DECOUPLING INEQUALITIES
IN HARMONIC ANALYSIS AND APPLICATIONS

NOTATION AND STATEMENT

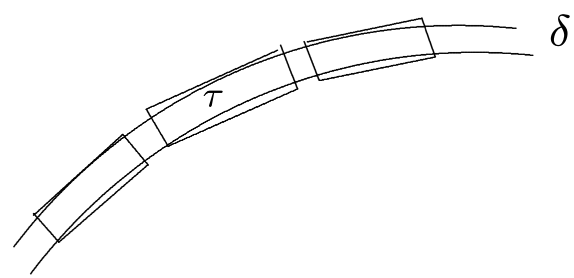
S compact smooth hypersurface in $\mathbb{R}^n$ with positive definite second fundamental form

(Examples: sphere S^{d-1}, truncated paraboloid

$$\mathcal{P}^{d-1} = \{(\xi, |\xi|^2) \in \mathbb{R}^{d-1} \times \mathbb{R}; |\xi| \leq 1\}\Big)$$

$S_\delta = \delta$-neighborhood of S

$S_\delta \subset \bigcup_\tau \tau$ where τ is a $\sqrt{\delta} \times \cdots \times \sqrt{\delta} \times \delta$ rectangular box in $\mathbb{R}^d$



For $f \in L^p(\mathbb{R}^d)$, $f_\tau = (\widehat{f}|\tau)^\vee$ is the Fourier restriction of f to τ

THEOREM (**B-Demeter, 014**)

Assume supp $\widehat{f} \subset S_\delta$. *Then*

$$\|f\|_p \ll_\varepsilon \delta^{-\varepsilon} \Big(\sum_\tau \|f_\tau\|_p^2 \Big)^{\frac{1}{2}} \ \text{for } 2 \le p \le \frac{2(d+1)}{d-1}$$

- Range $p \leq \frac{2d}{d-1}$ obtained earlier (**B, 013**)

- Exponent $p = \frac{2(d+1)}{d-1}$ is best possible

EXAMPLE $f = \widehat{1_{S_\delta}}$. Then $|f| \sim \frac{\delta}{(1+|x|)^{\frac{d-1}{2}}}$ and $\|f\|_p \sim \delta$

$|f_\tau| \sim \delta^{\frac{d+1}{2}} 1_{\overset{\circ}{\tau}}$ with $\overset{\circ}{\tau}$ the polar of $\tau \Rightarrow \|f_\tau\|_p \sim \delta^{\frac{d+1}{2}} \Big(\frac{1}{\delta}\Big)^{\frac{d+1}{2p}} = \delta^{\frac{d+3}{4}}$

$$\Big(\sum_\tau \|f_\tau\|_p^2\Big)^{\frac{1}{2}} \sim \Big(\frac{1}{\delta}\Big)^{\frac{d-1}{4}} \delta^{\frac{d+3}{4}} = \delta$$

- **Original motivations**

  PDE Smoothing for the wave equation (**T.Wolff, 2000**)

  Schrödinger equation on tori and irrational tori (**B, 92**$\longrightarrow$)

  Spectral Theory Eigenfunction bounds for spheres

- Diophantine applications, new approach to mean value theorems in Number Theory, bounds on exponential sums

MOMENT INEQUALITIES FOR TRIGONOMETRIC POLYNOMIALS WITH SPECTRUM IN CURVED HYPERSURFACES

$S \subset \mathbb{R}^d$ smooth with positive curvature

$\lambda \gg 1, \mathcal{E} = \mathbb{Z}^d \cap \lambda S$ = lattice points on λS

Let

$$\phi(x) = \sum_{\xi \in \mathcal{E}} a_\xi \, e^{2\pi i x.\xi} \Rightarrow \phi \text{ is 1-periodic}$$

Apply decoupling theorem to rescaled $f(y) = \sum_{\xi \in \mathcal{E}} a_\xi \, e^{2\pi i y.\frac{\xi}{\lambda}} \eta(\lambda^{-2} y)$

with $\delta = \lambda^{-2}$

$$\Rightarrow \|\phi\|_{L^p([0,1]^d)} \ll \lambda^\varepsilon \Big(\sum_{\xi \in \mathcal{E}} |a_\xi|^2 \Big)^{\frac{1}{2}} \text{ for } p \leq \frac{2(d+1)}{d-1}$$

COROLLARY $(S = S^{d-1})$

Eigenfunctions ϕ_E *of* d*-torus* $\mathbb{T}^d$*, i.e.* $-\Delta\phi_E = E\phi_E$*, satisfy*

$$\|\phi\|_p \ll E^{\varepsilon}\|\phi\|_2 \ \textit{for}\ p \le \frac{2(d+1)}{d-1}$$

(Known for $d = 3$ for arithmetical reason, new for $d > 3$)

SCHRODINGER EQUATIONS AND STRICHARTZ INEQUALITIES

Linear Schrödinger equation on $\mathbb{R}^d$

$$iu_t + \Delta u = 0, u(0) = \phi \in L^2(\mathbb{R}^d)$$

$$u(t) = e^{it\Delta}\phi = \int \widehat{\phi}(\xi) e^{i(x.\xi + t|\xi|^2)} d\xi$$

STRICHARTZ INEQUALITY: $\|e^{it\Delta}\phi\|_{L^p_{x,t}} \leq C\|\phi\|_2$ with $p = \frac{2(d+2)}{d}$

Local wellposedness fo Cauchy problem for NLS on $\mathbb{R}^d$

$$\begin{cases} iu_t + \Delta u + \alpha u|u|^{p-2} = 0 \\ u(0) = \phi \in H^s(\mathbb{R}^d) \end{cases}$$

$$s \geq s_0 \text{ with } s_0 \text{ defined by } p - 2 = \frac{4}{d - 2s_0}$$

$$(p - 2 \geq [s] \text{ if } p \notin 2\mathbb{Z})$$

TORI AND IRRATIONAL TORI

$$\phi \in L^2(\mathbb{T}^d) \qquad \phi(x) = \sum_{\xi \in \mathbb{Z}^d} a_\xi \, e^{2\pi i x.\xi}$$

$$e^{it\Delta}\phi = \sum_{\xi \in \mathbb{Z}^d} a_\xi \, e^{2\pi i (x.\xi + t|\xi|^2)}$$

More generally (irrational tori)

$$e^{it\Delta}\phi = \sum_{\xi \in \mathbb{Z}^d} a_\xi \, e^{2\pi i (x.\xi + tQ(\xi))} \text{ where } Q(\xi) = \alpha_1 \xi_1^2 + \cdots + \alpha_d \xi_d^2, \alpha_j > 0$$

PROBLEM **Which L^p-inequalities are satisfied**?

(periodic Strichartz inequalities)

EXAMPLES (**B, 92**)

• $\underline{d=1}$ Assume $\phi(x) = \sum_{\xi\in\mathbb{Z},|\xi|\leq R} a_\xi \, e^{2\pi i x\xi}$

$$\|e^{it\Delta}\phi\|_{L^6_{[|x|,|t|\leq 1]}} \ll R^\varepsilon \|\phi\|_2$$

Moreover, taking $a_\xi = \frac{1}{\sqrt{R}}$ for $|\xi| \leq R, a_\xi = 0$ for $|\xi| > R$,
$\|e^{it\Delta}\phi\|_6 \sim (\log R)^{1/6}$

(Classical Strichartz inequality fails in periodic case!)

•$\underline{d=2}$ $\|e^{it\Delta}\phi\|_4 \ll R^\varepsilon \|\phi\|_2$

Proven using arithmetical techniques (# lattice points on ellipses)

•$\underline{d\geq 3}$ Only partial results

Irrational tori No satisfactory results for $d \geq 2$

THEOREM *Let Q be a positive definite quadratic form on $\mathbb{R}^d$. Then*

$$\Big\| \sum_{\xi \in \mathbb{Z}^d, |\xi| \le R} a_\xi \, e^{2\pi i (x.\xi + tQ(\xi))} \Big\|_{L^p_{[|x|,|t| \le 1]}} \ll R^\varepsilon \Big(\sum |a_\xi|^2 \Big)^{\frac{1}{2}}$$

for $p \le \frac{2(d+2)}{d}$.

COROLLARY *Local wellposedness of NLS on $\mathbb{T}^d$ for $s > s_0$*

Recent results of **Killip-Visan**

Proof

Define $f(x', t') = \sum_{|\xi| \le R} a_\xi e^{2\pi i (x' . \frac{\xi}{R} + t' \frac{Q(\xi)}{R^2})} \eta(R^{-2} x', R^{-2} t')$.

$$\left(\frac{\xi}{R}, \frac{Q(\xi)}{R^2}\right) \in S = \{\big(y, Q(y)\big); y \in \mathbb{R}^d, |y| \le 1\} \subset \mathbb{R}^{d+1}$$

From decoupling inequality

$$\|f\|_{L^p(B_{R^2})} \ll R^\varepsilon R^{\frac{2(d+1)}{p}} \Big(\sum |a_\xi|^2\Big)^{1/2}$$

and setting $x' = Rx, t' = R^2 t$

$$\|f\|_{L^p(B_{R^2})} = R^{\frac{d+2}{p}} \Big\| \sum a_\xi \; e^{2\pi i (x.\xi + tQ(\xi))} \Big\|_{L^p_{|x|<R, |t|<1}}$$

$$= R^{\frac{2(d+1)}{p}} \Big\| \sum a_\xi \; e^{2\pi i (x.\xi + tQ(\xi))} \Big\|_{L^p_{|x|<1, |t|<1}}$$

MOMENT INEQUALITIES FOR EIGENFUNCTIONS

(M,g) compact, smooth Riemannian manifold of dimension d without boundary

$$\Delta\varphi_E = -E\varphi_E \qquad (E=\lambda^2)$$

THEOREM (Hormander, Sogge)

$$\|\varphi_E\|_p \le c\lambda^{\delta(p)}\|\varphi_E\|_2$$

where

$$\delta(p) = \begin{cases} \dfrac{d-1}{2}\Big(\dfrac{1}{2}-\dfrac{1}{p}\Big) & \text{if} \quad 2 \le p \le \dfrac{2(d+1)}{d-1} \\[2ex] d\Big(\dfrac{1}{2}-\dfrac{1}{p}\Big)-\dfrac{1}{2} & \text{if} \quad \dfrac{2(d+1)}{d-1} \le p \le \infty \end{cases}$$

Estimates are sharp for $M=S^d$

FLAT TORUS $\mathbb{T}^d$

CONJECTURE

$$\|\varphi_E\|_p \leq c_p \|\varphi_E\|_2 \quad \textit{if } p < \frac{2d}{d-2}$$

and

$$\|\varphi_E\|_p \leq c_p \lambda^{\left(\frac{d-2}{2} - \frac{d}{p}\right)} \|\varphi_E\|_2 \quad \textit{if } p > \frac{2d}{d-2}$$

THEOREM (**B–Demeter, 014**)

$$\|\varphi_E\|_p \ll \lambda^{\varepsilon}\|\varphi_E\|_2 \text{ if } d \geq 3 \text{ and } p \leq \frac{2(d+1)}{d-1} \qquad (*)$$

and

$$\|\varphi_E\|_p \leq C_p \lambda^{(\frac{d-2}{2}-\frac{d}{p})}\|\varphi_E\|_2 \text{ if } d \geq 4 \text{ and } p > \frac{2(d-1)}{d-3} \qquad (**)$$

$(**)$ follows from $(*)$ combined with distributional inequality

$$t^{2\frac{d-1}{d-3}}|[\varphi > t]| \ll \lambda^{\frac{2}{d-3}+\varepsilon} \text{ for } t > \lambda^{\frac{d-1}{4}} \qquad (\mathbf{B, 093})$$

(**Hardy-Littlewood + Kloosterman**)

ESTIMATES ON $\mathbb{T}^2$

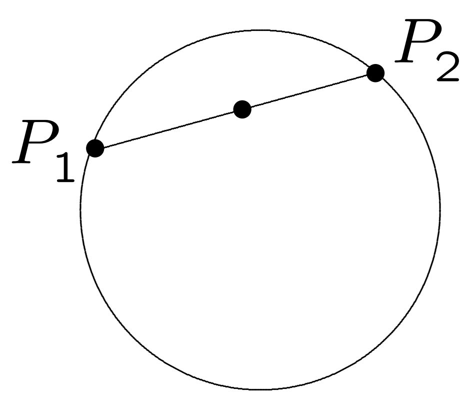
E = sum of 2 squares

multiplicity E = number N of representations of E as sum of two squares

$$E = \prod p_\alpha^{e_\alpha} \Rightarrow N = 4 \prod (1 + e_\alpha)$$

On average, $N \sim (\log E)^{1/2}$

THEOREM (Zygmund–Cook) $\|\varphi_E\|_4 \leq C \|\varphi_E\|_2$



Estimating the L^6-norm $\Leftrightarrow$

number of solutions of $P_1 + P_2 + P_3 = P_4 + P_5 + P_6$ with

$$P_i \in \mathcal{E} = \{P \in \mathbb{Z} + i\mathbb{Z}; |P|^2 = E\}$$

COMBINATORIAL APPROACH Use of incidence geometry

ANALYTICAL APPROACH: Use of the theory of elliptic curves (conditional)

THEOREM (Bombieri–B, 012)

- $\|\varphi_E\|_6 \ll N^{\frac{1}{12}+\varepsilon}\|\varphi_E\|_2$

- *For all* $p < \infty, \|\varphi_E\|_p \leq C\|\varphi_E\|_2$ *for 'most'* E

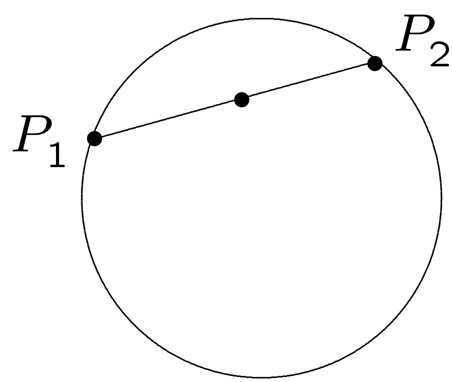
- *Conditional to GRH and* **Birch, Swinnerton-Dyer** *conjecture*

  $\|\varphi_E\|_6 \ll N^{\varepsilon}\|\varphi_E\|_2$ *for 'most' smooth numbers* E

THE COMBINATORIAL APPROACH

Consider equation $P_1 + P_2 + P_3 = (A, B)$ with $P_j = (x_j, y_j) \in \mathcal{E}$

Set $u = x_1 + x_2, v = y_1 + y_2$ establishing correspondence between $\mathcal{E} \times \mathcal{E}$ and $\mathcal{P} = \mathcal{E} + \mathcal{E}$



Introduce curves $C_{A,B} : (A - u)^2 + (B - v)^2 = E$

$\Rightarrow$ family $\mathcal{C}$ of circles of same radius $\sqrt{E}$ (pseudo-line system)

Need to bound

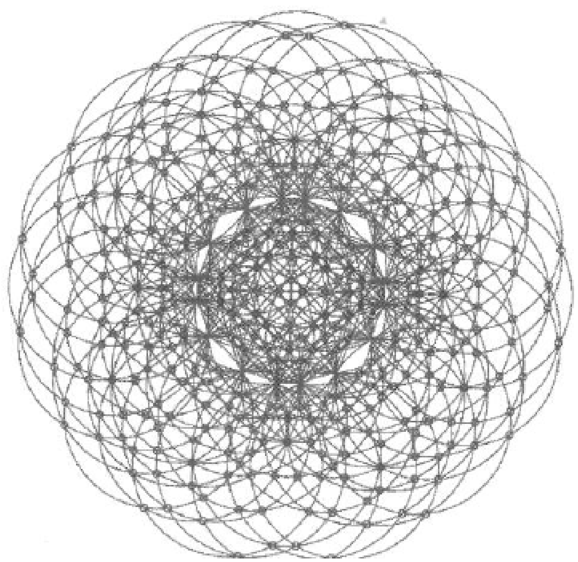
$$\sum_{A,B} |C_{A,B} \cap \mathcal{P}|^2$$

USE OF SZEMEREDI-TROTTER THEOREM
FOR PSEUDO-LINE SYSTEMS

$I(\mathcal{P}, \mathcal{C}) =$ number of incidences between point $\mathcal{P}$ and curves $\mathcal{C}$

$$\leq C(|\mathcal{P}|^{2/3}|\mathcal{C}|^{2/3} + |\mathcal{P}| + |\mathcal{C}|)$$

$\Rightarrow$ bound $N^{7/2}$



$$E = 65, |\mathcal{P}| = 112, |\mathcal{C}| = 372$$

REMARK Erdös unit distance conjecture would imply $N^{3+\varepsilon}$

PROBLEM *Establish uniform L^p-bound for some $p > 2$ for higher dimensional toral eigenfunctions*

Important to PDE's

Control for Schrodinger operators on tori

THEOREM (**B-Burq-Zworski, 012**)

Let $d \leq 3, V \in L^\infty(\mathbb{T}^d), \Omega \subset \mathbb{T}^d, \Omega \neq \phi$, an open set. Let $T > 0$. There is a constant $C = C(\Omega, T, V)$ such that for any $u_0 \in L^2(\mathbb{T}^d)$

$$\|u_0\|_{L^2(\mathbb{T}^d)} \leq C\|e^{it(\Delta+V)}u_0\|_{L^2(\Omega\times[0,T])}$$

(uses Zygmund's inequality)

More regular V: **Anantharaman-Macia (011)**

INGREDIENTS IN THE PROOF OF DECOUPLING THEOREM

- **Wave packet decomposition**

- **Parabolic rescaling**

- **Multilinear harmonic analysis**

$d = 2$ bilinear square function inequalities (**Cordoba 70's**)

$d \geq 3$ **Bennett, Carbery, Tao (06)**

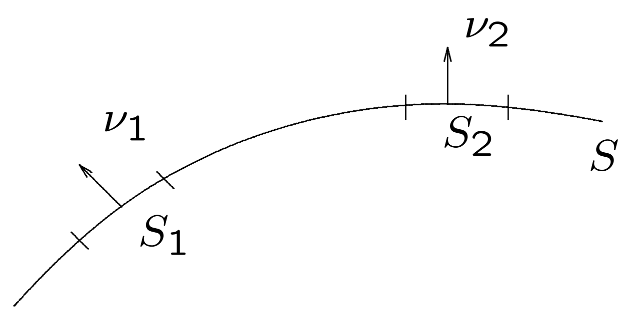
- **Multi-scale bootstrap**

BENNETT - CARBERRY - TAO THEOREM

$S \subset \mathbb{R}^d$ compact smooth hypersurface

$S_1, \ldots, S_d \subset S$ disjoint patches in transversal position

$$|\nu(\xi_1) \wedge \cdots \wedge \nu(\xi_d)| > c \text{ for all } \xi_1 \in S_1, \ldots, \xi_d \in S_d$$



Let $\mu_1, \ldots, \mu_d$ be measures on S, $\mu_j \ll \sigma$ = surface measure and supp$\mu_j \subset S_j$

$$d\mu_j = \varphi_j \, d\sigma, \varphi_j \in L^2(\sigma)$$

Take large ball $B_R \subset \mathbb{R}^d$. Then

$$\Big\| \prod_{j=1}^{d} |\widehat{\mu}_j(x)|^{\frac{1}{d}} \Big\|_{L^p(B_R)} \ll R^{\varepsilon} \prod_{j=1}^{d} \|\varphi_j\|_{L^2(\sigma)} \text{ with } p = \frac{2d}{d-1}$$

MEAN VALUE THEOREMS AND DIOPHANTINE RESULTS (FIRST APPLICATIONS)

(1) **Around Hua's inequality**

$$\int_0^1 \int_0^1 \Big| \sum_{1 \leq n \leq N} e(nx + n^3 y) \Big|^6 dx dy = I_6(N) < CN^3 (\log N)^c \quad \textbf{(Hua, 1947)}$$

Arithmetically, $I_6(N)$ is the number of integral points $n_1, \ldots, n_6 \leq N$ on the Segre cubic

$$\begin{cases} x_1 + x_2 + x_3 = x_4 + x_5 + x_6 \\ x_1^3 + x_2^3 + x_3^3 = x_4^3 + x_5^3 + x_6^3 \end{cases}$$

Vaughan-Wooley (95) $\quad I_6(N) = 6N^3 + U(N), U(N) = O\Big(N^2 (\log N)^5\Big)$

De La Bretèche (07) $\quad$ Precise asymptotic for $U(N)$

Applications of decoupling inequality to $S = \{(t, t^3), t \sim 1\} \subset \mathbb{R}^2$

$$\fint_{B(N^2)} \Big| \sum_{n\sim N} e\Big(\frac{n}{N}x + \Big(\frac{n}{N}\Big)^3 y\Big)\Big|^6 dxdy \ll N^{3+\varepsilon}$$

By change of variables and periodicity

$$\int_0^1 \int_0^1 \Big| \sum_{n\sim N} e\Big(nx + \frac{n^3}{N}y\Big)\Big|^6 = \int_0^1 \int_0^1 \Big| \sum_{n\sim N} e\Big(nNx + \frac{n^3}{N}y\Big)\Big|^6 \ll N^{3+\varepsilon}$$

which means that

$$\begin{cases} n_1 + n_2 + n_3 - n_4 - n_5 - n_6 = 0 \\ |n_1^3 + n_2^3 + n_3^3 - n_4^3 - n_5^3 - n_6^3| < N \end{cases}$$

has at most $O(N^{3+\varepsilon})$ solutions with $n_i \sim N$

$$\int_0^1 \int_0^1 \Big| \sum_{n \sim N} e\Big(nx + \frac{n^3}{N} y\Big)\Big|^6 dxdy \ll N^{3+\varepsilon}$$ is optimal

CONJECTURE

$$I_8(N) = \int_0^1 \int_0^1 \Big| \sum_{n \leq N} e\Big(nx + n^3 y\Big)\Big|^8 dxdy \ll N^{4+\varepsilon}$$

KNOWN

Hua (1947) $I_{10}(N) \ll N^{6+\varepsilon}$ (optimal)

Wooley (014) $I_9(N) \leq N^{5+\varepsilon}$ (optimal)

$$I_8(n) \ll N^{\frac{13}{9}+\varepsilon}$$

Consequence of (optimal) Vinogradov mean value theorem

$$\int_0^1 \int_0^1 \int_0^1 \Big| \sum e(nx + n^2 y + n^3 z)\Big|^{12} dxdydz \ll N^{6+\varepsilon}$$

(2) A MEAN VALUE THEOREM OF ROBERT AND SARGOS

THEOREM (**Robert-Sargos, 2000**)

$$\int_0^1 \int_0^1 \Big| \sum_{n\sim N} e\Big(n^2 x + \frac{n^4}{N^3} y\Big)\Big|^6 dx dy \ll N^{3+\varepsilon}$$

Proof based on Poisson summation (**Bombieri-Iwaniec** method)

Applications to exponential sums and Weyl's inequality

THEOREM (**B-D, 014**)

$$\int_0^1 \int_0^1 \Big| \sum_{n\sim N} e\Big(n^2 x + \frac{n^k}{N^{k-1}} y\Big)\Big|^6 dx dy \ll N^{3+\varepsilon}$$

PROOF

Apply DCT with $\delta = N^{-\frac{1}{2}}$ to

$$\iint_{\substack{|x|<N^2 \\ |y|<N}} \Big| \sum_{n\sim N} e\Big(\Big(\frac{n}{N}\Big)^2 x + \Big(\frac{n}{N}\Big)^k y\Big)\Big|^6 dxdy \qquad (*)$$

This gives a bound

$$N^\varepsilon \Big\{ \sum_I \Big(\int_0^1 \int_0^1 \Big| \sum_{n\in I} e(n^2 x + N^{-k+1} n^k y) \Big|^6 dxdy \Big)^{\frac{1}{3}} \Big\}^3$$

where $\{I\}$ is a partition of $\left[\frac{N}{2}, N\right]$ in intervals of size $\sqrt{N}$

If $I = [\ell, \ell + \sqrt{N}], n = \ell + m, m < \sqrt{N}$

$$\Big| \sum_{n\in I} e(n^2 x + N^{-k+1} n^k y) \Big| \sim \Big| \sum_{m<\sqrt{N}} e\Big(m(2\ell x + kN^{-k+1} \ell^{k-1} y) + m^2 x \Big) \Big|$$

and 6th moment bounded by $N^{\frac{3}{2}+\varepsilon}$

$$\Rightarrow (*) \ll N^\varepsilon \Big(\sqrt{N} \Big(N^{\frac{3}{2}} \Big)^{\frac{1}{3}} \Big)^3 = N^{3+\varepsilon}$$

DECOUPLING FOR CURVES IN HIGHER DIMENSION

$\Gamma \subset \mathbb{R}^d$ non-degenerate smooth curve, i.e.

$$\phi'(t) \wedge \cdots \wedge \phi^{(d)}(t) \neq 0$$

with $\phi : [0,1] \to \Gamma$ a parametrization

Let $\Gamma_1, \ldots, \Gamma_{d-1} \subset \Gamma$ be separated arcs and $(\Gamma_j)_\delta$ a δ-neighborhood of Γ_j

Assume supp $\widehat{f}_j \subset (\Gamma_j)_\delta$

Apply a version of the hypersurface DCT to

$$S = \Gamma_1 + \cdots + \Gamma_{d-1}$$

(Note that second fundamental form may be indefinite)

THEOREM ($*$) (B-D)

$$\Big\| \prod_{j=1}^{d-1} |f_j|^{\frac{1}{d-1}} \Big\|_{L_\#^{2(d+1)}(B_N)} \ll N^{\frac{1}{2(d+1)}+\varepsilon} \prod_{j=1}^{d-1} \Big[\sum \|f_{j,\tau}\|^p_{L^p_\#(B_N)} \Big]^{\frac{1}{2(d+1)}}$$

where $p = \frac{2(d+1)}{d-1}$ *and* $\{\tau\}$ *a partition of* Γ_δ *in size* $\delta^{\frac{1}{2}}$*-tubes,*
$N = \delta^{-1}$

THEOREM ($**$) (B)

Let d be even

$$\Big\| \prod_{j=1}^{d/2} |f_j|^{2/d} \Big\|_{L_\#^{3d}(B_N)} \ll N^{\frac{1}{6}+\varepsilon} \prod_{j=1}^{d/2} \Big[\sum \|f_{j,\tau}\|^6_{L^6_\#(B_N)} \Big]^{\frac{1}{3d}}$$

MEAN VALUE THEOREMS

Let $d \geq 3$ and $\varphi_3, \ldots, \varphi_d : [0,1] \to \mathbb{R}$ be real analytic such that

$$W\left(\varphi_3''', \ldots, \varphi_d'''\right) \neq 0$$

$I_1, \ldots, I_{d-1} \subset \left[\frac{N}{2}, N\right] \cap \mathbb{Z}$ intervals that are $\sim N$ separated

THEOREM

$$\Big\| \prod_{j=1}^{d-1} \Big| \sum_{k \in I_j} e\Big(kx_1 + k^2 x_2 + N\varphi_3\Big(\frac{k}{N}\Big)x_3 + N\varphi_4\Big(\frac{k}{N}\Big)x_4 + \cdots + N\varphi_d\Big(\frac{k}{N}\Big)x_d\Big)\Big|^{\frac{1}{d-1}} \Big\|_{L^{2(d+1)}([0,1]^d)}$$

$$\ll N^{\frac{1}{2}+\varepsilon}$$

COROLLARY (**Bombieri-Iwaniec, 86**)

Assume $\varphi : [0,1] \to \mathbb{R}$ *real analytic,* $\varphi''' > 0$

Then

$$\int_0^1 \int_0^1 \int_0^1 \Big| \sum_{k \sim N} e\Big(kx_1 + k^2 x_2 + N\varphi\Big(\frac{k}{N}\Big)x_3\Big)\Big|^8 dx_1 dx_2 dx_3 \ll N^{4+\varepsilon}$$

Essential ingredient in their work on $\zeta\left(\frac{1}{2} + it\right)$

THEOREM (**Bombieri-Iwaniec, 86**)

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \ll t^{\frac{9}{56}} \text{ for } |t| \to \infty$$

(later improvement by **Huxley, Kolesnik, Watt**.... using refinements of the method)

EXPONENTIAL SUMS (Bombieri-Iwaniec method)

(**Huxley** Area, lattice points and Exponential Sums, LMSM 1996)

GOAL Obtaining estimates on exponential sum

$$\sum_{m \sim M} e\Big(TF\Big(\frac{m}{M}\Big)\Big)$$

EXAMPLE By approximate functional equation, bounding $\zeta(\frac{1}{2} + iT)$ reduces to sums

$$\sum_{m \sim M} e\Big(T \log \frac{m}{M}\Big) \text{ with } M < T^{1/2}$$

STEP 1 Subdivide $\left[\frac{M}{2}, M\right]$ in shorter intervals I of size N on which $TF(\frac{m}{M})$ can be replaced by cubic polynomial

$\Rightarrow$ cubic exponential sums of the form

$$\sum_{n \leq N} e(a_1 n + a_2 n^2 + \mu n^3)$$

with a_1, a_2, μ depending on I and μ is small

STEP 2 Conversion to new exponential sum by Poisson summation (stationary phase) of the form

$$\sum_{h \leq H} e\Big(b_1 h + b_2 h^2 + b_3 h^{3/2} + b_4 h^{1/2}\Big)$$

with b_1, b_2, b_3, b_4 depending on I.

STEP 3 Understanding the distribution of $b_1(I), b_2(I), b_3(I), b_4(I)$ when I varies (the 'second spacing problem')

STEP 4 Use of the large sieve to reduce to mean value problems of the form

$$A_k(H) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \Big| \sum_{h \sim H} e\Big(x_1 h + x_2 h^2 + x_3 H^{\frac{1}{2}} h^{3/2} + x_4 H^{\frac{1}{2}} h^{1/2}\Big)\Big|^{2k} dx$$

$k = 4, 5, 6$

THEOREM (**B**) Assume $W(\varphi''', \psi''') \neq 0$ and $0 < \delta < \Delta < 1$. Then

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \Big| \sum_{h \sim H} e\Big(hx_1 + h^2 x_2 + \frac{1}{\delta}\varphi\Big(\frac{h}{H}\Big)x_3 + \frac{1}{\Delta}\psi\Big(\frac{h}{H}\Big)x_4\Big)\Big|^{10} dx_1 dx_2 dx_3 dx_4 \ll$$

$$[\delta \Delta^{3/4} H^7 + (\delta + \Delta) H^6 + H^5] H^{\varepsilon}$$

COROLLARY (**Huxley-Kolesnik, 1991**) $A_5(H) \ll H^{5+\varepsilon}$

THEOREM (**Huxley, 05**) $|\zeta(\frac{1}{2} + it)| \ll |t|^{\frac{32}{205}}, \frac{32}{205} = 0,1561\ldots$

PROBLEM (**Huxley**)

Obtain good bound on $A_6(H)$

THEOREM (**B, 014**)

$A_6(H) \ll H^{6+\varepsilon}$

This bound follows from THEOREM ($**$) and is optimal.

Combined with the work of **Huxley** on second spacing problem

COROLLARY

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll |t|^{\frac{53}{342}+\varepsilon}, \frac{53}{342} = 0,1549\dots$$

PROGRESS TOWARDS THE LINDELÖF HYPOTHESIS

$$\mu(\sigma) = \inf\left(\beta > 0; |\zeta(\sigma+it)| = 0\left(|t|^{\beta}\right)\right)$$

Bounds on $\mu(\frac{1}{2})$

$\frac{1}{4}$	**(Lindelöf, 1908)**	$\frac{173}{1067} = 0,16214\ldots$	**(Kolesnik, 1973)**
$\frac{1}{6}$	**(Hardy-Littlewood)**	$\frac{35}{216} = 0,16204$	**(Kolesnik, 1982)**
$\frac{163}{988} = 1,1650\ldots$	**(Walfisz, 1924)**	$\frac{139}{858} = 0,16201\ldots$	**(Kolesnik, 1985)**
$\frac{17}{164} = 0,1647\ldots$	**(Titchmarsh, 1932)**	$\frac{9}{56} = 0,1607\ldots$	**(Bombieri-Iwaniec, 1986)**
$\frac{229}{1392} = 0,164512\ldots$	**(Phillips, 1933)**	$\frac{17}{108} = 0,1574\ldots$	**(Huxley-Kolesnik, 1990)**
$\frac{19}{116} = 0,1638\ldots$	**(Titchmarsh, 1942)**	$\frac{89}{570} = 0,15614\ldots$	**(Huxley, 2002)**
$\frac{15}{92} = 0,1631\ldots$	**(Min, 1949)**	$\frac{32}{205} = 0,15609\ldots$	**(Huxley, 2005)**

SKETCH OF PROOF OF DECOUPLING IN 2D

$(*)$ $K_p(N) \ll C_\varepsilon N^{\gamma+\varepsilon}$ best bound in inequality

$$\|f\|_{L^p_{\#(B_N)}} \leq K_p(N)\Big(\sum_\tau \|f_\tau\|^2_{L^p_\#(B_N)}\Big)^{\frac{1}{2}}$$

for supp $\widehat{f} \subset \Gamma_{\frac{1}{N}}, \tau : \frac{1}{\sqrt{N}} \times \frac{1}{N}$ tube

There is bilinear reduction (B-G argument)

$$\|(|f^1|\,|f^2|)^{\frac{1}{2}}\|_{L^p_\#(B_N)} \leq K_p(N) \prod_{i=1}^{2} \Big(\sum_\tau \|f^i_\tau\|^2_{L^p_\#(B_N)}\Big)^{\frac{1}{4}}$$

if supp $\widehat{f^1} \subset (\Gamma^1)_{\frac{1}{N}}$, supp $\widehat{f^2} \subset (\Gamma^2)_{\frac{1}{N}}$ with $\Gamma^1, \Gamma^2 \subset \Gamma$ separated

By parabolic rescaling, if $M \leq N$ and $\tau' : \frac{1}{\sqrt{M}} \times \frac{1}{M}$ tube

$$\|f_{\tau'}\|_{L^p_\#(B_N)} \leq K_p\Big(\frac{N}{M}\Big)\Big(\sum_{\tau\subset\tau'} \|f_\tau\|^2_{L^p_\#(B_N)}\Big)^{\frac{1}{2}}$$

$(**)$ From bilinear L^4-inequality, for $M \leq \sqrt{N}$, $\tau' = \frac{1}{\sqrt{M}} \times \frac{1}{M}$ tubes

$$\Big\| \prod_{i=1}^{2} \Big(\sum_{\tau'} |f_{\tau'}|^2 \Big)^{\frac{1}{4}} \Big\|_{L^4_\#(B_M)} \leq \|f^1\|_{L^2_\#(B_M)} \|f^2\|_{L^2_\#(B_M)} =$$

$$\prod_i \Big(\sum_{\tau'} \|f^i_{\tau'}\|^2_{L^2_\#(B_M)} \Big)^{\frac{1}{4}}$$

Interpolation with trivial $L^\infty - L^\infty$ bound (using ware packet decomposition)

$$\Big\| \prod_i \Big(\sum_{\tau'} |f^i_{\tau'}|^2 \Big)^{\frac{1}{4}} \Big\|_{L^p_\#(B_M)} \ll M^\varepsilon \prod_i \Big(\sum_{\tau'} \|f^i_{\tau'}\|^2_{L^{p/2}_\#(B_M)} \Big)^{\frac{1}{4}}$$

$$\ll M^\varepsilon \prod_i \Big(\sum_{\tau'} \|f^i_{\tau'}\|^2_{L^p_\#(B_M)} \Big)^{\frac{\kappa}{4}} \prod_i \Big(\sum_{\tau} \|f^i_{\tau}\|^2_{L^2_\#(B_M)} \Big)^{\frac{1-\kappa}{4}}$$

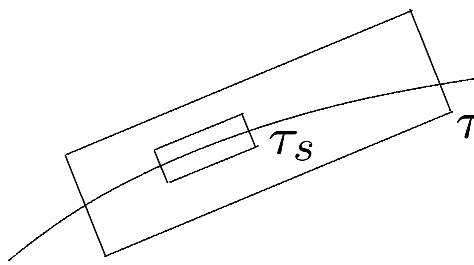
with $\kappa = \frac{p-4}{p-2}$ and $\tau : \frac{1}{M} \times \frac{1}{M^2}$ tubes

$$\ll M^\varepsilon \prod_i \Big(\sum_{\tau'} \|f^i_{\tau'}\|^2_{L^2_\#(B_M)} \Big)^{\frac{\kappa}{4}} \Big\| \prod_i \Big(\sum_{\tau} |f^i_{\tau}|^2 \Big)^{\frac{1}{4}} \Big\|^{1-\kappa}_{L^p_\#(B_M)}$$

Writing B_N as a union of M-balls

$$\Big\| \prod_i \Big(\sum_{\tau'} |f^i_{\tau'}|^2 \Big)^{\frac{1}{4}} \Big\|_{L^P_\#(B_N)} \ll N^\varepsilon \prod_i \Big(\sum_{\tau'} \|f^i_{\tau'}\|^2_{L^p_\#(B_N)} \Big)^{\frac{\kappa}{4}} \Big\| \prod_i \Big(\sum_{\tau} |f^i_{\tau}|^2 \Big)^{\frac{1}{4}} \Big\|^{1-\kappa}_{L^p_\#(B_N)}$$

$(***)$ Iterate, considering coverings $\{\tau_s\}_{1\leq s\leq \bar{s}}, \tau_s : N^{-2^{-s}} \times N^{-2^{-s+1}}$ tubes



$$\Rightarrow K_p(N) \ll N^\varepsilon N^{2^{-\bar{s}-1}} K_p(N^{1-2^{-\bar{s}}})^\kappa\, K_p(N^{1-2^{-\bar{s}+1}})^{\kappa(1-\kappa)} \dots K_p(1)^{(1-\kappa)^{\bar{s}}}$$

$$\Rightarrow \gamma \leq 2^{-\bar{s}-1} + \kappa\gamma \Big\{ \frac{1-(1-\kappa)^{\bar{s}}}{\kappa} - 2^{-\bar{s}} \, \frac{1-(2(1-\kappa))^{\bar{s}}}{2\kappa-1} \Big\} \Big(2(1-\kappa) = \frac{4}{p-2} < 1 \text{ for } p > 6 \Big)$$

$$\Rightarrow \; 0 \leq \frac{1}{2} - \frac{\kappa}{2\kappa-1} \Big(1 - (2(1-\kappa))^{\bar{s}} \Big) \gamma$$

Taking $\bar{s}$ large enough $\Rightarrow \gamma \leq \frac{2\kappa-1}{\kappa} \Rightarrow \gamma = 0$ for $p = 6$