Dynamics of energy-critical wave equation

Carlos Kenig Thomas Duyckaerts Franke Merle

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Outline

Critical wave equation

$$\partial_t^2 u - \Delta u = |u|^{\frac{4}{N-2}} u, \quad x \in \mathbb{R}^N$$
$$\vec{u}_{\restriction t=0} = (u_0, u_1) \in \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$$

where $N \in \{3, 4, 5\}$.

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$$E(\vec{u}) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla_x u(t)|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |\partial_t u(t)|^2 - \frac{N-2}{2N} \int |u(t)|^{\frac{2N}{N-2}}.$$

and momentum

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Invariant by the scaling of the equation

$$u_\lambda(t, x) = \lambda^{\frac{N-2}{2}} u(\lambda t, \lambda x).$$

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Denote by $T_+(u)$ the maximal time of existence of u.
Goal: describe the asymptotics of u as $t \to T_+(u)$, for large solutions.

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a) Scattering: $T_+(u) = +\infty$ and

$$\lim_{t\to+\infty} \|\nabla_{t,x} u - \nabla_{t,x} u_L(t,x)\|_{\dot{H}^1\times L^2} = 0$$

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b) Type I blow-up: $T_+(u) < \infty$ and

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$$-\Delta Q = |Q|^{\frac{4}{N-2}} Q, \quad Q \in \dot{H}^1(\mathbb{R}^N)$$

Existence, with arbitrarily large energy: [W.Y. Ding 1986], [Del Pino, Musso, Pacard, Pistoia 2013].

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$$W = \frac{1}{\left(1 + \frac{|x|^2}{N(N-2)}\right)^{\frac{N}{2}-1}}.$$

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c') Travelling waves: , $p = |\mathbf{p}| < 1$:

$$Q_{\mathbf{p}}(t,x) = Q\left(\left(-\frac{t}{\sqrt{1-p^2}} + \frac{1}{p^2}\left(\frac{1}{\sqrt{1-p^2}} - 1\right)\mathbf{p}\cdot x\right)\mathbf{p} + x\right)$$

$$Q_{\mathbf{p}}(t,x) = Q_{\mathbf{p}}(0, x - t\mathbf{p}).$$

Soliton resolution conjecture

Conjecture:
Let u a solution which does not scatter and such that

$$\liminf_{t \to T_+(u)} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} < \infty.$$

Then there exist $J \geq 1$ *and*

- v_L *s.t.* $\partial_t^2 v_L - \Delta v_L = 0$,
- *Travelling waves* $Q^j_{\mathbf{p}_j}$, $j = 1 \ldots J$,
- *Parameters* $x_j(t) \in \mathbb{R}^N$, $\lambda_j(t) > 0$,

such that

$$u(t) = v_L(t) + \sum_{j=1}^{J} \frac{1}{\lambda_j^{\frac{N-2}{2}}(t)} Q^j_{\mathbf{p}_j}\left(0, \frac{x - x_j(t)}{\lambda_j(t)}\right) + r(t),$$

where

$$\lim_{t \to T_+(u)} \left\|\vec{r}(t)\right\|_{\dot{H}^1 \times L^2} = 0.$$

Remarks on the conjecture

a) Existence of solutions with $J = 1$ and $Q = W$:

- $T_+(u) < \infty$ [Krieger, Schlag, Tataru 2009], [Hillairet, Raphaël 2012], [Krieger, Schlag 2012]
- $T_+(u) = \infty$, $\lambda(t) = 1$ [Krieger-Schlag 2007], $\lambda(t) \approx t^\eta$, $|\eta|$ small [Donninger, Krieger 2013]

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b) Soliton resolution was only known for smooth solutions of completely integrable equations, using the method of inverse scattering: for example KdV [Eckhaus, Schuur 1985].

c) The dynamics described in the conjecture is believed to be unstable, at the threshold between type I blow-up and scattering. However, it is the stable dynamics for more geometric equations (wave maps...).

References on energy-critical waves

Defocusing equation (scattering for all solutions) [Grillakis 90, 92], [Shatah Struwe, 93, 94], [Kapitanski 94], [Ginibre Velo 95], [Nakanishi 95], [Bahouri Shatah 98], [Bahouri Gérard 99], [Tao 06].

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Focusing equation
Below the ground-state energy [Kenig Merle]
At the ground state energy [TD Merle]
Slightly above the ground state energy (center stable manifold): [Krieger Nakanishi Schlag 2013]
Remarks on Type II blow-up [Krieger Wong]
Dimension $N \geq 6$: [Bulut Czubak Li Pavlović Zhang]
In this talk: soliton resolution for radial data $N = 3$, [TD Kenig Merle 2013] and weaker results in the nonradial case [TD Kenig Merle 2012, 2014]
Some results for radial data in space dimension 4: [Côte Lawrie Kenig Schlag 2013].

Outline

Soliton resolution

Theorem. *Assume* $N = 3$. *Let* u *be a* *radial* *solution which does not scatter and such that*

$$\liminf_{t\to T_+(u)} \|\vec{u}(t)\|_{\dot{H}^1\times L^2} < \infty.$$

Then there exist $J \geq 1$ *and:*

- v_L *such that* $\partial_t^2 v_L - \Delta v_L = 0$,
- *signs* $\iota_j \in \{\pm 1\}$, $j = 1\ldots J$,
- *parameters* $\lambda_j(t)$, $0 < \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_J(t)$,

such that

$$u(t) = v_L(t) + \sum_{j=1}^{J} \frac{\iota_j}{\lambda_j^{\frac{N-2}{2}}(t)} W\left(\frac{x}{\lambda_j(t)}\right) + r(t),$$

where $\lim_{t\to T_+(u)} \left\|\vec{r}(t)\right\|_{\dot{H}^1\times L^2} = 0.$

Proof of the resolution I: general strategy

Let u be a non-scattering solution. Assume $\exists \{t_n\}_n$ such that

$$t_n \underset{n\to\infty}{\longrightarrow} T_+, \quad t_n < T_+$$

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Using the profile expansion of [Bahouri Gérard] and approximation results, we have (for small τ or $|x| \geq |\tau| + R$, R large)

$$u(t_n+\tau) \approx v_L(t_n+\tau) + \sum_{j=1}^{J} U_n^j(\tau,x) + w_n^J(\tau)$$

where v_L is a fixed solution of the linear wave equation, w_n^J is a dispersive remainder and U_n^j are nonlinear profiles:

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$$U_n^j(\tau,x) = \frac{1}{\lambda_{j,n}^{\frac{N-2}{2}}} U^j\left(\frac{\tau - t_{j,n}}{\lambda_{j,n}}, \frac{x}{\lambda_{j,n}}\right).$$

$$\partial_t^2 U^j - \Delta U^j = |U^j|^{\frac{4}{N-2}} U^j, \quad x_{j,n} \in \mathbb{R}^N, \quad \lambda_{j,n} > 0$$

Proof of the resolution II: exterior energy

Goal: prove that the only possible profiles U^j are W (up to scaling and sign change).

Main tool:

Assume $N = 3$. Let u be a radial, nonstationary solution of the equation. Then there exists $r_0 > 0$, $\eta > 0$ and a small, global solution $\tilde{u}$ such that

$$\tilde{u}(t,r) = u(t,r) \text{ if } r \geq r_0 + t \quad t \text{ in the domain of existence of } u \qquad (1)$$

and

$$\forall t \geq 0 \text{ or } \forall t \leq 0, \quad \int_{|t|+r_0}^{+\infty} |\nabla_{t,x}\tilde{u}(t,x)|^2\,dx \geq \eta, \qquad (2)$$

Proof: use exterior energy estimates for radial solutions of the linear wave equation in dimension 3 and small data theory.

Proof of the resolution III: channels of energy.

Assume to fix ideas $T_+(u) < \infty$. Then

$$\vec{u}(t_n) \xrightharpoonup[n\to\infty]{} (v_0, v_1) \text{ in } \dot{H}^1 \times L^2$$

and $\vec{u}(t) - (v_0, v_1)$ will concentrate in a light cone.

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Channels of energy method:

Consider a profile expansion for a sequence $\{u(t_n)\}_n$, with nonlinear profiles U^j. If one of the profiles U^j is not equal to W, then by the two preceding slides, he will send energy outside of the light-cone, at the blow-up time, or arbitrarily close to the boundary of the light cone, at the initial time, yielding a contradiction.

Outline

$N \geq 4$ or nonradial data

New difficulties

- Weaker exterior energy estimates for the linear equation.
- In the nonradial case, no classification of stationary solutions.
- Technical difficulties given by new geometric transformations: space translations, rotations, Lorentz transformation, Kelvin transformation for the elliptic equation.

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We have proved partial results:

- A “small data” result ($\dot{H}^1 \times L^2$ norm close to the $\dot{H}^1$-norm of W).
- Local strong convergence up to the transformations of the equation analog to the result of [Struwe 2003] for wave maps.
- Rigidity theorem for solutions with the compactness property.

Compact solution:

Definition. A solution u has the compactness property if there exist $\lambda(t)$, $x(t)$ such that

$$K = \left\{ \left(\frac{1}{\lambda^{\frac{N-2}{2}}(t)} u\left(t, \frac{x - x(t)}{\lambda(t)}\right), \frac{1}{\lambda^{\frac{N}{2}}(t)} \partial_t u\left(t, \frac{x - x(t)}{\lambda(t)}\right) : t \in (T_-(u), T_+(u)) \right\}$$

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Rigidity conjecture for solutions with the compactness property: *the only such solutions are* 0 *and solitary waves* $Q_{\mathbf{p}}$.

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Let $a \in \mathbb{R}^N$. Conjugating the Kelvin transformation and the translation with respect to a, we obtain that the equation is also invariant by

$$Q \mapsto \left|\frac{x}{|x|} - a|x|\right|^{2-N} Q\left(\frac{x - a|x|^2}{1 - 2\langle a, x\rangle + |a|^2|x|^2}\right).$$

Nondegeneracy assumption

To study the (conditional) stability of Q, consider the linearized operator $L_Q := -\Delta - \frac{N+2}{N-2}|Q|^{\frac{4}{N-2}}$. Then

$$\sigma(Q) = \{-\omega_p^2 \leq \ldots \leq -\omega_1^2 < 0\} \cup [0, +\infty).$$

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Let $\mathcal{Z}_Q = \left\{ f \in \dot{H}^1 \ : \ L_Q f = 0 \right\}$. Each of the preceding transformation generates an element of $\mathcal{Z}_Q$: let

$$\widetilde{\mathcal{Z}}_Q = \operatorname{span}\Big\{ (2-N)x_j Q + |x|^2 \partial_{x_j} Q - 2x_j x \cdot \nabla Q, \partial_{x_j} Q,\ 1 \leq j \leq N,$$
$$(x_j \partial_{x_k} - x_k \partial_{x_j})Q,\ 1 \leq j < k \leq N,\ \frac{N-2}{2} Q + x \cdot \nabla Q \Big\}.$$

Then $\widetilde{\mathcal{Z}}_Q \subset \mathcal{Z}_Q$.

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Then $\widetilde{\mathcal{Z}}_Q \subset \mathcal{Z}_Q$. **Nondegeneracy assumption:** $\mathcal{Z}_Q = \widetilde{\mathcal{Z}}_Q$.

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- True for W.
- [Musso Wei 2014]: true for the solutions of [Del Pino & al].

Nonradial result

Theorem *Let u be a nonzero solution with the compactness property, with maximal time of existence* (T_-, T_+). *Then*

1. *There exist a sequence of time* $\{t_n\}_n$, *and a travelling wave* $Q_{\mathbf{p}}$ *such that* $\lim_{n\to+\infty} t_n = T_+$ *and*

$$\lim_{n\to\infty} \left\| \lambda^{\frac{N}{2}-1}(t_n)\, u\left(t_n, \lambda(t_n)\cdot + x(t_n)\right) - Q_{\mathbf{p}}(t_n) \right\|_{\dot{H}^1} + \left\| \lambda^{\frac{N}{2}}(t_n)\, \partial_t u\left(t_n, \lambda(t_n)\cdot + x(t_n)\right) - \partial_t Q_{\mathbf{p}}(t_n) \right\|_{L^2} = 0.$$

2. *Assume that Q satisfies the non-degeneracy assumption. Then* $u = \widetilde{Q}_{\mathbf{p}}$, *where* $\widetilde{Q} \approx Q$ *up to translation and scaling.*

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Theorem *Let u be a nonzero solution with the compactness property, with maximal time of existence (T_-, T_+). Then*

1. *There exist a sequence of time $\{t_n\}_n$, and a travelling wave $Q_{\mathbf{p}}$ such that $\lim_{n\to+\infty} t_n = T_+$ and*

$$\lim_{n\to\infty} \left\| \lambda^{\frac{N}{2}-1}(t_n)\, u(t_n, \lambda(t_n)\cdot + x(t_n)) - Q_{\mathbf{p}}(t_n) \right\|_{\dot{H}^1} + \left\| \lambda^{\frac{N}{2}}(t_n)\, \partial_t u(t_n, \lambda(t_n)\cdot + x(t_n)) - \partial_t Q_{\mathbf{p}}(t_n) \right\|_{L^2} = 0.$$

2. *Assume that Q satisfies the non-degeneracy assumption. Then $u = \widetilde{Q}_{\mathbf{p}}$, where $\widetilde{Q} \approx Q$ up to translation and scaling.*

Proof of point (1) "classical": monotonicity formulas and nonexistence of self-similar compact solutions from [Kenig Merle 08].

More energy channels

Main new idea of the proof of point (2): consider the equation

$$\partial_t^2 v + L_Q v = 0, \quad L_Q = -\Delta - \frac{N+2}{N-2}|Q|^{\frac{4}{N-2}}.$$

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Proposition (Exterior energy for eigenfunctions). *Let*

$$v(t,x) = e^{-\omega t} Y(x), \quad L_Q Y = -\omega^2 Y, \quad Y \not\equiv 0$$

Then if $r_0 \gg 1$, we have the following exterior energy property

$$\lim_{t \to -\infty} \int_{|x| \geq r_0 + |t|} |\nabla_{t,x} v(t,x)|^2 \, dx = \varepsilon(r_0) > 0$$

where $\varepsilon(r_0) > 0$, $\lim_{r_0 \to \infty} \varepsilon(r_0) = 0$.

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Proof. In radial coordinates $x = r\theta$, we have [Agmon 82], [Meshkov 91], for $r \to +\infty$,

$$Y(r,\theta) \approx \frac{e^{-\omega r}}{r^{\frac{N-1}{2}}} V(\theta), \quad V \in C^0(S^{N-1}) \setminus \{0\}.$$

Independent of the dimension!

Further references:

Channels of energy method:

- wave maps: [Côte, Kenig, Lawrie, Schlag 2012], [Kenig, Lawrie, Schlag 2013], [Côte 2013], [Liu, Kenig, Lawrie, Schlag 2014].
- energy-supercritical wave equations: [TD, Kenig, Merle 2012], [Dodson, Lawrie 2014].
- energy-subcritical wave equations: [Ruipeng Shen 2012], [Dodson, Lawrie 2014].
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Failure of the radial exterior energy estimate in even space dimension: [Côte, Kenig, Lawrie, Schlag 2012]

Exterior energy estimate in odd space dimension: [Liu, Kenig, Lawrie, Schlag 2014]

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- wave maps: [Côte, Kenig, Lawrie, Schlag 2012], [Kenig, Lawrie, Schlag 2013], [Côte 2013], [Liu, Kenig, Lawrie, Schlag 2014].
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Open questions. Full resolution in dimension $N \geq 4$, and in dimension 3 for nonradial data. Existence of solutions with several stationary profiles. Classification of stationary solutions. Nondegeneracy assumptions for these stationary solutions. Other equations.