

The Gross-Zagier formula, 40 years later

There and back again

Tony Feng, UC Berkeley



Notational (and moral) conventions

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Big Picture View

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GROSS-ZAGIER

“Height pairing of CM point $\sim$ central derivative of base change L -function”

$$\mathcal{P}^K := \text{Heegner point}, \\ \langle \mathcal{P}_f^K, \mathcal{P}_f^K \rangle \sim L'\left(f_K, \frac{1}{2}\right).$$

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$$\begin{aligned} \mathcal{P}^K &:= \text{Heegner point,} \\ \langle \mathcal{P}_f^K, \mathcal{P}_f^K \rangle &\sim L'\left(f_K, \frac{1}{2}\right). \end{aligned}$$

The right sides of these formulas bear an obvious relation. [What about the left sides?](#)

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Both the Waldspurger and Gross–Zagier formulas can be viewed as incarnations of the following template.

- Reductive group $G \rightsquigarrow$ an “automorphic space” $[G]$.

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SLOGAN

$\langle \phi_*[T], \phi_*[T] \rangle_{[G]}$ is related to the base change L -function.

Waldspurger's formula

SETUP

Let $G = \mathrm{PGL}_2/\mathbb{Q}$. Let $K/\mathbb{Q}$ be a quadratic extension $\rightsquigarrow T = K^\times/\mathbf{G}_m \xrightarrow{\phi} G$.

	WALDSPURGER	TEMPLATE	
$\left \int_{[T]} f(t) dt \right ^2$	PGL_2	G	$\rightsquigarrow \langle (\phi_*[T]) f, (\phi_*[T]) f \rangle_{[G]}$
$\sim L\left(f_K, \frac{1}{2}\right)$	$(\mathrm{Res}_{K/\mathbb{Q}} \mathbf{G}_m)/\mathbf{G}_m$	T	$\sim L\left(f_K, \frac{1}{2}\right)$
	$G(\mathbb{Q}) \backslash G(\mathbf{A}_{\mathbb{Q}})$	$[G]$	
	$\int_{[T]} \phi^* f$	$\int_{[G]} (\phi_*[T]) f$	

Gross–Zagier formula

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GROSS–ZAGIER TEMPLATE

$\mathcal{P}^K :=$ Heegner point,	$\rightsquigarrow$	PGL_2	G	$\rightsquigarrow$	$\langle (\phi_*[T])_f, (\phi_*[T])_f \rangle$
$\langle \mathcal{P}_f^K, \mathcal{P}_f^K \rangle \sim L'(f_K, \frac{1}{2})$		$\mathrm{Res}_{K/\mathbb{Q}} \mathbf{G}_m$	T		$\sim L'(f_K, \frac{1}{2})$
		Sh_G	$[G]$		
		$\mathcal{P}^K$	$[T]$		

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		Sh_G	$[G]$		
		$\mathcal{P}^K$	$[T]$		

View the Heegner point $\mathcal{P}^K$ as the **integral** Shimura variety for T ; its Néron–Tate height is a self-intersection within Sh_G , the **integral** Shimura variety of G .



Waldspurger and Gross–Zagier unified

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$$\langle (\phi_*[T])_f, (\phi_*[T])_f \rangle_{[G]} \sim L(f_K, \tfrac{1}{2})$$

GROSS–ZAGIER

$$\langle (\phi_*[T])_f, (\phi_*[T])_f \rangle_{[G]} \sim L'(f_K, \tfrac{1}{2})$$

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QUESTION

Why does the *value* appear on the left while the *derivative* appears on the right?

Whence the first derivative?

ANSWER

The notion of *automorphic space* is different.

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WALDSPURGER

$r = 0$

$$[G]_{\mathbf{0}} = G(\mathbb{Q}) \backslash G(\mathbf{A}_{\mathbb{Q}})$$
$$\langle \phi_*[T]_{\mathbf{0}}, \phi_*[T]_{\mathbf{0}} \rangle_{[G]_{\mathbf{0}}} \sim L^{(0)}\left(f_K, \frac{1}{2}\right)$$

GROSS-ZAGIER

$r = 1$

$$[G]_{\mathbf{1}} = \mathrm{Sh}_G \longrightarrow \mathrm{Spec} \mathbb{Z}$$
$$\langle \phi_*[T]_{\mathbf{1}}, \phi_*[T]_{\mathbf{1}} \rangle_{[G]_{\mathbf{1}}} \sim L^{(1)}\left(f_K, \frac{1}{2}\right)$$

Whence the first derivative?

ANSWER

The notion of *automorphic space* is different.

WALDSPURGER

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GROSS-ZAGIER

$r = 1$

$$[G]_1 = \mathrm{Sh}_G \longrightarrow \mathrm{Spec} \mathbb{Z} \\ \langle \phi_*[T]_1, \phi_*[T]_1 \rangle_{[G]_1} \sim L^{(1)}\left(f_K, \frac{1}{2}\right)$$

SLOGAN The first derivative appears because $[G]_1$ is fibered over the base curve $\mathrm{Spec} \mathbb{Z}$.

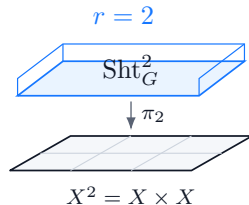
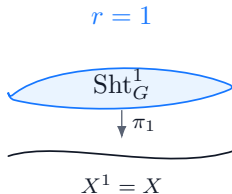
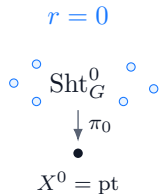
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Let X be any smooth projective curve over a finite field $\mathbb{F}_q$.

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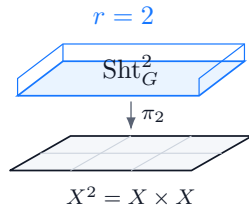
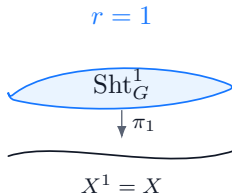
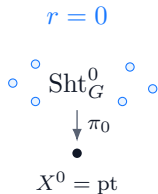
Let G be a reductive group over X . For every $r \geq 0$, there is a moduli space of shtukas Sht_G^r fibering over X^r .



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SLOGAN We can take $[G]_r = [\mathrm{Sht}_G^r]$ as our automorphic space.

Yun–Zhang formula

SETUP

Let $X' \rightarrow X$ be an étale double cover of curves over $\mathbb{F}_q$

$$\rightsquigarrow T = (\mathrm{Res}_{X'/X} \mathbf{G}_m) / \mathbf{G}_m \xrightarrow{\phi} G = \mathrm{PGL}_2.$$

YUN–ZHANG	TEMPLATE
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$$\mathrm{Sht}_G^r$$

$$[G]_r$$

$$\mathrm{Sht}_T^r$$

$$[T]_r$$

THEOREM (YUN–ZHANG)

Let f be an automorphic form on G . For every $r = 0, 1, 2, \dots$,

$$\langle [\mathrm{Sht}_T^r]_f, [\mathrm{Sht}_T^r]_f \rangle_{[\mathrm{Sht}_G^r]} \sim L^{(r)}\left(f_{X'}, \frac{1}{2}\right).$$

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Towards higher theta functions



Theta functions

A classical theta function is a Fourier series whose coefficients encode counts of lattice vectors.

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EXAMPLE

Let Λ be a lattice with quadratic form Q , and let

$$\mathcal{Z}_\Lambda(n) = \{\lambda \in \Lambda : Q(\lambda) = n\}$$

The associated theta series is

$$\theta_\Lambda = \sum_{n \geq 0} \# \mathcal{Z}_\Lambda(n) e^{2\pi i n z}.$$

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This series turns out to have more structure than initially visible!

Modularity

$\theta_Q(z)$ is defined by a Fourier series, which has an obvious *translational* symmetry. Less obviously, it has more symmetry: it defines a **modular form**!

Modularity

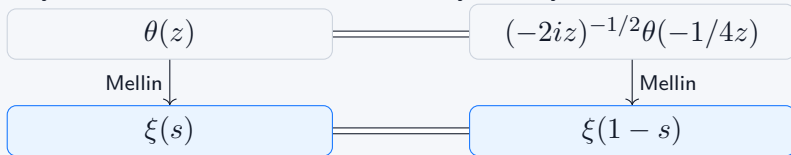
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EXAMPLE

Let $\Lambda = \mathbb{Z}$, with theta function

$$\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad q = e^{2\pi iz}.$$

The modularity of θ amounts to the additional symmetry



Siegel–Weil formula

EXAMPLE

Let $\Lambda = \mathbb{Z}^2$ with $Q(a, b) = a^2 + b^2$. Then it turns out that

$$\#\mathcal{Z}_\Lambda(n) = \#\{n = a^2 + b^2\} = 4 \sum_{d|n} \chi(d).$$

Corollary: an odd prime p is a sum of two squares iff $p \equiv 1 \pmod{4}$.

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Corollary: an odd prime p is a sum of two squares iff $p \equiv 1 \pmod{4}$.

In general, $\theta \in \text{Fun}([G] \times [H])$ for a dual reductive pair (G, H) , e.g. $(\text{SO}_m, \text{Sp}_{2n})$ or (U_m, U_n) .

SIEGEL–WEIL FORMULA

For the Siegel–Eisenstein series Eis_H ,

$$\int_{[G]} \theta(g, h) dg \sim \text{Eis}_H(s_0).$$

Three distinctive features of theta functions

Let (G, H) be a dual reductive pair, e.g., $(\mathrm{Sp}_{2n}, \mathrm{SO}_m)$ or (U_n, U_m) .

Let $P \subset H$ be the Siegel parabolic. A priori, $\theta \in \mathrm{Fun}([G] \times [P])$ is a Fourier series.

MODULARITY

θ descends to $\mathrm{Fun}([H])$

SIEGEL-WEIL FORMULA

$$\int_{[G]} \theta(h, g) dg \sim \mathrm{Eis}_H(h, s_0)$$

RALLIS INNER PRODUCT FORMULA

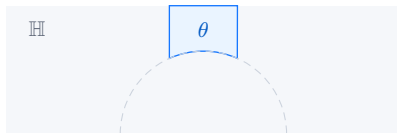
$$\langle \theta(f), \theta(f) \rangle_{[G]} \sim L(f, \tfrac{1}{2})$$

Arithmetic theta functions

CLASSICAL THETA FUNCTION

Fourier series

$$\theta: \text{Fun}([P]_0) \longrightarrow \text{Fun}([G]_0)$$

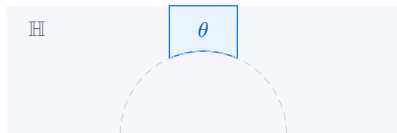


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ARITHMETIC THETA FUNCTION (KUDLA)

Fourier series

$$\Theta: \text{Fun}([P]_0) \longrightarrow \text{CH}_*([G]_1)$$



Three distinctive features of (arithmetic) theta functions

CLASSICAL THETA

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ARITHMETIC THETA (KUDLA PROGRAM)

MODULARITY

Θ descends to $\text{Fun}([H])$

[BHKRY; Howard–Madapusi]

ARITHMETIC SIEGEL–WEIL FORMULA

$$\int_{[G]_1} \Theta(h, g) \sim \text{Eis}'_H(h, s_0)$$

[Li–Zhang]

ARITHMETIC INNER PRODUCT

$$\langle \Theta(f), \Theta(f) \rangle_{[G]_1} \sim L'(f, \tfrac{1}{2})$$

[Li–Liu]

Biodiversity of theta functions

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Theta functions over function fields

Let X be a smooth projective curve over $\mathbb{F}_q$. We want to make an analogous story of theta functions.

NUMBER FIELD	FUNCTION FIELD
lattice Λ	vector bundle $\mathcal{F}$ on X
quadratic form $Q: \Lambda \rightarrow \Lambda^\vee$	$Q: \mathcal{F} \rightarrow \mathcal{F}^\vee$
vector $\lambda \in \Lambda$	section $s \in H^0(X - \{\infty\}, \mathcal{F})$
tuple of lattice vectors	$s \in \text{Hom}(\mathcal{E}, \mathcal{F})$, $\mathcal{E}$ a vector bundle

Representation numbers over function fields

What is the analogue of the representation number $\#\mathcal{Z}_\Lambda(n)$?

Given $s: \mathcal{E} \rightarrow \mathcal{F}$, define

$$Q(s) := (\mathcal{E} \xrightarrow{s} \mathcal{F} \xrightarrow{Q} \mathcal{F}^\vee \xrightarrow{s^\vee} \mathcal{E}^\vee) \in \text{Hom}(\mathcal{E}, \mathcal{E}^\vee).$$

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FUNCTION-FIELD THETA SERIES

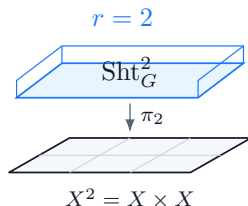
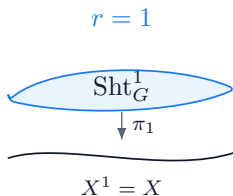
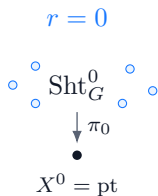
For $a \in \text{Hom}(\mathcal{E}, \mathcal{E}^\vee)$, set

$$\mathcal{Z}_\mathcal{E}(a) = \{s \in \text{Hom}(\mathcal{E}, \mathcal{F}) : Q(s) = a\}$$

Define a theta function by assembling the counts $\#\mathcal{Z}_\mathcal{E}(a)$ into a Fourier series.

Recall: there are moduli of shtukas for every r

For every $r \geq 0$, there is a moduli space of shtukas Sht_G^r fibering over X^r .



SLOGAN We can take $[G]_r = [\mathrm{Sht}_G^r]$ as our automorphic space.

Higher theta series

We define **higher special cycles** $\mathcal{Z}_{\mathcal{E}}^r \rightarrow \mathrm{Sht}_G^r$ for every $r = 0, 1, \dots$

EXAMPLE ($r = 0$)

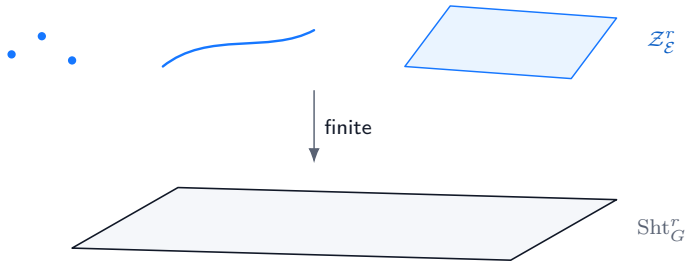
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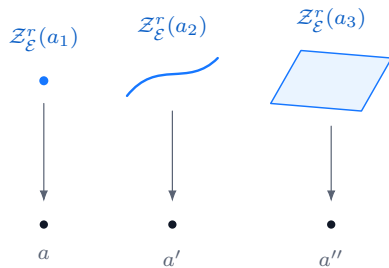
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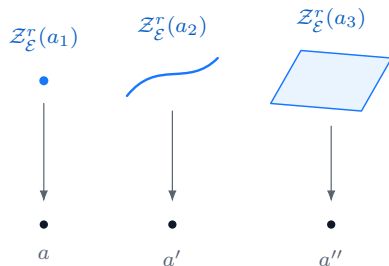
The special cycles decompose over “Fourier space” $a \in \text{Hom}(\mathcal{E}, \sigma^* \mathcal{E}^\vee)$

$$\mathcal{Z}_{\mathcal{E}}^r = \coprod_a \mathcal{Z}_{\mathcal{E}}^r(a).$$



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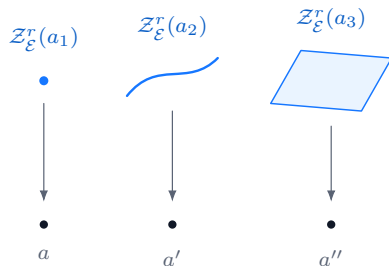


THE DUAL SIDE

Implicitly there is a dual reductive group H , with Siegel parabolic $P \subset H$. Here $[H]$ is the stack of pairs $(\mathcal{E} \hookrightarrow \mathcal{G})$ where $\mathcal{G}$ is a symplectic bundle and $\mathcal{E} \subset \mathcal{G}$ is a Lagrangian.

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The higher theta series Θ^r are obtained by assembling the cycle classes into a Fourier series,

$$\Theta^r(\mathcal{E} \subset \mathcal{G}) = \text{Fourier Series}([\mathcal{Z}_{\mathcal{E}}^r(a)]).$$

Virtual fundamental classes

PROBLEM

The $\mathcal{Z}_{\mathcal{E}}^r(a)$ have varying dimension, so we must construct **virtual fundamental classes** $[\mathcal{Z}_{\mathcal{E}}^r(a)]^{\text{vir}}$ before defining

$$\Theta^r(\mathcal{E} \subset \mathcal{G}) = \text{Fourier Series} \left([\mathcal{Z}_{\mathcal{E}}^r(a)]^{\text{vir}} \right).$$

How to define $[\mathcal{Z}_{\mathcal{E}}^r(a)]^{\text{vir}}$? [FYZ, “Higher theta series...”]

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CONCRETE BUT UNWIELDY

$$[\mathcal{Z}_{\mathcal{E}}^r(a)]^{\text{vir}} = \sum_{\mathcal{K} \subset \mathcal{E}} [\mathcal{Z}_{\mathcal{E}/\mathcal{K}}^r(\bar{a})] \prod_{i=1}^r c_1(\omega_{\mathcal{K},i}) \dots$$

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$$\Theta^r(\mathcal{E} \subset \mathcal{G}) = \text{Fourier Series} \left([\mathcal{Z}_{\mathcal{E}}^r(a)]^{\text{vir}} \right).$$

How to define $[\mathcal{Z}_{\mathcal{E}}^r(a)]^{\text{vir}}$? [FYZ, “Higher theta series...”]

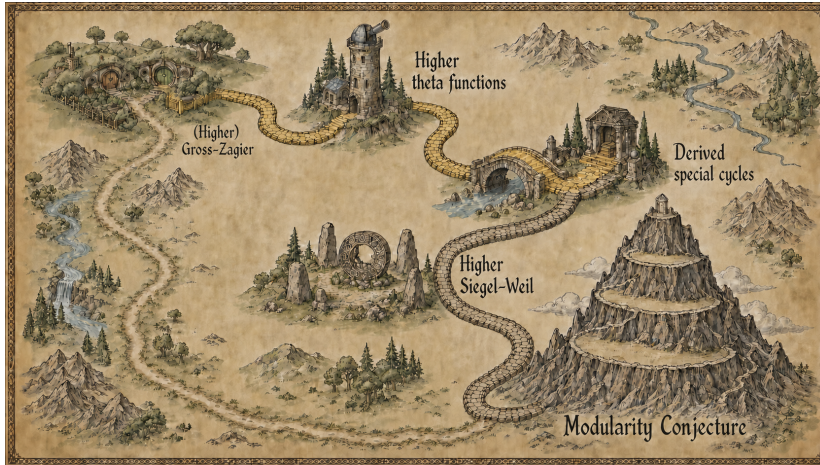
CONCRETE BUT UNWIELDY

$$[\mathcal{Z}_{\mathcal{E}}^r(a)]^{\text{vir}} = \sum_{\mathcal{K} \subset \mathcal{E}} [\mathcal{Z}_{\mathcal{E}/\mathcal{K}}^r(\bar{a})] \prod_{i=1}^r c_1(\omega_{\mathcal{K},i}) \dots$$

ABSTRACT BUT CONCEPTUAL

$[\mathcal{Z}_{\mathcal{E}}^r(a)]^{\text{vir}}$ is the (naive) fundamental class of a **derived** stack ${}^{\mathbf{L}}\mathcal{Z}_{\mathcal{E}}^r(a)$.

Derived special cycles



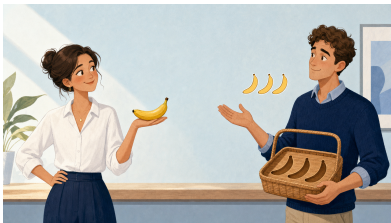
Explain Like I'm 5: a toy analogy



“Natural” numbers describe a quantity of bananas you could eat.

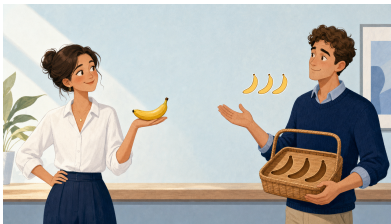
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Suppose Alice has 1 banana and then gives 3 bananas to Bob. How many bananas does Alice have?



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WHAT IS VISIBLE

$$\max(1 - 3, 0) = 0$$

A negative banana cannot sit on the table.

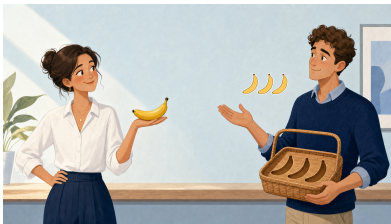
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$$1 - 3 = -2$$

Alice owes two bananas.

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WHAT THE LEDGER RECORDS

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It does not make physical sense to eat -2 bananas, so negative numbers are “unnatural” (?). But without negative numbers we would have to carry around “auxiliary stories” in all our calculations.

Motivating derived algebraic geometry

Naive definition of (relative) dimension: “number of variables — number of equations.”

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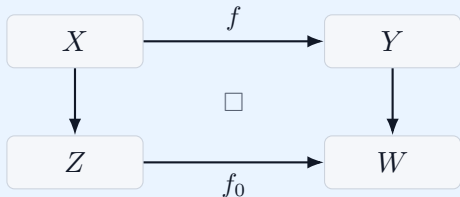
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A map is a *local complete intersection* (LCI) if its true relative dimension equals its naive relative dimension.

REFINED GYSIN PULLBACK (FULTON)

Suppose $f: X \rightarrow Y$ can be presented as the pullback of some map f_0 which is LCI:

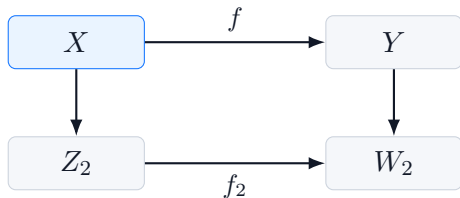
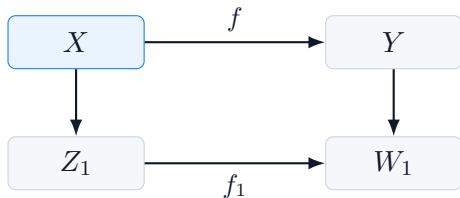


Then there exists a “refined Gysin pullback”

$$f_0^!: \mathrm{CH}_*(Y) \longrightarrow \mathrm{CH}_{*+\dim f_0}(X)$$

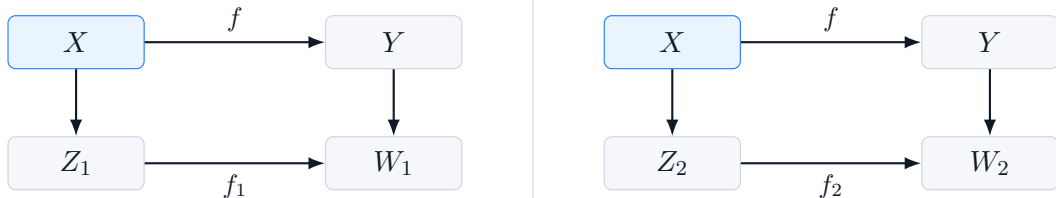
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Such a presentation is an “auxiliary story”. Two different presentations may lead to two different Gysin pullbacks which are hard to compare.



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This feels like the same contrivedness that we encounter if we try to avoid negative numbers.

What is derived algebraic geometry?

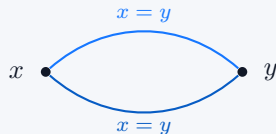
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KEY IDEA FROM HOMOTOPY THEORY

Equality $x = y$ is replaced by a *path* from x to y . Imposing $x = y$ twice creates a non-trivial *loop*.

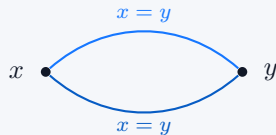


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ACTUAL ANSWER

Rebuild algebraic geometry with commutative rings of homotopy-theoretic nature, e.g. “simplicial commutative rings” or “animated rings”.

The Upshot

Concrete but unwieldy construction: to define $[\mathcal{Z}_{\mathcal{L}}^r]^{\text{vir}}$, devise auxiliary stories until the dimension looks right.

EXAMPLE

$\mathcal{Z}_{\mathcal{L}}^r(a) \rightarrow \text{Sht}_G$ is LCI for $\text{rank}(\mathcal{L}) = 1$ and $a \neq 0$.

$$\begin{array}{ccc} \mathcal{Z}_{\mathcal{L}_1 \oplus \mathcal{L}_2}^r \left(\begin{smallmatrix} a_1 & a_2 \end{smallmatrix} \right) & \longrightarrow & \mathcal{Z}_{\mathcal{L}_1}^r(a_1) \\ \downarrow & & \downarrow \\ \mathcal{Z}_{\mathcal{L}_2}^r(a_2) & \longrightarrow & \text{Sht}_G^r \end{array}$$

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CONCEPTUAL CONSTRUCTION

“Repeating” the definition of the special cycle $\mathcal{Z}_{\mathcal{E}}^r$ in derived algebraic geometry gives a derived special cycle ${}^{\mathbf{L}}\mathcal{Z}_{\mathcal{E}}^r$ which has the correct dimension and is LCI in the derived sense $\implies$ take $[\mathcal{Z}_{\mathcal{E}}^r]^{\text{vir}} = [{}^{\mathbf{L}}\mathcal{Z}_{\mathcal{E}}^r]^{\text{naive}}$.

Higher Siegel–Weil formulas



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If $G = U(n)$ and $\text{rank}(\mathcal{E}) = n$, then $\dim[\mathcal{Z}_{\mathcal{E}}(a)^r]^{\text{vir}} = 0$. There is a Siegel–Eisenstein series Eis on the “Howe dual” group $H = U(2n)$.

THEOREM (F–YUN–ZHANG)

If a has rank n , then

$$\int_{[\text{Sht}_G^r]} [\mathcal{Z}_{\mathcal{E}}(a)^r]^{\text{vir}} \sim \left. \frac{d^r}{ds^r} \right|_{s=\frac{1}{2}} \text{Eis}_a(s) \quad (a\text{th Fourier coefficient}).$$

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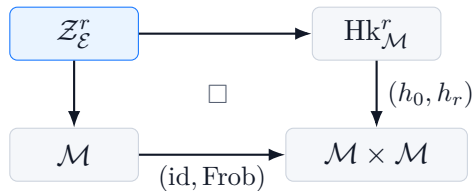
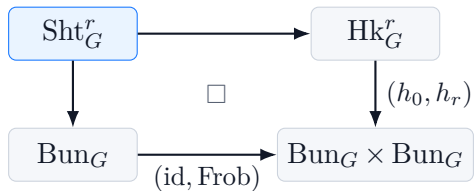
THEOREM (F–HOWARD–MKRTCHYAN)

If a has rank $n - 1$, then

$$\int_{[\text{Sht}_G^r]} [\mathcal{Z}_{\mathcal{E}}(a)^r]^{\text{vir}} \sim \left. \frac{d^r}{ds^r} \right|_{s=\frac{1}{2}} \mathcal{L}(s) \text{Eis}_a\left(\frac{1}{2}\right) \quad (a\text{th Fourier coefficient}).$$

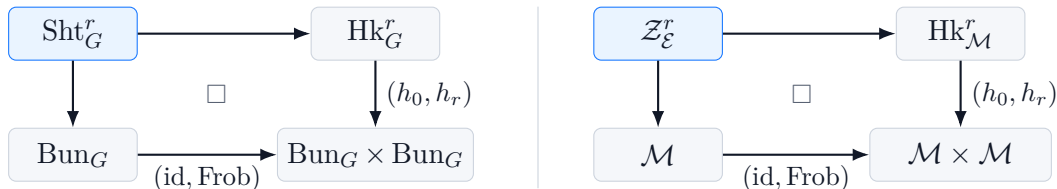
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Key: Sht_G^r and $\mathcal{Z}_{\mathcal{E}}^r$ themselves arise as “fixed points” of correspondences:



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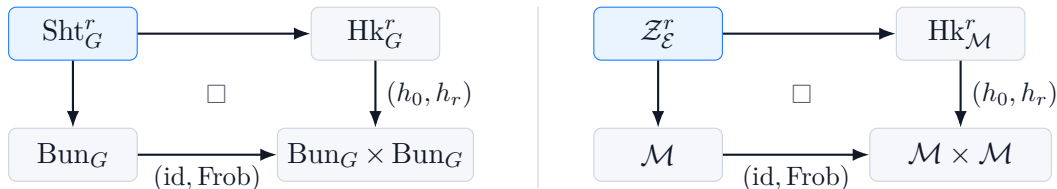


CONSEQUENCE

We can compute $\int_{[G]_r} [\mathcal{Z}_{\mathcal{E}}^r(a)]$ using the Grothendieck–Lefschetz trace formula.

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We can compute $\int_{[G]_r} [\mathcal{Z}_{\mathcal{E}}^r(a)]$ using the Grothendieck–Lefschetz trace formula.

This requires computing the trace of a complicated endomorphism of a complicated sheaf. A key point is to use the theory of perverse sheaves to extend the computation from “simple” a to all a .

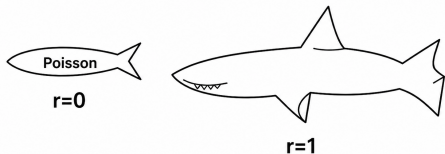
Three tiers of modularity

TIER	NUMBER FIELDS	FUNCTION FIELDS
Full modularity	BHKRY, Howard–Madapusi	FYZ: “Modularity of higher theta series III”
Chow group of the generic fiber	Borcherds, YZZ, Z, Bruinier–Westerholt–Raum	F–Khan: “Modularity of higher theta series II”
Cohomology of the generic fiber	Kudla–Millson	FYZ: “Modularity of higher theta series I”

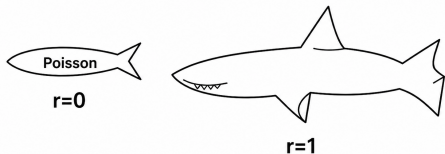


Comparison to the classical case

- Classical theta functions: modularity proved by [Poisson summation](#).

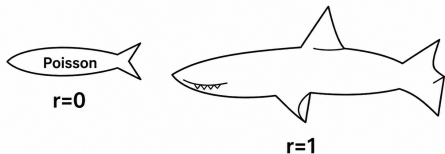


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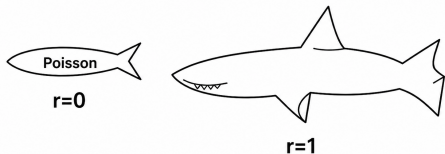
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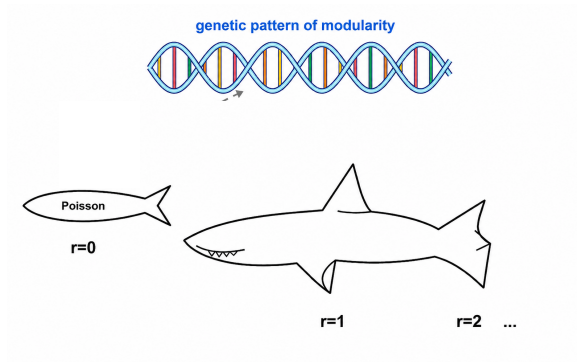
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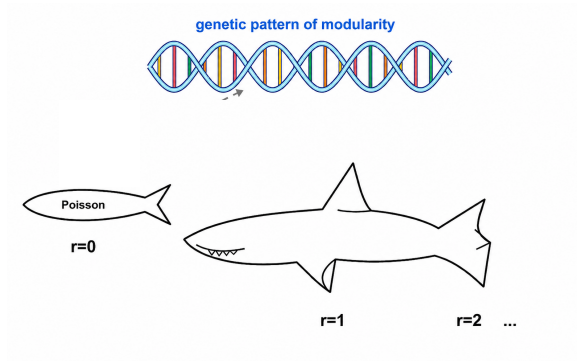
We need a new strategy!

Step 1: Study Poisson ...



We believe in a uniform explanation, where the modularity of Θ^r for all r can be seen as different **phenotypes** of a common **genotype**.

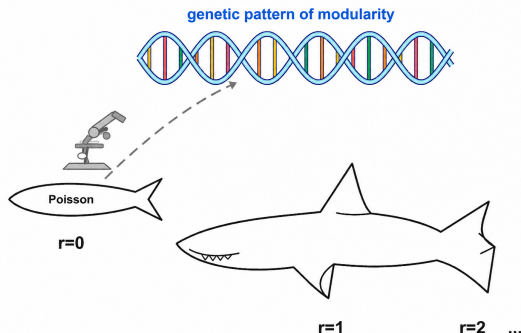
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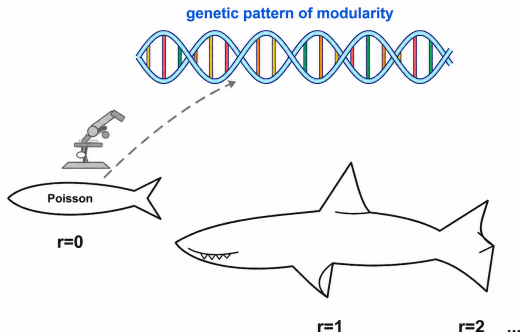
If this is true, then the genotype can already be found within Poisson.

Step 2: Derived Fourier analysis . . .



Poisson's argument is Fourier analysis for functions on vector spaces. We categorify this via Fourier analysis for sheaves on (derived) vector bundles.

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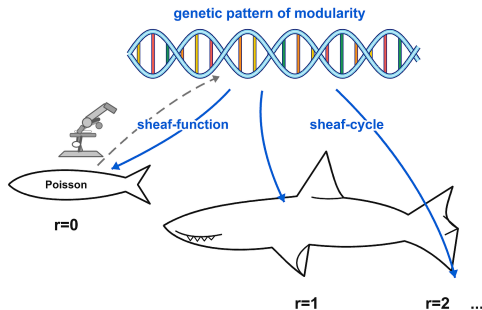


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We develop an extension of the Deligne–Laumon Fourier transform, called **derived Fourier analysis**, to do this.

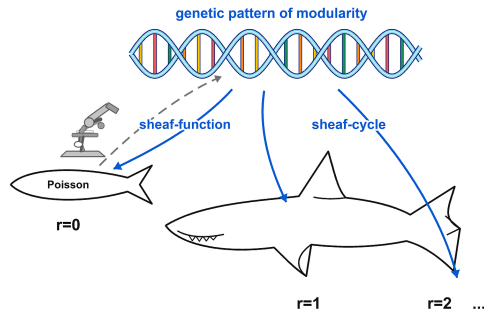
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Step 3. The sheaf-cycle correspondence



We develop a [sheaf-cycle correspondence](#) extending the sheaf-function correspondence.

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We develop a **sheaf-cycle correspondence** extending the sheaf-function correspondence. Taking the “trace” extracts modularity of all

Θ^r uniformly.

[F–Khan: “Modularity of higher theta series II”]

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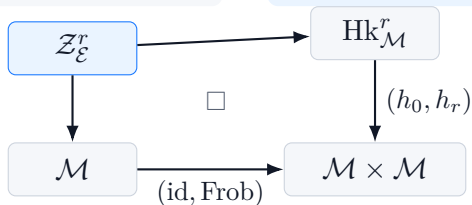
The sheaf-cycle correspondence

SHEAF-FUNCTION

$\alpha \in \text{Hom}(\mathcal{K}, \mathcal{K}) \xrightarrow{\text{Tr}} \text{function on fixed points}$

SHEAF-CYCLE

$\alpha \in \text{Ext}^i(\mathcal{K}, \mathcal{K}) \xrightarrow{\text{Tr}} \text{cycle on fixed locus}$



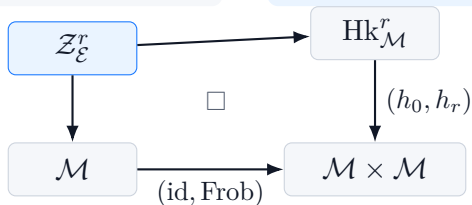
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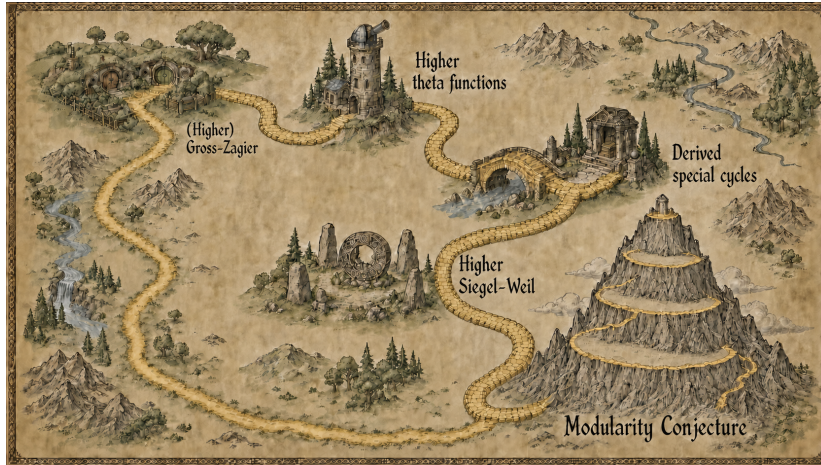
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TRACE CONJECTURE (F-HARRIS)

$[\mathcal{Z}_{\mathcal{E}}^r]^{\text{vir}}$ equals the “trace” of some cohomological correspondence on $\mathcal{M}$.

... and back again



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QUESTION

How much of this story can be taken back to number fields?

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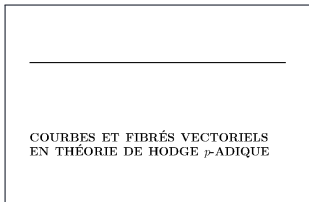
...or is there?

Metamorphoses of Bun_G

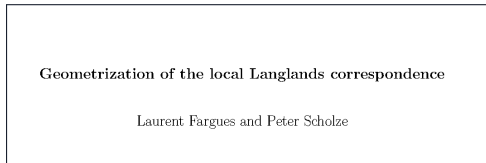
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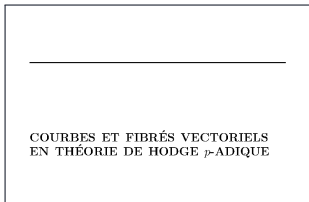
Fargues–Fontaine (Astérisque 406)



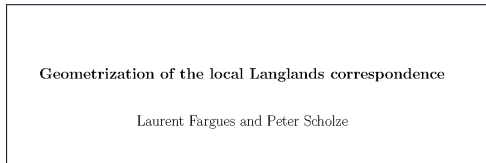
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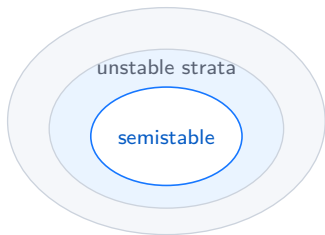
FARGUES–FONTAINE

Over $\mathbb{Q}_p$ there is “the” Fargues–Fontaine curve X_{FF} . This gives a moduli stack Bun_G .

Two incarnations of Bun_G

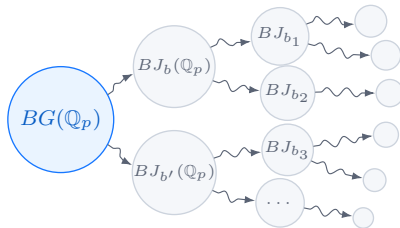
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Bun_G for $X/\mathbb{F}_q$



an onion of Harder–Narasimhan layers

Bun_G for X_{FF}



classifying stacks $BJ_b(\mathbb{Q}_p)$, glued together

The template

NATURAL CONSTRUCTION An object on Bun_P , where $P \subset H$ is the Siegel parabolic.

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$\Theta(\mathcal{E} \subset \mathcal{G})$ is independent of the choice of $\mathcal{E}$.

FARGUES-FONTAINE

The sheaf descends along $\mathrm{Bun}_P \rightarrow \mathrm{Bun}_H$;
equivalently, the representation extends to $\tilde{H}(\mathbb{Q}_p)$.

JEOPARDY!

What is a canonical representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{Q}_p)$ that has an obvious action of $P(\mathbb{Q}_p)$?

ANSWER

Answer: the Weil representation

Thanks for your attention!

