Branching to maximal compact subgroups

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Outline

Something to do during the talk

k_v local field, $G_v = G(k_v)$ reductive, $\mathfrak{g}_v = \mathrm{Lie}(G_v)$.

$\mathfrak{g}_v^* =$ lin fnls on $\mathfrak{g}_v$, $\mathcal{O}_v = G_v \cdot x_v$ coadjt orbit.

$N(\mathcal{O}_v) =_{\mathrm{def}} \overline{k_v \cdot \mathcal{O}_v} \cap \mathcal{N}_v^*$ asymp nilp cone of $\mathcal{O}_v$.

k global, $\pi = \otimes_v \pi_v$ automorphic rep of G reductive.

Conjecture

1. $\exists$ coadjt orbit $G(k) \cdot x \subset \mathfrak{g}(k)^*$, $N(G_v \cdot x) = \mathrm{WF}(\pi_v)$.
2. $\exists$ global version of local char expansions for π_v.

Says $G(k) \cdot x \rightsquigarrow$ asymp of K-types at each place.

$\mathcal{O}_{\overline{k}} =_{\mathrm{def}} G(\overline{k}) \cdot x \rightsquigarrow N(\mathcal{O}_{\overline{k}}) = \overline{\overline{k} \cdot \mathcal{O}_{\overline{k}}} \cap \mathcal{N}_{\overline{k}}^*$

$N(\mathcal{O}_{\overline{k}}) =$ closure of one nilp orbit $\mathcal{M}$.

$N(G_v \cdot x)$ contained in $N(\mathcal{O}_{\overline{k}})$, but may not meet $\mathcal{M}$.

Setting

Compact groups K are relatively easy...

Noncompact groups G are relatively hard.

Harish-Chandra *et al.* idea:

$$\text{understand } \pi \in \widehat{G} \leftsquigarrow \text{understand } \pi|_K$$

(nice compact subgroup $K \subset G$).

Get an invariant of a repn $\pi \in \widehat{G}$:

$$m_\pi \colon \widehat{K} \to \mathbb{N}, \qquad m_\pi(\mu) = \text{mult of } \mu \text{ in } \pi|_K.$$

1. What's the support of m_π? (subset of $\widehat{K}$)
2. What's the rate of growth of m_π?
3. What functions on $\widehat{K}$ can be m_π?

Examples

1. $G = GL(n,\mathbb{C})$, $K = U(n)$. Typical restriction to K is
$$\pi|_K = \mathrm{Ind}_{U(1)^n}^{U(n)}(\gamma) = \sum_{\mu \in \widehat{U(n)}} m_\mu(\gamma)\gamma \quad (\gamma \in \widehat{U(1)^n}) :$$
$m_\pi(\mu)$ = mult of μ is $m_\mu(\gamma)$ = dim of γ wt space.
2. $G = GL(n,\mathbb{R})$, $K = O(n)$. Typical restriction to K is
$$\pi|_K = \mathrm{Ind}_{O(1)^n}^{O(n)}(\gamma) = \sum_{\mu \in \widehat{O(n)}} m_\mu(\gamma) :$$
$m_\pi(\mu)$ = mult of μ in π is $m_\mu(\gamma)$ = mult of γ in μ.
3. G split of type E_8, $K = Spin(16)$. Typical res to K is
$$\pi|_{Spin(16)} = \mathrm{Ind}_M^{Spin(16)}(\gamma) = \sum_{\mu \in \widehat{Spin(16)}} m_\mu(\gamma)\gamma;$$
here $M \subset Spin(16)$ subgp of order 512, cent ext of $(\mathbb{Z}/2\mathbb{Z})^8$.

Moral: may compute m_π using compact groups.

Machinery to use

Roger's approach to these questions:

Roger's results on classical groups

Our approach today:

Use fundamental tools

Ask George and Roman for advice

Get new results on general groups

Plan for today

Work with real reductive Lie group $G(\mathbb{R})$.

Describe (old) associated cycle $\mathcal{AC}(\pi)$ for irr rep $\pi \in \widehat{G(\mathbb{R})}$: geometric shorthand for approximating restriction to $K(\mathbb{R})$ of π.

Describe (new) algorithm for computing $\mathcal{AC}(\pi)$.

A *real* algorithm is one that's been implemented on a computer. This one has not, but should be possible soon.

Assumptions

$G(\mathbb{C}) = G =$ cplx conn reductive alg gp.

$G(\mathbb{R}) =$ group of real points for a real form.

Could allow fin cover of open subgp of $G(\mathbb{R})$, so allow nonlinear.

$K(\mathbb{R}) \subset G(\mathbb{R})$ max cpt subgp; $K(\mathbb{R}) = G(\mathbb{R})^\theta$.

$\theta =$ alg inv of G; $K = G^\theta$ possibly disconn reductive.

Harish-Chandra idea:

∞-diml reps of $G(\mathbb{R}) \leftrightsquigarrow$ alg gp $K \curvearrowright$ cplx Lie alg $\mathfrak{g}$

$(\mathfrak{g}, K)$-module is vector space V with

1. repn π_K of algebraic group K: $V = \sum_{\mu \in \widehat{K}} m_V(\mu)\mu$
2. repn $\pi_\mathfrak{g}$ of cplx Lie algebra $\mathfrak{g}$
3. $d\pi_K = \pi_\mathfrak{g}|_\mathfrak{k}$, $\qquad \pi_K(k)\pi_\mathfrak{g}(X)\pi_K(k^{-1}) = \pi_\mathfrak{g}(\mathrm{Ad}(k)X)$.

In module notation, cond (3) reads $k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v)$.

Geometrizing representations

$G(\mathbb{R})$ real reductive, $K(\mathbb{R})$ max cpt, $\mathfrak{g}(\mathbb{R})$ Lie alg

$\mathcal{N}^* =$ cone of nilpotent elements in $\mathfrak{g}^*$.

$\mathcal{N}^*_{\mathbb{R}} = \mathcal{N}^* \cap i\mathfrak{g}(\mathbb{R})^*$, finite # $G(\mathbb{R})$ orbits.

$\mathcal{N}^*_{\theta} = \mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*$, finite # K orbits.

Goal 1: Attach orbits to representations in theory.

Goal 2: Compute them in practice.

"In theory there is no difference between theory and practice. In practice there is." Jan L. A. van de Snepscheut (or not).

$(\pi, \mathcal{H}_\pi)$ irr rep of $G(\mathbb{R})$	$\mathcal{H}_\pi^K$ irr $(\mathfrak{g}, K)$-module
$\downarrow$ Howe wavefront	$\downarrow$ assoc var of gr
$\mathrm{WF}(\pi) = G(\mathbb{R})$ orbs on $\mathcal{N}^*_{\mathbb{R}}$	$\mathcal{AC}(\pi) = K$ orbits on $\mathcal{N}^*_\theta$

Columns related by HC, Kostant-Rallis, Sekiguchi, Schmid-Vilonen.

So Goal 1 is completed. Turn to Goal 2. . .

Associated varieties

$\mathcal{F}(\mathfrak{g}, K)$ = finite length $(\mathfrak{g}, K)$-modules...

noncommutative world we care about.

$\mathcal{C}(\mathfrak{g}, K) =$ f.g. $(S(\mathfrak{g}/\mathfrak{k}), K)$-modules, support $\subset \mathcal{N}_\theta^*$...

commutative world where geometry can help.

$$\mathcal{F}(\mathfrak{g}, K) \overset{\mathrm{gr}}{\rightsquigarrow} \mathcal{C}(\mathfrak{g}, K)$$

gr not quite a functor (choice of good filts), but

Prop. gr induces surjection of Grothendieck groups

$$K\mathcal{F}(\mathfrak{g}, K) \xrightarrow{\mathrm{gr}} K\mathcal{C}(\mathfrak{g}, K);$$

image records restriction to K of HC module.

So restrictions to K of HC modules sit in equivariant coherent sheaves on nilp cone in $(\mathfrak{g}/\mathfrak{k})^*$

$$K\mathcal{C}(\mathfrak{g}, K) =_{\mathrm{def}} K^K(\mathcal{N}_\theta^*),$$

equivariant K-theory of the K-nilpotent cone.

Goal 2: compute $K^K(\mathcal{N}_\theta^*)$ and the map **Prop.**

Equivariant K-theory

Setting: (complex) algebraic group K acts on (complex) algebraic variety X.

Originally K-theory was about vector bundles, but for us coherent sheaves are more useful.

$\mathrm{Coh}^K(X) =$ abelian categ of coh sheaves on X with K action.

$K^K(X) =_{\mathrm{def}}$ Grothendieck group of $\mathrm{Coh}^K(X)$.

Example: $\mathrm{Coh}^K(\mathrm{pt}) = \mathrm{Rep}(K)$ (fin-diml reps of K).

$K^K(\mathrm{pt}) = R(K) =$ rep ring of K; free $\mathbb{Z}$-module, basis $\widehat{K}$.

Example: $X = K/H$; $\mathrm{Coh}^K(K/H) \simeq \mathrm{Rep}(H)$

$E \in \mathrm{Rep}(H) \rightsquigarrow \mathcal{E} =_{\mathrm{def}} K \times_H E$ eqvt vector bdle on K/H

$K^K(K/H) = R(H)$.

Example: $X = V$ vector space.

$E \in \mathrm{Rep}(K) \rightsquigarrow$ proj module $\mathcal{O}_V(E) =_{\mathrm{def}} \mathcal{O}_V \otimes E \in \mathrm{Coh}^K(X)$

proj resolutions $\implies K^K(V) \simeq R(K)$, basis $\{\mathcal{O}_V(\tau)\}$.

Doing nothing carefully

Suppose $K \curvearrowright X$ with finitely many orbits:

$X = Y_1 \cup \cdots \cup Y_r, \qquad Y_i = K \cdot y_i \simeq K/K^{y_i}.$

Orbits partially ordered by $Y_i \geq Y_j$ if $Y_j \subset \overline{Y_i}$.

$(\tau, E) \in \widehat{K^{y_i}} \rightsquigarrow \mathcal{E}(\tau) \in \mathrm{Coh}^K(Y_i)$.

Choose (always possible) K-eqvt coherent extension

$$\widetilde{\mathcal{E}}(\tau) \in \mathrm{Coh}^K(\overline{Y_i}) \rightsquigarrow [\widetilde{\mathcal{E}}] \in K^K(\overline{Y_i}).$$

Class $[\widetilde{\mathcal{E}}]$ on $\overline{Y}_i$ unique modulo $K^K(\partial Y_i)$.

Set of all $[\widetilde{\mathcal{E}}(\tau)]$ (as Y_i and τ vary) is basis of $K^K(X)$.

Suppose $M \in \mathrm{Coh}^K(X)$; write class of M in this basis

$$[M] = \sum_{i=1}^{r} \sum_{\tau \in \widehat{K^{y_i}}} n_\tau(M)[\widetilde{\mathcal{E}}(\tau)].$$

Maxl orbits in Supp(M) = maxl Y_i with some $n_\tau(M) \neq 0$.

Coeffs $n_\tau(M)$ on maxl Y_i ind of choices of exts $\widetilde{\mathcal{E}}(\tau)$.

Our story so far

We have found

1. homomorphism

$$\text{virt } G(\mathbb{R}) \text{ reps } \; K\mathcal{F}(\mathfrak{g}, K) \xrightarrow{\text{gr}} K^K(\mathcal{N}_\theta^*) \; \text{ eqvt } K\text{-theory}$$

2. geometric basis $\left\{[\widetilde{\mathcal{E}(\tau)}]\right\}$ for $K^K(\mathcal{N}_\theta^*)$, indexed by irr reps of isotropy gps
3. expression of $[\text{gr}(\pi)]$ in geom basis $\rightsquigarrow \mathcal{AC}(\pi)$.

Problem is expressing ourselves...

Teaser for the next section: Kazhdan and Lusztig taught us how to express π using std reps $I(\gamma)$:

$$[\pi] = \sum_\gamma m_\gamma(\pi)[I(\gamma)], \qquad m_\gamma(\pi) \in \mathbb{Z}.$$

$\{[\text{gr } I(\gamma)]\}$ is another basis of $K^K(\mathcal{N}_\theta^*)$.

Last goal is compute change of basis matrix.

The last goal

Studying cone $\mathcal{N}^*_\theta =$ nilp lin functionals on $\mathfrak{g}/\mathfrak{k}$.

Found (for free) basis $\left\{[\widetilde{\mathcal{E}(\tau)}]\right\}$ for $K^K(\mathcal{N}^*_\theta)$, indexed by orbit K/K^i and irr rep τ of K^i.

Found (by rep theory) second basis $\{[\mathrm{gr}\, I(\gamma)]\}$, indexed by (parameters for) std reps of $G(\mathbb{R})$.

To compute associated cycles, enough to write

$$[\mathrm{gr}\, I(\gamma)] = \sum_{\text{orbits}} \sum_{\substack{\tau \text{ irr for} \\ \text{isotropy}}} N_\tau(\gamma)[\widetilde{\mathcal{E}}(\tau)].$$

Equivalent to compute inverse matrix

$$[\widetilde{\mathcal{E}}(\tau)] = \sum_\gamma n_\gamma(\tau)[\mathrm{gr}\, I(\gamma)].$$

Need to relate geom of nilp cone to geom std reps: parabolic subgroups. Use Springer resolution.

Introducing Springer

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ Cartan decomp, $\mathcal{N}_\theta^* \simeq \mathcal{N}_\theta =_{\text{def}} \mathcal{N} \cap \mathfrak{s}$ nilp cone in $\mathfrak{s}$.

Kostant-Rallis, Jacobson-Morozov: nilp $X \in \mathfrak{s} \rightsquigarrow Y \in \mathfrak{s}$, $H \in \mathfrak{k}$

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H,$$

$$\mathfrak{g}[k] = \mathfrak{k}[k] \oplus \mathfrak{s}[k] \qquad (\text{ad}(H) \text{ eigenspace}).$$

$\rightsquigarrow \mathfrak{g}[\geq 0] =_{\text{def}} \mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ $\quad \theta$-stable parabolic.

Theorem (Kostant-Rallis) Write $\mathcal{O} = K \cdot X \subset \mathcal{N}_\theta$.

1. $\mu \colon \mathcal{O}_Q =_{\text{def}} K \times_{Q \cap K} \mathfrak{s}[\geq 2] \to \overline{\mathcal{O}}$, $\quad (k, Z) \mapsto \text{Ad}(k)Z$ is proper birational map onto $\overline{\mathcal{O}}$.
2. $K^X = (Q \cap K)^X = (L \cap K)^X (U \cap K)^X$ is a Levi decomp; so $\widehat{K^X} = [(L \cap K)^X]\widehat{\ }$.

So have resolution of singularities of $\overline{\mathcal{O}}$:

$$\begin{array}{ccc} & K \times_{Q\cap K} \mathfrak{s}[\geq 2] & \\ {\scriptstyle \text{vec bdle}} \swarrow & & \searrow {\scriptstyle \mu} \\ K/Q \cap K & & \overline{\mathcal{O}} \end{array}$$

Use it (*i.e.*, copy McGovern, Achar) to calculate equivariant K-theory. . .

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Using Springer to calculate K-theory

$X \in \mathcal{N}_\theta$ represents $\mathcal{O} = K \cdot X$.

$\mu\colon \mathcal{O}_Q =_{\mathrm{def}} K \times_{Q \cap K} \mathfrak{s}[\geq 2] \to \overline{\mathcal{O}}$ Springer resolution.

Theorem Recall $\widehat{K^X} = [(L \cap K)^X]^{\frown}$.

1. $K^K(\mathcal{O}_Q)$ has basis of eqvt vec bdles:
$$(\sigma, F) \in \mathrm{Rep}(L \cap K) \rightsquigarrow \mathcal{F}(\sigma).$$
2. Get extension of $\mathcal{E}(\sigma|_{(L\cap K)^X})$ on $\mathcal{O}$
$$[\overline{\mathcal{F}}(\sigma)] =_{\mathrm{def}} \sum_i (-1)^i [R^i \mu_*(\mathcal{F}(\sigma))] \in K^K(\overline{\mathcal{O}}).$$
3. Compute (very easily) $[\overline{\mathcal{F}}(\sigma)] = \sum_\gamma n_\gamma(\sigma)[\mathrm{gr}\, I(\gamma)]$.
4. Each irr $\tau \in [(L \cap K)^X]^{\frown}$ extends to (virtual) rep $\sigma(\tau)$ of $L \cap K$; can choose $\overline{\mathcal{F}(\sigma(\tau))}$ as extension of $\mathcal{E}(\tau)$.

Now we're done

Recall $X \in \mathcal{N}_\theta \rightsquigarrow \mathcal{O} = K \cdot X$; $\tau \in [(L \cap K)^X]^\frown$.

Now we know formulas

$$[\widetilde{\mathcal{E}}(\tau)] = [\overline{\mathcal{F}(\sigma(\tau))}] = \sum_\gamma n_\gamma(\tau)[\mathrm{gr}\, I(\gamma)].$$

Here's why this does what we want:

1. inverting matrix $n_\gamma(\tau) \rightsquigarrow$ matrix $N_\tau(\gamma)$ writing $[\widetilde{\mathcal{E}}(\tau)]$ in terms of $[\mathrm{gr}\, I(\gamma)]$.
2. multiplying $N_\tau(\gamma)$ by Kazhdan-Lusztig matrix $m_\gamma(\pi)$ $\rightsquigarrow$ matrix $n_\tau(\pi)$ writing $[\mathrm{gr}\, \pi]$ in terms of $[\widetilde{\mathcal{E}}(\tau)]$.
3. Nonzero entries $n_\tau(\pi) \rightsquigarrow \mathcal{AC}(\pi)$.

Side benefit: algorithm (for $G(\mathbb{R})$ cplx) also computes bijection (conj by Lusztig, estab by Bezrukavnikov)

$$\text{(dom wts)} \leftrightarrow \text{(pairs } (\tau, \mathcal{O}))$$

Mirror, mirror, on the wall

Who's the fairest one of all?

The winner and still champion!

HAPPY
BIRTHDAY
ROGER!