Automatic convergence for modular forms

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Siegel modular forms

Symplectic group

Let $n \geq 1$ be an integer and set $J_n = \left(\begin{smallmatrix} 0_n & 1_n \\ -1_n & 0_n \end{smallmatrix}\right)$. One sets

$$\mathrm{Sp}_{2n} := \{g \in \mathrm{GL}_{2n} : g^t J_n g = J_n\}.$$

Siegel upper half space

Let

$$\mathcal{H}_n = \{Z = X + iY \in M_n(\mathbf{C}) : Z^t = Z, Y > 0\}.$$

If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbf{R})$ and $Z \in \mathcal{H}_n$, set

$$g \cdot Z = (aZ + b)(cZ + d)^{-1}.$$

This defines an action $\mathrm{Sp}_{2n}(\mathbf{R}) \times \mathcal{H}_n \to \mathcal{H}_n$.

Set $j(g, Z) = \det(cZ + d)$.

Fourier expansion

Siegel modular forms

Fix $\ell \in \mathbf{Z}_{\geq 0}$ and $\Gamma \subseteq \mathrm{Sp}_{2n}(\mathbf{R})$ arithmetic subgroup. A function $f : \mathcal{H}_n \to \mathbf{C}$ is said to be a **Siegel modular form** of weight ℓ and level Γ is f is holomorphic, of moderate growth, and $f(\gamma Z) = j(\gamma, Z)^\ell f(Z)$ for all $\gamma \in \Gamma$.

Suppose for simplicity that $\Gamma = \mathrm{Sp}_{2n}(\mathbf{Z})$.

Fourier coefficients

If f is a Siegel modular form, then

$$f(Z) = \sum_{T \geq 0} a_f(T) e^{2\pi i \,\mathrm{tr}(TZ)},$$

for some Fourier coefficients $a_f(T) \in \mathbf{C}$. The sum ranges over T half-integral and symmetric.

Fourier and Fourier-Jacobi coefficients

Suppose f is a Siegel modular form with Fourier coefficients $a_f(T)$.

Basic fact

If $\gamma \in \mathrm{SL}_n(\mathbf{Z})$, then $a_f(\gamma^t T \gamma) = a_f(T)$.

Let $n = 2$ for simplicity. Fix integers $m \geq 1$ and μ.

Definition

Suppose f is a Siegel modular form. Define $h_{m,\mu}(D) = a_f(T)$ if $T = \left(\begin{smallmatrix} m & \mu/2 \\ \mu/2 & * \end{smallmatrix}\right)$ and $4\det(T) = D$.

Proposition 1

Notation as above. Then, for every fixed μ,

$$\sum_{D \geq 0} h_{m,\mu}(D) q^D \in M_{\ell - 1/2}.$$

(The level depends explicitly upon m.)

Formal modular forms

Names in alphabetical order, some in collaboration with others, and some with multiplicity > 1:

Definition: Aoki, Bruinier, Ibukiyama, Pollack, Poor, Raum, Yuen

A function $A : 2^{-1}M_2(\mathbf{Z}) \to \mathbf{C}$ is said be a **formal modular form** if it satisfies the following conditions:

1. $A(T) \neq 0$ implies $T \geq 0$;
2. $A(\gamma^t T\gamma) = a(T)$ for all $\gamma \in \mathrm{SL}_2(\mathbf{Z})$;
3. For every m, μ, define $h_{m,\mu}(D)$ as above in terms of $A(T)$. Then
$$\sum_{D\geq 0} h_{m,\mu}(D)q^D$$
is the Fourier expansion of a modular form in $M_{\ell-1/2}(\Gamma_0(4m))$.

Sending a Siegel modular form to its Fourier coefficients defines an injection

$$M_\ell(\Gamma) \to M_\ell^f(\Gamma)$$

Automatic convergence

Authors on previous slide, in differing degrees of generality:

Theorem 2

The injection $M_\ell(\Gamma) \to M_\ell^f(\Gamma)$ is an isomorphism. In particular, if A is a formal modular form, then $|A(T)|$ grows polynomially with T.

Convergence is the whole point

Suppose A is a formal modular form. **Once one knows** that the $A(T)$ grow slowly, then one can define $f(Z) = \sum_T A(T) e^{2\pi i \,\mathrm{tr}(TZ)}$. The sum converges to a holomorphic function on $\mathcal{H}_2$. The conditions satisfied by A imply that f is a Siegel modular form.

Automatic convergence is an algebraicity result

One way of interpreting "automatic convergence" theorems is that they prove that (certain types of) modular forms can be defined algebraically, modulo modular forms on lower rank groups.

Some context

Geometric Kudla conjecture

This result was studied by Aoki, Ibukiyama-Poor-Yuen and Aoki-Ibukiyama-Poor in the context of computing Siegel modular forms, and by Bruinier, Raum in the context of the geometric Kudla conjecture.

The proof of the the above theorems by Aoki-Ibukiyama-Poor and Bruinier-Raum

1. uses somewhat delicate complex analysis on the tube domains;
2. the ring structure of Siegel modular forms.

Earlier special cases used exact dimension formulas for certain spaces of modular forms.

Proof sketch

Proof: (Argument of P).

1. Suppose A is a cuspidal formal modular form, i.e., $A(T) \neq 0$ implies $T > 0$.
2. **Induct** on floor of $\log(\log(\det(T)))$.
3. Suppose $\ell^2 \cdot 2^{2^r} \leq \det(T) \leq \ell^2 \cdot 2^{2^{r+1}}$, and one has bound on $|A(S)|$ for all S with $\det(S) \leq \ell^2 \cdot 2^{2^r}$.
4. **Reduction theory**: There exists $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ so that if $T' = \gamma^t T \gamma$, then $m' := m(T') \leq \det(T)^{1/2} \leq \ell \cdot 2^{2^r}$.
5. The number $A(T) = A(T')$ appears as a Fourier coefficient of $f_{m'} \in M_{\ell-1/2}(\Gamma_0(4m'))$.
6. By a **Sturm bound**, the modular form $f_{m'}$ is determined by the Fourier coefficients $h_{m',\mu}(D)$ with $D \leq \ell \cdot m' \leq \ell^2 \cdot 2^{2^r}$.
7. The $h_{m',\mu}(D)$'s are $A(S)$ for some S. One can **propagate** the bound on the $A(S)$ to a bound on $A(T)$.

□

Proof of quantitative Sturm bound

Proposition 3

Suppose a weight ℓ_1 is fixed. There are constants $A, B, D > 0$ so that $f \in S_{\ell_1}(\mathrm{Mp}_2, K(m))$ has Fourier coefficients c_n with $|c_n(k)| \leq \epsilon n^{\ell_1/2}$ for all $n \leq A\log(m)$, and $k \in K_f = \mathrm{Mp}_2(\widehat{\mathbf{Z}})$, then $|c_n(g_f)| \leq B \cdot m^D \cdot \epsilon \cdot n^{\ell_1/2}$ for all n and $g_f \in \mathrm{Mp}_2(\mathbf{A}_f)$.

Proof sketch

1. Let $\varphi \in \mathcal{A}_0(\mathrm{Mp}_2)$ be the automorphic form corresponding to f and $||\varphi||$ its supremum.
2. By reduction theory, there is $z_1 \in \mathcal{F} \subseteq \mathfrak{h}$ in the standard fundamental domain and $k \in K_f$ so that $|\varphi(kz_1)| = ||\varphi||$.
3. Have
$$||\varphi|| = |\varphi(z_1 k)| \leq \sum_{n>0} |c_n(k)| y_1^{\ell_1/2} e^{-2\pi y_1 n}.$$

Proof continued

Proof sketch continued

4. One has $|c_n(g_f)| \leq ||\varphi|| \cdot n^{\ell_1/2}$ for all $n > 0$ all $g_f \in \mathrm{Mp}_2(\mathbf{A}_f)$ and $|c_n(k)| \leq \epsilon n^{\ell_1/2}$ for n small.
5. Thus $||\varphi||$ is bounded above by

$$\epsilon \left(\sum_{n \text{ small}} (ny_1)^{\ell_1/2} e^{-2\pi y_1 n} \right) + ||\varphi|| \left(\sum_{n \text{ large}} (ny_1)^{\ell_1/2} e^{-2\pi y_1 n} \right).$$

6. Using that y_1 is bounded below, one can write

$$||\varphi|| \leq \epsilon \cdot S_1(m) + ||\varphi|| \cdot S_2(m)$$

where $S_1(m)$ and $S_2(m)$ are explicit. Then

$$||\varphi|| \leq \frac{\epsilon \cdot S_1(m)}{1 - S_2(m)} \quad \text{if } 1 - S_2(m) > 0.$$

Some other work

1. Work of **Xia**, and then recently **Raum**, generalizing automatic convergence to quasisplit unitary groups that have tube structure
2. There has been work of **Bruinier-Raum**, and then **Marco Flores**, **Haocheng Fan**, investigating automatic convergence from the point of view of the coherent cohomological dimension of Shimura varieties (i.e., the maximal d so that $H^d(X, \mathcal{F}) \neq 0$ for some quasicoherent sheaf $\mathcal{F}$ on a Shimura variety X.)
3. There has been some work of **Haowu Wang and Brandon Williams** on automatic convergence for finite list of orthogonal groups.
4. Relatively recent work of **Aoki-Ibukiyama-Poor** for paramodular forms on Sp(4): If the Siegel modular form has paramodular level, and is an eigenvector for a Fricke involution, then one only needs to assume modularity of the Fourier-Jacobi coefficients at a single cusp.

Fourier coefficient symmetries

Familiar fact about Fourier coefficient of automorphic form

Setup

Suppose

1. G is a reductive group over $\mathbf{Q}$,
2. $P \subseteq G$ is a parabolic subgroup with $P = MN$
3. $\chi \in \widehat{N}$, $\chi : N(\mathbf{Q})\backslash N(\mathbf{A}) \to \mathbf{C}^\times$ is an automorphic character
4. φ is an automorphic form on G
5. $\gamma \in M(\mathbf{Q})$

One sets $\chi \cdot \gamma \in \widehat{N}$ as $(\chi \cdot \gamma)(n) = \chi(\gamma n \gamma^{-1})$.

The χ Fourier coefficient of φ is

$$\varphi_\chi(g) := \int_{[N]} \chi^{-1}(n)\varphi(ng)\,dn.$$

If $\chi \in M(\mathbf{Q})$, then

$$\varphi_\chi(\gamma g) = \varphi_{\chi\cdot\gamma}(g).$$

Fourier-Jacobi coefficients

Setup

1. $Q \subseteq G$ is another parabolic subgroup, $Q = LV$
2. The unipotent radical V is two-step, $V \supseteq Z \supseteq 0$
3. $\psi : Z(\mathbf{Q})\backslash Z(\mathbf{A}) \to \mathbf{C}^\times$ is a non-degenerate character
4. $V/Z = X \oplus Y$ is a Lagrangian decomposition for the adjoint action of L
5. $Q' = L'V$ is the stabilizer of ψ
6. $\phi \in S(X(\mathbf{A}))$ is a Schwartz-Bruhat function

One defines a theta function. For $h \in Q'(\mathbf{A})$ or a double cover,

$$\theta_\phi(h) = \sum_{\xi \in X(\mathbf{Q})} (\omega_\psi(h)\phi)(\xi).$$

For $g \in L'(\mathbf{A})$, the Fourier-Jacobi coefficient is

$$\mathrm{FJ}(\varphi; \phi, \psi)(g) := \int_{[V]} \theta_\phi(vg)\varphi(vg)\, dv.$$

It is an automorphic form on L'.

To generalize

Suppose J is one of the following

1. $J = H_n(\mathbf{Q}) = \mathrm{Sym}^2(\mathbf{Q}^n)$; $G = \mathrm{Sp}_{2n}$.
2. $J = H_n(K)$ with K an imaginary quadratic field; $G = U(n,n)$.
3. $J = H_n(B)$ with B a quaterion algebra ramified at ∞; $G = \mathrm{SO}^*(4n)$;
4. $J = H \oplus V_0$ with $H = \mathbf{Q}^2$ a hyperbolic plane and V_0 a negative-definite rational quadratic space; $G = \mathrm{SO}(2, n+2)$.
5. $J = H_3(\mathbb{O})$ with $\mathbb{O}$ the unique octonion division algebra over $\mathbf{Q}$; $G = E_7$.

Tube domain

For J as above, let $\mathcal{H} = \{X + iY : X, Y \in J(\mathbf{R}), Y > 0\}$. Then $\mathcal{H}$ is the symmetric space for $G(\mathbf{R})$.

There is a completely analogous notion of Fourier coefficients and formal modular forms for all of the pairs G, J above.

Theorem statement

Theorem 4 (Automatic convergence)

Suppose the rank n of G is at least two. Then the natural map $M_\ell(G,U) \to M_\ell^f(G,U)$ sending a holomorphic modular form φ of weight ℓ to its Fourier coefficients is a $\mathbf{C}$-linear isomorphism.

Suppose $a \in M_\ell^f(G,U)$. As soon as one knows a growth property for the $a(T)$, one can write down the inverse map $M_\ell^f(G,U) \to M_\ell(G,U)$ as

$$a(\bullet) \mapsto F, \quad F(Z) = \sum_{T \geq 0} a(T) e^{2\pi i (T,Z)}.$$

Work in progress

Generalizing further the theorem above to arbitrary J over a totally real field, corresponding to groups with tube domains, and rank at least 2.

Quaternionic modular forms

Non-holomorphic modular forms

The exceptional groups $G_2, F_{4,4}, E_{6,4}, E_{7,4}, E_{8,4}$, and the classical groups $\mathrm{SO}(4,n)$, $n \geq 3$, have a class of **non-holomorphic** automorphic forms called **quaternionic modular forms**. These automorphic forms have a semi-classical notion of Fourier coefficients, like the holomorphic modular forms.

One has

$$G_2 \subseteq \mathrm{Spin}(4,3) \subseteq \mathrm{Spin}(4,4) \subseteq F_4 \subseteq E_{6,4} \subseteq E_{7,4} \subseteq E_{8,4}.$$

- We write G for a semisimple group over $\mathbf{Q}$ with $G(\mathbf{R})$ one of the above Lie groups.
- Let $K \subseteq G(\mathbf{R})$ be a maximal compact subgroup. One has $K = (\mathrm{SU}(2) \times L)/\Delta(\mu_2)$ for some compact group L. E.g., if $G = F_4$, $L = \mathrm{Sp}(6)$.

Quaternionic modular forms (QMF)

Definition

Let $\ell \geq 1$ be an integer. Set $\mathbf{V}_\ell = Sym^{2\ell}(\mathbf{C}^2)$ as a representation of $K \to \mathrm{SU}(2)/\mu_2 = \mathrm{SO}(3)$. A **quaternionic modular form** of weight ℓ is an automorphic form $\varphi \in \mathcal{A}(G) \otimes \mathbf{V}_\ell$ satisfying

1. $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $k \in K$
2. φ satisfies a particular special linear differential equation, $D_\ell \varphi \equiv 0$

- The definition (i.e., what is a good differential equation to require) goes back to work of Gross-Wallach and Gan-Gross-Savin
- If $\iota : H_1 \subseteq H_2$ is an inclusion of quaternionic exceptional groups as above, and φ is a QMF on H_2, then its pullback $\iota^*(\varphi)$ is a QMF on H_1.

The Fourier expansion of quaternionic modular forms

Fix $\ell \geq 1$ an integer, G a quaternionic exceptional group, W a certain rational vector space depending on G.

Theorem 5 (Wallach, P.)

For every $\lambda \in W \otimes \mathbf{R}$, there is a completely explicit function $\mathcal{W}_\lambda(g) : G \to \mathbf{V}_\ell$ satisfying

1. $\mathcal{W}_\lambda(gk) = k^{-1}\mathcal{W}_\lambda(g)$ *for all* $k \in K$
2. $D_\ell \mathcal{W}_\lambda(g) = 0$.

If φ is a cuspidal modular form of weight ℓ, then φ has an expansion in terms of the $\mathcal{W}_\lambda(g)$: If $g_f \in G(\mathbf{A}_f)$ and $g \in G(\mathbf{R})$ then

$$\varphi(g_f g) \text{``}=\text{''} \sum_{\lambda \in W} a_\varphi(\lambda; g_f)\mathcal{W}_\lambda(g).$$

- The locally constant functions $a_\varphi(\lambda) : G(\mathbf{A}_f) \to \mathbf{C}$ are called the Fourier coefficients of φ

Notation

Suppose $G \in \{F_{4,4}, E_{6,4}, E_{7,4}, E_{8,4}\}$. If $R \subseteq \mathbf{C}$, let $S_\ell(G, U; R)$ denote the cuspidal quaternionic modular forms of weight ℓ on G of level $U \subseteq G(\mathbf{A}_f)$, with all Fourier coefficients in R.

Theorem 6

Suppose $G \in \{F_{4,4}, E_{6,4}, E_{7,4}, E_{8,4}\}$ has real and rational rank equal to 4. Then $S_\ell(G, U; \mathbf{C}) = S_\ell(G, U; \overline{\mathbf{Q}}) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$.

One defines a notion of formal modular forms on G, and proves $S_\ell(G, U) \to S_\ell^f(G, U)$ is an isomorphism via automatic convergence.

Proof sketch

1. The groups in the theorem have rational rank four.
2. One of the maximal parabolic subgroups, the Heisenberg parabolic P, gives rise to the Fourier coefficients $a_{\varphi}(\lambda)$.
3. Using two of the other conjugacy classes of maximal parabolic subgroups, Q and R, one can define a notion of Fourier-Jacobi coefficients of $\varphi \in S_{\ell}(G, U)$.
4. The Fourier-Jacobi coefficients corresponding to Q are automorphic forms on Mp_2. One proves they are holomorphic modular forms, and calculates their Fourier coefficients in terms of the $a_{\varphi}(\lambda)(g_f)$.
5. The Fourier-Jacobi coefficients corresponding to R are automorphic forms on a group isogenous to $\mathrm{SO}(2, n+2)$, n depending on G. One proves that they too are holomorphic modular forms, and calculates their Fourier coefficients in terms of the $a_{\varphi}(\lambda)(g_f)$.

Formal modular forms

Proof sketch: continued

6. One then can define formal modular forms on G as collections of locally constant functions $a_\lambda : G(\mathbf{A}_f) \to \mathbf{C}$ that satisfy all the algebraic relations imposed by the bullets on the previous slide, called P, Q, and R symmetries.
7. One then proves an automatic convergence theorem for these formal modular forms.

No multiplication

There is no multiplication on the spaces of quaternionic modular forms. Consequently, one cannot deduce the general case from the cuspidal case as was done in the case of holomorphic modular forms.

Thank you

Thank you for your attention!