

Iwasawa Main Conjectures and Arithmetic Applications

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The Gross—Zagier Formula, 40 Years Later (MIT)

The Birch and Swinnerton-Dyer conjecture

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Moreover,

$$\frac{L^{(r_{\text{an}})}(E, 1)}{r_{\text{an}}! \text{Reg}(E) \cdot \Omega_E} = \frac{\#\text{III}(E)\text{Tam}(E)}{\#E(\mathbb{Q})_{\text{tors}}^2}.$$

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Recall for any p prime:

$$0 \rightarrow E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \text{Sel}_{p^\infty} \rightarrow \text{III}(E)[p^\infty] \rightarrow 0$$

Iwasawa theory

Let F be a number field (e.g. $F = \mathbb{Q}, K$ quadratic imaginary, ...)

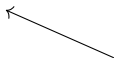
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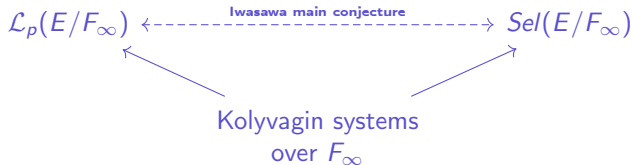
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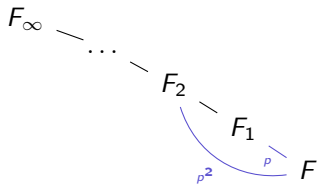
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Iwasawa theory and BSD: rank zero

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$$\begin{array}{ccc}
 \text{III}(E/\mathbb{Q})[p^\infty] & \xleftrightarrow{\text{BSD conjecture}} & L(E/\mathbb{Q}, s) \\
 \uparrow \text{control} & & \uparrow \text{ } \\
 \text{theorems} & & \mathcal{L}(E/\mathbb{Q}_\infty)(0) \sim \\
 & & \mathcal{L}(E/\mathbb{Q}, 1)_p \\
 \\
 \text{Sel}_\Lambda^{\text{ord}} & \xleftrightarrow{\text{Mazur's cyclotomic}} & \mathcal{L}(E/\mathbb{Q}_\infty) \in \Lambda \\
 \text{main conjecture} & &
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- IMC: $\text{Char}((\text{Sel}_\Lambda^{\text{ord}})^\vee) = (\mathcal{L}(E/\mathbb{Q}_\infty)) \subset \Lambda$;
- control theorem: $\text{ord}_p(f(0)) \sim \text{ord}_p(\#\text{III}(E/\mathbb{Q})[p^\infty])$;

Iwasawa theory and BSD: rank one

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 $\Gamma = \text{Gal}(K_\infty/K)$ anticyclotomic, $p = v \cdot \bar{v}$

$$\begin{array}{ccc}
 \text{III}(E/K)[p^\infty] & \xleftrightarrow{\text{BSD conjecture}} & L(E/K, s) \\
 \uparrow \text{control} & & \uparrow \text{Gross-Zagier} \\
 \text{theorems} & & \text{formulae} \\
 \text{Sel}_\Lambda^{\text{str,rel}} & \xleftrightarrow{\text{Greenberg's anticyclotomic}} & \mathcal{L}(E/K_\infty) \\
 & \text{main conjecture} &
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- $\text{ord}_p(f(0)) \sim \text{ord}_p(\#\text{III}(E/K)[p^\infty]) \cdot \text{ord}_p\left(\frac{\log_{\omega_E} Q}{[E(K):\mathbb{Z} \cdot P_K]^2}\right);$
- Gross-Zagier formula: $L'(E/K, 1) \sim [E(K) : \mathbb{Z} \cdot P_K]^2;$
 p -adic GZ (BDP): $\mathcal{L}(E/K_\infty)(0) \sim \log_{\omega_E}(P_K)$

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- ⊆ • converse divisibility using Eisenstein congruences for larger groups
 - (in the residually reducible case) use of “smaller rank” IMCs

Some (old) results: IMCs for Eisenstein primes

Theorem (Castella-G.-Lee-Skinner, Castella-G.-Skinner)

*Let p be an odd prime of good ordinary reduction such that E has a rational p -isogeny whose kernel $\mathbb{F}_p(\phi)$ satisfies $\phi|_{G_{\mathbb{Q}_p}} \neq 1, \chi_{\text{cyc}},$
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- *(anticyclotomic IMC) $\text{Sel}_{\Lambda_{K_\infty}}^{\text{str,rel}}$ is Λ_{K_∞} -torsion and*

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- (cyclotomic IMC) $\text{Sel}_{\Lambda_{\mathbb{Q}_\infty}}^{\text{ord}}$ is $\Lambda_{\mathbb{Q}_\infty}$ -torsion and

$$\text{char}(\text{Sel}_{\Lambda_{\mathbb{Q}_\infty}}^{\text{ord}}) = (\mathcal{L}^{\text{MSD}}(E/\mathbb{Q}_\infty)).$$

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Theorem (Gross–Zagier '86, Kolyvagin '89)

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- see also: Bertolini–Darmon '05, W. Zhang '14, Venerucci '16, Bertì–Bertolini–Venerucci '17

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- The Iwasawa main conjecture away from $T = 0$ (over K_∞ and $\mathbb{Q}_\infty$ respectively) is one of the key ingredients in the proof.

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- $\mathbb{Q}_\infty$ Kato, Skinner-Urban

- K_∞ Generalised Heegner cycles on Kuga-Sato varieties

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- Daniel’s talk: $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$, quasi-split unitary (Disegni, Disegni-Zhang, Disegni-Liu, Li-Liu)

How to prove the main conjecture(s): Eisenstein case

$$\text{Char}(Sel_{\Lambda}^{\vee}) \stackrel{?}{=} (\mathcal{L}(E))$$

- existence of Λ -adic Euler systems (Kato's ES, Heegner points)
 - with explicit reciprocity law
- converse divisibility using Eisenstein congruences for larger groups
 - (in the residually reducible case) use of “smaller rank” IMCs

How to *use* “smaller rank” IMCs

$$(\mathcal{F}(E/F_\infty)) \stackrel{?}{=} (\mathcal{L}(E/F_\infty)) \subset \Lambda_{F_\infty}$$

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$$\begin{array}{ccccc}
 & & H^1_{\text{ord}}(K_v)/(\mathcal{L}_1) & \longrightarrow & \text{Sel}_{\text{Gr}} \\
 & \nearrow \text{loc}_v & & & \searrow \\
 0 \rightarrow H^1(K)/(\mathcal{Z}^+) & & & & \text{Sel}_{\text{ord, str}} \rightarrow 0 \\
 & \searrow \text{loc}_{\bar{v}} & & & \nearrow \\
 & & H^1_{/\text{ord}}(K_{\bar{v}})/(\mathcal{L}_2) & \longrightarrow & \text{Sel}_{\text{ord}}
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Beilinson–Flach classes
[BDR,LLZ,KLZ]

The diagram illustrates the Rankin-Selberg main conjecture. It features two terms at the top: $L(f \otimes g, s)$ on the left and $Sel_{BK}(T^*)$ on the right. A horizontal dashed arrow labeled "Bloch–Kato" points from the Selmer group to the L-function. Below this, the text "Beilinson–Flach classes [BDR,LLZ,KLZ]" is centered. Two blue arrows originate from this text: one points to the L-function and the other points to the Selmer group, indicating that these classes provide a map from the Selmer group to the L-function.

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Moreover, Iwasawa main conjecture over $\mathbb{Q}_\infty$ involving Hida's Rankin–Selberg p -adic L -function $\mathcal{L}_p(f \otimes g)$.

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$$L(f \otimes g, s) \overset{\text{Bloch-Kato}}{\longleftrightarrow} \text{Sel}_{\text{BK}}(T^*)$$

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- Opposite divisibility from congruences by Hsieh–Z. Liu (large image and $\epsilon_f \epsilon_g = 1$)

Rankin–Selberg and smaller-rank IMCs

Assume $p \geq 5$, ordinary and p -distinguished hypotheses at p , and suppose that g is of weight 2 and residually Eisenstein:

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(and corresponding congruence for $\text{char}_\Lambda(\text{Sel}_{\text{Gr}}^\vee)$) hold when $k \geq 2$

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$$\mathcal{L}_p^\Sigma(f \otimes g) \equiv \mathcal{L}_p^\Sigma(f \otimes \psi) \cdot \mathcal{L}_p^\Sigma(f \otimes \psi^{-1}\chi_{\text{cyc}}) \pmod{p}$$

(and corresponding congruence for $\text{char}_\Lambda(\text{Sel}_{\text{Gr}}^\vee)$) hold when $k \geq 2$

- Assume *twisted* IMC holds (Kato $\checkmark$)

Rankin–Selberg and smaller-rank IMCs

Assume $p \geq 5$, ordinary and p -distinguished hypotheses at p , and suppose that g is of weight 2 and residually Eisenstein:

$$T_g/pT_g \simeq \begin{pmatrix} \psi & * \\ 0 & \psi^{-1}\chi_{\text{cyc}} \end{pmatrix}.$$

Then

$$(T_f \otimes T_g)^{\text{ss}} \equiv (T_f \otimes \psi) \oplus (T_f \otimes \psi^{-1}\chi_{\text{cyc}}) \pmod{p}.$$

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- Assume vanishing of μ -invariants

Rankin–Selberg and smaller-rank IMCs

Recall strategy:

$$\checkmark \begin{cases} \text{congruences} \pmod{p} \\ \text{GL}_2\text{-IMC and } \mu = 0 \end{cases} \Rightarrow \text{IMC mod } p \text{ (Jha-Shekhar-Vangala)}.$$

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$$\text{in progress: } \begin{cases} \text{ES divisibility argument} \\ \text{explicit reciprocity law and global duality} \end{cases} \Rightarrow \begin{matrix} \text{divisibility} \\ \text{in } \Lambda[\frac{1}{p}] \end{matrix}$$

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the cyclotomic IMC for $E \otimes \rho_f$ holds.

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- HOPE: use p -adic variation to achieve $\text{wt}(f) < \text{wt}(g)$, especially weight $(1,2)$