Iwasawa Main Conjectures and Arithmetic Applications

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The Gross—Zagier Formula, 40 Years Later (MIT)

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Recall for any p prime:

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to Sel_{p^\infty} \to \text{Ш}(E)[p^\infty] \to 0$$

Let F be a number field (e.g. $F = \mathbb{Q}, K$ quadratic imaginary, ...)

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Iwasawa theory

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$$\begin{tikzcd}
\mathcal{L}_p(E/F_\infty) \arrow[rr, leftrightarrow, dashed, "\text{Iwasawa main conjecture}"] & & Sel(E/F_\infty) \\
& \begin{array}{c}\text{Kolyvagin systems} \\ \text{over } F_\infty\end{array} \arrow[ul] \arrow[ur] &
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Sel_{p^\infty} \arrow[r, leftrightarrow, "\text{BSD conjecture}"] & L(E/F,s) \\
Sel_\Lambda \arrow[u, "\substack{\text{control}\\ \text{theorems}}"] \arrow[r, leftrightarrow, "\substack{\text{Iwasawa main}\\ \text{conjecture}}"] & \mathcal{L}(E) \in \Lambda
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- $\boxed{\textbf{Iwasawa main conjecture}\text{: } (f(T)) = (\mathcal{L}(E)) \subset \Lambda}$;
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Iwasawa theory and BSD: rank zero

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$$\begin{tikzcd}
Ш(E/\mathbb{Q})[p^\infty] \arrow[rr, leftrightarrow, "\text{BSD conjecture}"] & & L(E/\mathbb{Q}, s) \\
Sel_\Lambda^{\mathrm{ord}} \arrow[u, "\substack{\text{control}\\ \text{theorems}}"] \arrow[rr, leftrightarrow, "\substack{\text{Mazur's cyclotomic}\\ \text{main conjecture}}"] & & \mathcal{L}(E/\mathbb{Q}_\infty) \in \Lambda \arrow[u, "\substack{\mathcal{L}(E/\mathbb{Q}_\infty)(\mathbf{0}) \sim\\ L(E/\mathbb{Q},\mathbf{1})_p}"']
\end{tikzcd}$$

- IMC: $\mathrm{Char}((Sel_\Lambda^{\mathrm{ord}})^\vee) = (\mathcal{L}(E/\mathbb{Q}_\infty)) \subset \Lambda$;
- control theorem: $\mathrm{ord}_p(f(0)) \sim \mathrm{ord}_p(\#Ш(E/\mathbb{Q})[p^\infty])$;

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$\Gamma = Gal(K_\infty/K)$ anticyclotomic, $p = v \cdot \bar{v}$

$$\begin{tikzcd}
Ш(E/K)[p^\infty] \arrow[rr, leftrightarrow, "\text{BSD conjecture}"] & & L(E/K,s) \\
Sel_\Lambda^{\mathrm{str,rel}} \arrow[u, "\substack{\text{control}\\\text{theorems}}"] \arrow[rr, leftrightarrow, "\substack{\text{Greenberg's anticyclotomic}\\\text{main conjecture}}"] & & \mathcal{L}(E/K_\infty) \arrow[u, "\substack{\text{Gross–Zagier}\\\text{formulae}}"']
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- $\mathrm{ord}_p(f(0)) \sim \mathrm{ord}_p(\#Ш(E/K)[p^\infty]) \cdot \mathrm{ord}_p(\frac{\log_{\omega_E} Q}{[E(K):\mathbb{Z}\cdot Q]^2})$;
- Gross-Zagier formula: $L'(E/K,1) \sim [E(K) : \mathbb{Z} \cdot P_K]^2$;

  p-adic GZ (BDP): $\mathcal{L}(E/K_\infty)(0) \sim \log_{\omega_E}(P_K)$

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$\subseteq$
- converse divisibility using Eisenstein congruences for larger groups
- (in the residually reducible case) use of "smaller rank" IMCs

Some (old) results: IMCs for Eisenstein primes

Theorem (Castella-G.-Lee-Skinner, Castella-G.-Skinner)

Let p be an odd prime of good ordinary reduction such that E has a rational p-isogeny whose kernel $\mathbb{F}_p(\phi)$ satisfies $\phi_{|G_{\mathbb{Q}_p}} \neq 1, \chi_{cyc}$, $r \in \{0, 1\}$.

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- *(anticyclotomic IMC) $Sel^{\mathrm{str,rel}}_{\Lambda_{K_\infty}}$ is Λ_{K_∞}-torsion and*

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- *(cyclotomic IMC) $Sel_{\Lambda_{\mathbb{Q}_\infty}}^{\mathrm{ord}}$ is $\Lambda_{\mathbb{Q}_\infty}$-torsion and*

$$\mathrm{char}(Sel_{\Lambda_{\mathbb{Q}_\infty}}^{\mathrm{ord}}) = (\mathcal{L}^{\mathrm{MSD}}(E/\mathbb{Q}_\infty)).$$

Iwasawa theory and BSD

Theorem (Gross–Zagier '86, Kolyvagin '89)

Let $E/\mathbb{Q}$ be an elliptic curve and p a prime, $r \in \{0, 1\}$, then

$$\mathrm{ord}_{s=1} L(E, s) = r \Rightarrow \mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = r \text{ and } \#\text{Ш}(E/\mathbb{Q})[p^\infty] < \infty.$$

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- see also: Bertolini–Darmon '05, W. Zhang '14, Venerucci '16, Berti–Bertolini–Venerucci '17

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Let $E/\mathbb{Q}$, p a prime of good ordinary reduction and K a quadratic imaginary field such that p and the primes of bad reduction split in K and $E(K)[p] = \{0\}$, then the Kolyvagin system of Heegner points for E/K is not trivial

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- Previous results: W. Zhang '14, Sweeting '21, C.-H. Kim '22 (only when $E[p]$ irreducible)

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- The Iwasawa main conjecture away from $T = 0$ (over K_∞ and $\mathbb{Q}_\infty$ respectively) is one of the key ingredients in the proof.

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  - K_∞ Generalised Heegner cycles on Kuga-Sato varieties
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    - Daniel's talk: $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$, quasi-split unitary (Disegni, Disegni-Zhang, Disegni-Liu, Li-Liu)

How to prove the main conjecture(s): Eisenstein case

$$\boxed{\mathrm{Char}(Sel_\Lambda^\vee) \stackrel{?}{=} (\mathcal{L}(E))}$$

$\supseteq$
- existence of Λ-adic Euler systems (Kato's ES, Heegner points)
- with explicit reciprocity law

$\subseteq$
- converse divisibility using Eisenstein congruences for larger groups
- (in the residually reducible case) use of "smaller rank" IMCs

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and $\begin{cases} 0 \to \mathbb{F}_p(\phi) \to E[p] \to \mathbb{F}_p(\chi_{cyc} \cdot \phi^{-1}) \to 0 \\ \mathrm{GL}_1\text{-IMC and } \mu = 0 \end{cases} \Rightarrow$ (2)

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More precisely, for $F = K$ in [CGLS] and for $F = \mathbb{Q}$ under some conditions for ϕ in [Greenberg-Vatsal], show

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$$\begin{tikzcd}
 & H^1_{\mathrm{ord}}(K_v)/(\mathcal{L}_1) \arrow[r] & Sel_{\mathrm{Gr}} \arrow[dr] & \\
0 \to H^1(K)/(\mathcal{Z}^+) \arrow[ur, "loc_v"] \arrow[dr, "loc_{\bar{v}}"'] & & & Sel_{\mathrm{ord,str}} \to 0 \\
 & H^1_{/\mathrm{ord}}(K_{\bar{v}})/(\mathcal{L}_2) \arrow[r] & Sel_{\mathrm{ord}} \arrow[ur] &
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- Opposite divisibility from congruences by Hsieh–Z. Liu (large image and $\epsilon_f \epsilon_g = 1$)

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Assume $p \geq 5$, ordinary and p-distinguished hypotheses at p, and suppose that g is of weight 2 and residually Eisenstein:

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in progress: $\begin{cases} \text{ES divisibility argument} \\ \text{explicit reciprocity law and global duality} \end{cases} \Rightarrow$ divisibility in $\Lambda[\frac{1}{p}]$

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in progress: $\begin{cases} \text{ES divisibility argument} \\ \text{explicit reciprocity law and global duality} \end{cases}$ $\Rightarrow$ divisibility in $\Lambda[\frac{1}{p}]$

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the cyclotomic IMC for $E \otimes \rho_f$ holds.

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Rankin–Selberg and smaller-rank IMCs

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- HOPE: use p-adic variation to achieve $\mathrm{wt}(f) < \mathrm{wt}(g)$, especially weight (1,2)