorphism for all $x \in M$.

Cap strictly smaller than the northern hemisphere

$$S^{n-1}_{>\alpha_0} = \{x \in S^{n-1} : x_n > \alpha_0\}, \quad 0 < \alpha_0 < 1,$$

is a simple manifold.

Geodesic ray transform is injective on simple manifolds and there are stability estimates !

Complex geometrical optics solutions

Approximate solutions of the Schrödinger equation of the form

$$u = \mathrm{e}^{-\left(s+\frac{n-2}{2}\right)t} v_s(\omega), \quad s = \tau + i\lambda, \quad \lambda > 0$$

where v_s is a quasimode of $\widehat{\Delta}_{S^{n-1}}$

$$\big(\widehat{\Delta}_{S^{n-1}} + s^2\big) v_s = \mathcal{O}_{L^2(S^{n-1}_+)}(1).$$

With this choice, we have

$$\Delta u = \mathrm{e}^{-\left(s+\frac{n+2}{2}\right)t} \big(\widehat{\Delta}_{S^{n-1}} + s^2\big) v_s$$

and therefore u is an approximate solution of the Schrödinger equation

$$\big\| |x-x_0|^{s+\frac{n-2}{2}} (-\Delta + q) u \big\|_{L^2(\Omega)} = \mathcal{O}(1).$$

Quasimodes

Pick $y \in \partial S^{n-1}_{+}$ on the equator

$$\widehat{\Delta}_{S^{n-1}} + s^2 = (\sin\theta)^{-\frac{n-2}{2}} \left(\partial_\theta^2 + s^2 \frac{1}{\sin^2\theta} \widetilde{\Delta}_{S^{n-2}} \right) (\sin\theta)^{\frac{n-2}{2}}$$

$\theta = d_{S^{n-1}}(\theta, \omega)$ is the distance to y, i.e. the colatitude with respect to the pole y,
$\eta \in S^{n-2}$, remaining angular variables,
Quasimode on the sphere

$$v_s(\omega) = (\sin\theta)^{-\frac{n-2}{2}} \mathrm{e}^{is\theta} b(\eta), \quad \omega = \exp_y(\theta\eta)$$

b is any smooth function

Approximate solutions can be upgraded to exact solutions of the Schrödinger equation of the form

$$u = \mathrm{e}^{-\left(s + \frac{n-2}{2}\right)t} \big(v_s(\omega) + \mathcal{O}(\tau^{-1}) \big)$$

thanks to Carleman estimates.

Semiclassical measure

Choose u_1, u_2 to be the solutions

$$u_1(x) = \mathrm{e}^{-\left(s+\frac{n-2}{2}\right)t}\big(v_s(\omega) + \mathcal{O}(\tau^{-1})\big),$$
$$u_2(x) = \mathrm{e}^{-\left(-\bar{s}+\frac{n-2}{2}\right)t}\big(v_{-\bar{s}}(\omega) + \mathcal{O}(\tau^{-1})\big)$$

with $s = \tau + i\lambda$, we have

$$\int q u_1 u_2 \,\mathrm{d}x = \int_{S^{n-1}_+} \underbrace{\int_{-\infty}^{\infty} \mathrm{e}^{2i\lambda t}\mathrm{e}^{2t} q(x_0 + \mathrm{e}^t\omega)\,\mathrm{d}t}_{=Q(\lambda,\omega)} \; v_s v_{-\bar{s}}\,\mathrm{d}\omega + \mathcal{O}(\tau^{-1})$$

and passing to the limit as $\tau \to \infty$ from $\int q u_1 u_2 \,\mathrm{d}x$ we finally control

$$\int_{S^{n-2}_+} \left(\int_0^{\pi} Q(-2\lambda, \exp_y(\theta\eta))\mathrm{e}^{-2\lambda\theta}\,\mathrm{d}\theta\right) b^2(\eta)\,\mathrm{d}\eta$$

The attenuated geodesic ray transform

Varying b, $\int q u_1 u_2 \, \mathrm{d}x$ we control the attenuated geodesic ray transform

$$T_{2\lambda} Q(y, \eta) = \int_0^{\pi} Q(\lambda, \exp_y(\theta\eta)) \mathrm{e}^{-2\lambda\theta} \, \mathrm{d}\theta$$

which is injective (and for which we have stability estimates) when λ is small.

The maximal principle allows to recover large frequencies λ of $Q(\lambda, \cdot)$.

Remarks

What have we used in this construction?

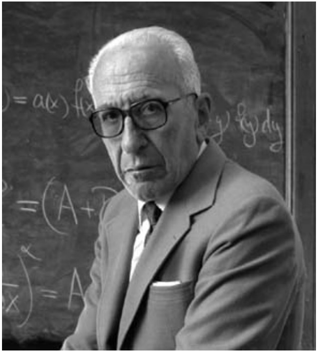
1. the warped structure of $(\Omega, \mathrm{d}x^2) \subset (\mathbf{R}^+ \times S^{n-1}_{>\alpha_0}, \mathrm{e}^{2t}(\mathrm{d}t^2 \times g_{S^{n-1}}))$,
2. the existence of Carleman estimates to upgrade approximate solutions to exact ones,
3. the explicit computation of semiclassical measures and the relation with an attenuated geodesic ray transform.

One can therefore adapt this construction to Riemannian manifolds with boundary presenting a similar structure (DSF, Kenig, Salo, Uhlmann).

Outline

The Calderón problem

On an inverse boundary value problem,
Seminar on Numerical Analysis and its Applications to Continuum Physics, Rio de Janeiro,
Editors W.H. Meyer and M.A. Raupp,
Sociedade Brasileira de Matematica (1980), 65–73.



http://en.wikipedia.org/wiki/Alberto_Calderón

In a foundational paper of 1980, A. Calderón asked the following question:
Is it possible to determine the electrical conductivity of a body by making current and voltage measurements at the boundary?

Riemannian rigidity

In fact, one can state the inverse problem with a geometric flavour.

Let (M,g) be a compact Riemannian manifold with boundary ∂M of dimension $n \geq 3$ and q a bounded measurable function. Consider the Dirichlet problem

$$\begin{cases} (\Delta_g + q)u = 0 \\ u|_{\partial M} = f \in H^{\frac{1}{2}}(\partial M) \end{cases}$$

and define the associated Dirichlet-to-Neumann map (under a natural spectral assumption)

$$\Lambda_{g,q} u = \partial_\nu u|_{\partial M}$$

where ν is a unit normal to the boundary.
If $q = 0$, we use $\Lambda_g = \Lambda_{g,0}$ as a short notation.

Riemannian rigidity

The inverse problem is whether the DN map determines the metric g.
There is a gauge invariance, that is by isometries which leave the boundary points unchanged:

$$\Lambda_{\varphi^* g} = \Lambda_g, \quad \varphi|_{\partial M} = \mathrm{Id}_{\partial M}$$

Inverse problem: Does the Dirichlet-to-Neumann map $\Lambda_{g,q}$ determine the potential q and the metric g modulo such isometries?

If $n \geq 3$ and $q = 0$ this is a generalization of the anisotropic conductivity problem and one passes from one to the other by

$$\gamma^{jk} = \sqrt{\det g}\, g^{jk}, \quad g^{jk} = (\det \gamma)^{-\frac{2}{n-2}} g^{jk}.$$

Conformal metrics

There is a conformal gauge transformation

$$\Delta_{cg} u = c^{-1}(\Delta_g + q_c)(c^{\frac{n-2}{4}} u), \quad q_c = c^{\frac{n+2}{4}} \Delta_g (c^{\frac{n-2}{4}})$$

which translates at the boundary into

$$\Lambda_{cg,q} f = c^{-\frac{n+2}{4}} \Lambda_{g,q+q_c}(c^{\frac{n-2}{4}} u) + \frac{n-2}{4} c^{-\frac{1}{2}} \partial_\nu c f.$$

So if one knows $c, \partial_\nu c$ at the boundary (boundary determination) then one can deduce one DN map from the other.

A more reasonable inverse problem: $\Lambda_{cg} = \Lambda_g \Rightarrow c = 1$.
Note that there is no isometry gauge invariance in this case.

Some references $n \geq 3$

1987 Sylvester-Uhlmann: isotropic case

1989 Lee-Uhlmann: boundary determination, analytic metrics, no potential, determination of the metric

2001 Lassas-Uhlmann: improvement on topological assumptions

2007 Kenig-Sjöstrand-Uhlmann: small subsets of the boundary, $n \geq 3$, global Carleman estimates with logarithmic weights, introduction of limiting Carleman weights.

2009 Guillarmou-Sa Baretto: Einstein manifolds, no potential, determination of the metric, unique continuation argument

2009 DSF-Kenig-Salo-Uhlmann: fixed admissible geometries, determination of a smooth potential, CGOs

2011 DSF-Kenig-Salo: fixed admissible geometries, determination of an unbounded potential, CGOs

Remarks

1. Analytic metrics case fairly well understood. The smooth case remains a challenging problem.
2. There are limitations in the method using CGO construction: the existence of limiting Carleman weights
3. We will concentrate on the case of identifiability of the metric within a conformal class

$$\Lambda_{cg} = \Lambda_g \Rightarrow c = 1.$$

4. With boundary determination

$$\Lambda_{cg} = \Lambda_g \Rightarrow c|_{\partial M} = 1, \quad \partial_\nu c|_{\partial M} = 0.$$

it is enough to solve the inverse problem on the Schrödinger equation with a fixed metric

$$\Lambda_{g,q_1} = \Lambda_{g,q_2} \Rightarrow q_1 = q_2.$$

A density property

Green's formula and the fact that $\Lambda_{g,q}^* = \Lambda_{g,\bar{q}}$ yield

$$\int_M (q_1 - q_2) u_1 \overline{u_2} \, \mathrm{d}V = \int_{\partial M} (\Lambda_{g,q_1} - \Lambda_{g,q_2}) u_1 \overline{u_2} \, \mathrm{d}S = 0$$

for all pairs (u_1, u_2) of solutions of the Schrödinger equations $(-\Delta_g + q_1)u_1 = 0$, $(-\Delta_g + \overline{q_2})u_2 = 0$. So we are reduced to the following density property

Is the linear span of products $u_1\overline{u_2}$ of solutions u_1, u_2 to two Schrödinger equations with two potentials q_1 and $\overline{q_2}$ dense in, say, $L^1(M)$?

Outline

Warped products

Let (M_0, g_0) be a compact Riemannian manifold (with boundary) and let ψ be a smooth function on $\mathbf{R}$, the warped product $(\mathbf{R}, \mathrm{d}t^2) \times_{\mathrm{e}^{2\psi}} (M_0, g_0)$ is the manifold $\mathbf{R} \times M_0$ endowed with the following metric

$$\mathrm{d}t^2 + \mathrm{e}^{2\psi(t)} g_0.$$

Our fixed metric g will be conformal to such a warped product. Two reasons for this choice:

- Warped metrics admit limiting Carleman weights.
- This is the natural setting to solve equations by separation of variables.

Unwarping warped products

An open manifold which is conformally imbedded in the warped product of the Euclidean line $\mathbf{R}$ and a Riemannian manifold (M_0, g_0) of dimension $n-1$ admits limiting Carleman weights. Indeed the metric on $\mathbf{R} \times_{\mathrm{e}^{2\psi}} M_0$ is conformal to

$$\mathrm{e}^{-2\psi(t)}\mathrm{d}t^2 + g_0$$

and one can make the change of variable

$$t' = \int_0^t \mathrm{e}^{-\psi(t'')}\,\mathrm{d}t''$$

to reduce the metric to

$$\mathrm{d}t'^2 + g_0$$

where $\varphi(t') = t'$ is a natural limiting Carleman weight.

Limiting Carleman weights

Definition

A limiting Carleman weight on an open Riemannian manifold is smooth real-valued function without critical points such that

$$\frac{1}{i}\{\overline{p_\varphi}, p_\varphi\} = 0 \quad \text{on } p_\varphi^{-1}(0)$$

where $p_\varphi(x,\xi) = |\xi|_g^2 - |\mathrm{d}\varphi|_g^2 + 2i\langle \xi, \mathrm{d}\varphi\rangle_g$ *is the semiclassical principal symbol of the conjugated operator* $\mathrm{e}^{\tau\varphi}(-\Delta_g)\mathrm{e}^{-\tau\varphi}$.

This notion was introduced by Kenig-Sjöstrand-Uhlmann.
In a product metrics $g = \mathrm{d}t^2 + g_0$, $\varphi = t$ is a limiting weight:

$$p_\varphi = (\tau + i)^2 + g_0 = \tau^2 + g_0 - 1 + 2i\tau$$

$$\frac{1}{i}\{\overline{p_\varphi}, p_\varphi\} = 4\{\tau^2 + g_0 - 1, \tau\} = \frac{\partial g_0}{\partial t} = 0.$$

Carleman estimates

Theorem

There exist two constants $C, \tau_0 > 0$ such that

$$|\operatorname{Re} s| \, \|e^{st} w\|_{L^2} + \|e^{st} dw\|_{L^2} \leq C \|e^{st}(\partial_t^2 + \Delta_{g_0} + q) w\|_{L^2}$$

for all $w \in C_0^\infty(\mathbf{R} \times M_0)$ and all $s \in \mathbf{C}$ such that $|\operatorname{Re} s| \geq \tau_0$.

By Hahn-Banach, one can construct a correction term r_s such that

$$(\partial_t^2 + \Delta_g + q)(e^{st} r_{s,q}) = e^{st} f$$
$$\|r_{s,q}\|_{L^2} \leq |\operatorname{Re} s|^{-1} \|f\|_{L^2}.$$

Remark: There exist L^p versions of Carleman estimates with limiting weights in this context (DSF, Kenig, Salo) which can be viewed as version of estimates of Jerison and Kenig and of Kenig, Ruiz and Sogge on manifolds.

Outline

The geodesic ray transform

The unit sphere bundle :

$$SM_0 = \bigcup_{x \in M_0} S_x, \qquad S_x = \big\{(x,\xi) \in T_x M_0\,;\, |\xi|_g = 1\big\}.$$

Boundary: $\partial(SM_0) = \{(x,\xi) \in SM_0\,;\, x \in \partial M_0\}$ union of inward and outward pointing vectors:

$$\partial_\pm(SM_0) = \big\{(x,\xi) \in SM_0\,;\, \pm\langle \xi, \nu\rangle \leq 0\big\}.$$

Denote by $t \mapsto \gamma(t,x,\xi)$ the unit speed geodesic starting at x in direction ξ, and let $\tau(x,\xi)$ be the time when this geodesic exits M_0.

Geodesic ray transform :

$$T_0 f(x,\xi) = \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi))\,\mathrm{d}t, \qquad (x,\xi) \in \partial_+(SM_0).$$

Simple manifolds

Definition

A compact manifold (M_0, g_0) with boundary is simple if for any $p \in M_0$ the exponential map $\exp_p$ with its maximal domain of definition is a diffeomorphism onto M_0, and if ∂M_0 is strictly convex (that is, the second fundamental form of $\partial M_0 \hookrightarrow M_0$ is positive definite).

1. Simple manifolds are non-trapping.
2. Simple manifolds are diffeomorphic to a ball.
3. A hemisphere is not simple.

Injectivity of the ray transform

Injectivity of the ray transform is known to hold for

1. Simple manifolds of any dimension.
2. Manifolds of dimension ≥ 3 that have strictly convex boundary and are globally foliated by strictly convex hypersurfaces (Uhlmann-Vasy).
3. A class of non-simple manifolds of any dimension such that there are sufficiently many geodesics without conjugate points and the metric is close to a real-analytic one (Stefanov-Uhlmann).
4. There are counterexamples to injectivity of the ray transform.

Main results in a conformal class

Theorem

Let (M, g) be a compact manifold with boundary which can be embedded in the warped product of the Euclidean line and another manifold. Let $q_1, q_2 \in C(M)$ such that 0 is not a Dirichlet eigenvalue of the corresponding Schrödinger operators. Assume in addition that the ray transform in the transversal manifold is injective. If $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, then $q_1 = q_2$.

Corollary

Let (M, g) be a compact manifold with boundary which can be embedded in the warped product of the Euclidean line and another manifold. Assume in addition that the ray transform in the transversal manifold is injective. If $\Lambda_{cg} = \Lambda_g$, then $c = 1$.

Outline

Separation of variables

We are interested in the density of the set of products of solutions of Schrödinger equations. Let us start by harmonic functions u using separation of variables

$$\Delta_g(w(t)v(x)) = (\partial_t^2 + \Delta_{g_0})(w(t)v(x)) = 0$$

and for $s \in \mathbf{C}$ take

$$(D_t^2 - s^2)w = 0, \quad (\Delta_{g_0} + s^2)v = 0$$

i.e. $u = \mathrm{e}^{st}v_s, \quad (\Delta_{g_0} + s^2)v_s = 0.$

By Hahn-Banach, one can construct a correction term r_s such that

$$\begin{aligned}(\partial_t^2 + \Delta_g + q)(\mathrm{e}^{st}r_{s,q}) &= -\mathrm{e}^{st}(\Delta_{g_0} + s^2 + q)v_s \\ \|r_{s,q}\|_{L^2} &\leq |\operatorname{Re} s|^{-1}\|(\Delta_{g_0} + s^2 + q)v_s\|_{L^2}.\end{aligned}$$

So it suffices to require $(\Delta_{g_0} + s^2)v_s = o_{L^2}(\operatorname{Re} s)$ and $v_s = O_{L^2}(1)$ to obtain a solution of the Schrödinger equation of the form

$$u = \mathrm{e}^{st}(v_s + r_{s,q}), \quad r_{s,q} = o_{L^2}(1).$$

Quasimodes

We take $h = (\mathrm{Re}\, s)^{-1}$ as a semiclassical parameter and write $s = h^{-1} + i\lambda$. We consider the family of products $u_1\overline{u_2}$ of $u_1 = \mathrm{e}^{-st}(v_s + r_{s,q_1})$ and $u_2 = \mathrm{e}^{st}(v_s + r_{s,\overline{q_2}})$ when $s \in \mathbf{C}$

$$\mathrm{e}^{-2i\lambda t}|v_s|^2 + o(1)$$

where v_s is a quasimode

$$(h^2\Delta_{g_0} + (1 + i\lambda h)^2)v_s = o_{L^2}(h).$$

The density property we are looking for will be satisfied if we can find quasimodes such that

$$\lim_{h\to 0}\int_M q\mathrm{e}^{-2i\lambda t}|v_s|^2\,\mathrm{d}t\,\mathrm{d}V_{g_0} = \lim_{h\to 0}\int_{M_0}\hat{q}(2\lambda, x)|v_s|^2\,\mathrm{d}V_{g_0} = 0$$

for $q \in C_0^0(\mathbf{R} \times M_0)$ implies $q = 0$.

Semiclassical measures

In fact, one is lead to study the measures which are limits of $|v_s|^2 \, \mathrm{d}V_{g_0}$ when $\operatorname{Re} s \to \infty$ where v_s is a quasimode.

If one lifts those measures to the cotangent bundle T^*M_0, one obtains semiclassical defect measures. These are in fact supported in the cosphere bundle S^*M_0 and satisfy a transport equation (loosely speaking, they are invariant under the cogeodesic flow when $\lambda = 0$).

In a previous work, we (Carlos Kenig, Mikko Salo, Gunther Uhlmann and myself) were able to construct quasimodes in a simple manifold which concentrate on a geodesic γ, thus obtaining

$$T_\lambda(\hat{q}(0,\cdot))(\gamma) = \int_0^L \mathrm{e}^{-2\lambda r} \hat{q}(2\lambda, \gamma(r)) \, \mathrm{d}r = 0$$

for all geodesics γ and all $\lambda \geq 0$.

Getting rid of the attenuation

Although it seems that one needs the injectivity of an attenuated geodesic ray transform to conclude $q = 0$, it suffices to deal with the case $\lambda = 0$. Indeed from

$$\int_0^{L(x,\theta)} \hat{q}(0, \exp_x(r\theta))\, \mathrm{d}r = 0$$

one gets $\hat{q}(0, \cdot) = 0$. Differentiating

$$\int_0^{L(x,\theta)} \mathrm{e}^{-2\lambda r} \hat{q}(\lambda, \exp_x(r\theta))\, \mathrm{d}r = 0$$

with respect to λ at $\lambda = 0$, one gets

$$\int_0^{L(x,\theta)} \partial_\lambda \hat{q}(0, \exp_x(r\theta))\, \mathrm{d}r = 0$$

hence $\partial_\lambda \hat{q}(0, \cdot) = 0$. Etc. Since $\hat{q}$ is analytic with respect to λ we get $q = 0$.

Quasimodes

One of the contribution of our work is the construction of such quasimodes — concentrating on a non-closed geodesic γ — in the general case. This construction can be done in two ways:

- using Gaussian beams ▸ Gauss,
- by a microlocal approach ▸ Micro.

To simplify, we will take $\lambda = 0$.

Microlocal construction

There is a canonical transformation ς which sends the cogeodeisc Γ to the line $\Lambda = \mathbf{R} \times \{0, \varepsilon_m\}$ ($\varepsilon_m = (0, \dots, 0, 1)$ with $m = \dim M_0 = n - 1$). Quantizing such a transformation by semiclassical Fourier integral operators one gets

$$V_h\Big(\sqrt{-h^2\Delta_{g_0}} - 1\Big)U_h = hD_1 + O(h^\infty)$$

Our quasimode will be

$$v_h = U_h(1 \otimes w_h(x')), \quad \mathrm{WF}_{\mathrm{sc}}(w_h) = \{(0, \varepsilon_m\},$$

this is a quasimode microlocally near Γ and also away from Γ since

$$\mathrm{WF}_{\mathrm{sc}}(v_h) \subset \Gamma.$$

The choice for w_h is a wave packet (or coherent state).

An application of Egorov's theorem

The semiclassical defect measure associated to the quasimode v_h can be computed using Egorov's theorem:

$$\int_{M_0} (\mathrm{Op}_h\, a) v_h \overline{v_h}\, \mathrm{d}V_{g_0} = \int_{\mathbf{R}^m} U_h^* (\mathrm{Op}_h\, a) U_h w_h \overline{w_h}\, \mathrm{d}x_1 \mathrm{d}x' = \int_{\mathbf{R}^m} (\mathrm{Op}_h\, \varsigma^* a) w_h \overline{w_h}\, \mathrm{d}x_1 \mathrm{d}x' + O(h)$$

and since the defect measure associated to wave packets concentrated near $(0, \varepsilon_n)$ is $\delta_{(0,\varepsilon)}$ passing to the limit we get

$$\int_\Lambda \varsigma^* a = \int_\Gamma a.$$

Hence the semiclassical measure is the measure on Γ (and its projection on M_0 the measure on the geodesic γ.

Gaussian beams

We look for quasimodes of the form $v_h = \mathrm{e}^{\frac{i}{h}\Phi} a$, we have

$$(h^2\Delta_{g_0} - 1)v_h = \mathrm{e}^{\frac{i}{h}\Phi}\big(h^0(1 - |\mathrm{d}\Phi|^2_{g_0})a + h(2L_{\nabla\Phi} + \Delta_{g_0}\varphi)a + h^2\Delta_{g_0} a\big)$$

- Eikonal equation: $|\mathrm{d}\Phi|^2_{g_0} = 1$, a solution given by $\Phi_{g_0} = d_{g_0}(x, x_0)$ if there are no conjugate points,
- Transport equation : $(L_{\nabla_{g_0}\Phi} + \frac{1}{2}\Delta_{g_0}\Phi)a = 0$, again explicit solutions in simple manifolds.

In the case of general manifolds, an alternative is to solve approximately those equations close to a geodesic and use a complex phase Φ.

Gaussian beams in Euclidean space

Our geodesic: $\gamma = \{(x_1, 0) : x_1 \in [0, T]\}$

Approximate eikonal equation: $(\mathrm{d}\Phi)^2 = 1 + \mathcal{O}(|x'|^4)$
Approximate transport equation: $L_{\nabla\Phi} a + \frac{1}{2}\Delta\Phi a = \mathcal{O}(|x'|^2)$
Phase: $\Phi(x) = x_1 + \frac{1}{2}\langle Q(x_1)x', x'\rangle$ with $\operatorname{Im} Q(x_1) > c\mathrm{I}_n$ so that

$$(\mathrm{d}\Phi)^2 - 1)\mathrm{e}^{\frac{i}{h}\Phi} a = \mathcal{O}(|x'|^4)\mathrm{e}^{-\frac{c}{h}|x'|^2} = \mathcal{O}(h^2)$$

$$\left(L_{\nabla\Phi} a + \frac{1}{2}\Delta\Phi a\right)\mathrm{e}^{\frac{i}{h}\Phi} = \mathcal{O}(|x'|^2)\mathrm{e}^{-\frac{c}{h}|x'|^2} = \mathcal{O}(h)$$

Solving the approximate eikonal equation

$$\begin{aligned}(\mathrm{d}\Phi)^2 - 1 &= \langle \dot{Q}(x_1)x', x'\rangle + (Q(x_1)x')^2 + \mathcal{O}(|x'|^4)\\ &= \langle (\dot{Q} + Q^2)x', x'\rangle + \mathcal{O}(|x'|^4)\end{aligned}$$

leads to a Riccati equation on the matrix $\dot{Q} + Q^2 = 0$

Gaussian beams in Euclidean space

Solving the approximate eikonal equation

$$\left(L_{\nabla\Phi} + \frac{1}{2}\Delta\Phi\right)a = \partial_{x_1} a + \frac{1}{2}\operatorname{Tr} Q(x_1)a + \langle Qx', \nabla_{x'} a\rangle + \mathcal{O}(|x'|^2)$$

leads if a is independent on x' to

$$a = c\exp\left(-\frac{1}{2}\int_0^{x_1} \operatorname{Tr} Q(t)\,\mathrm{d}t\right)$$

An explicit Gaussian beam:

$$v_h = h^{-\frac{n-2}{4}} \mathrm{e}^{-\frac{1}{2}\int_0^{x_1} \operatorname{Tr} Q(t)\,\mathrm{d}t}\, \mathrm{e}^{i\frac{x_1}{h}}\, \mathrm{e}^{\frac{i}{2h}\langle Q(x_1)x', x'\rangle},$$

$$\dot{Q} + Q^2 = 0.$$

Modulus of a Gaussian beam

$$|v_h|^2 = h^{-\frac{n-2}{2}} \mathrm{e}^{-\int_0^{x_1} \operatorname{Tr}\, \operatorname{Re} Q(t)\,\mathrm{d}t}\, \mathrm{e}^{-\frac{1}{2}\langle \operatorname{Im} Q(x_1)x', x'\rangle},$$

Limit measure

For Riccati matrices, we have $\mathrm{e}^{-\int_0^{x_1} \operatorname{Tr} \operatorname{Re} Q(t)\,\mathrm{d}t} = \sqrt{\dfrac{\det \operatorname{Im} Q(x_1)}{\det \operatorname{Im} Q(0)}}$

therefore passing to the limit in

$$\int \hat{q}(0,x)|v_h|^2\,\mathrm{d}x =$$

$$\int_{-\infty}^{\infty} \left(h^{\frac{n-2}{2}} \int_{\mathbf{R}^{n-2}} \sqrt{\frac{\det \operatorname{Im} Q(x_1)}{\det Q(0)}} \mathrm{e}^{-\frac{1}{2}\langle \operatorname{Im} Q(x_1)x', x'\rangle} \hat{q}(0,x)\,\mathrm{d}x' \right) \mathrm{d}x_1$$

gives

$$\lim_{h\to 0} \int \hat{q}(0,x)|v_h|^2\,\mathrm{d}x = c_n \int_{-\infty}^{\infty} \hat{q}(0,x_1,0)\,\mathrm{d}x_1 = c_n T_0(\hat{q}(0,\cdot))(\gamma)$$

an information on the geodesic ray transform T_0 of $\hat{q}(0,\cdot)$.