Cycle integrals of meromorphic modular forms

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Abstract

These are the notes of my talk at the conference *The Gross—Zagier Formula, 40 Years Later*. The talk is based upon joint work with Baptiste Depouilly, Paul Kiefer, and Markus Schwagenscheidt (arXiv:2406.03465) and on work in progress with Jens Funke, Sebastian Herrero, Özlem Imamoglu, and Paul Kiefer.

1 Motivation

The automorphic Green's function is given by

$$G_s(z_1, z_2) = -2 \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} Q_{s-1}\left(1 + \frac{|z_1 - \gamma z_2|^2}{2\,\mathrm{Im}(z_1)\,\mathrm{Im}(\gamma z_2)}\right),$$

where

$$Q_{s-1}(t) = \int_0^\infty \left(t + \sqrt{t^2-1} \cdot \cosh(u)\right)^{-s} du$$

is the classical Legendre function of the 2nd kind.

We summarize some of its essential properties:

- $G_s(z_1, z_2)$ converges absolutely for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$ and $z_1, z_2 \in \mathbb{H}$, $z_1 \notin \mathrm{SL}_2(\mathbb{Z})z_2$.
- $G_s(z_1, z_2)$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ in both variables and descends to a function on $(X \times X) \setminus Z(1)$, where $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and $Z(1)$ denotes the diagonal.
- $G_s(z_1, z_2)$ has a logarithmic singularity along $Z(1)$.
- $G_s(z_1, z_2)$s an eigenfunction of the hyperbolic Laplacian in both variables.
- $G_s(z_1, z_2)$ has a meromorphic continuation in s to $\mathbb{C}$ and satisfies a functional equation relating the values at s and $1 - s$.
- The constant term in the Laurent expansion at $s = 1$ of $G_s(z_1, z_2)$ is essentially given by $\log|j(z_1) - j(z_2)|$ which can be used to derive explicit formulas for the norms of singular moduli.

In their seminal work, Gross and Zagier conjectured that (certain linear relations) of higher Green's functions are log-algebraic at $s = k$.

There was partial progress on this conjecture by: Gross–Kohnen–Zagier, Zhang, Mellit, Viazovska, Li, Bruinier–Ehlen–Yang. A full proof was given by Bruinier–Li–Yang.

We briefly review the approach of Bruinier–Ehlen–Yang: They showed that certain linear combinations of

$$G_{k+1}\left(\underbrace{C(d_1)}_{\text{average of CM pts of disc } d_1}, \underbrace{z_2}_{\text{CM point of disc } d_2}\right)$$

are log-algebraic (up to constants).

Here,

$$C(d_1) = \sum_{Q \in \mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{w_Q} z_Q,$$

where $\mathcal{Q}_{d_1}$ denotes the set of positive definite integral binary quad. forms of discriminant d_1, w_Q the order of the stabilizer, and z_Q the root of $Q(z, 1) = 0$ in $\mathbb{H}$.

In other words: Taking the CM trace of the Green's function in one variable and evaluating at a CM point in the other is log algebraic.

Question: What if we take the trace not over CM points but over geodesics?
That is

$$\sum_{\substack{Q \in \mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z}) \\ d_1 > 0}} \int_{\Gamma_Q \backslash c_Q} G_s(\tau_1, \tau_2) \frac{d\tau_1}{Q(\tau_1, 1)},$$

where
$$c_Q = \left\{ z \in \mathbb{H} \,\middle|\, a|z|^2 + b\,\mathrm{Re}(z) + c = 0 \right\}$$
and where we let τ_2 be a CM point.

Observation (Alfes–Funke–Herrero–Imamoglu, work in progress): For even k this trace equals the trace of geodesic cycle integrals of the function
$$f_{k,A}(z) = \frac{|d|^{\frac{k+1}{2}}}{\pi} \sum_{Q \in A} Q(z,1)^{-k},$$
where $A \in \mathcal{Q}_d/\mathrm{SL}_2(\mathbb{Z})$ is a fixed equivalence class of quadratic forms of discriminant $d < 0$.

The reason for this is that the iterated raising operator applied to the Green's function (when evaluating it in a CM point in one variable) essentially gives
$$R^k G_k \left(\tau, \underbrace{z_A}_{\text{CM pt}} \right) = f_{k,A}.$$
Moreover, the cycle integrals of $G_k(\cdot, z_A)$ and $R^k G_k(\cdot, z_A)$ are equal up to a constant (Alfes–Schwagenscheidt).

In fact, certain linear combinations of these cycle integrals are rational.

Theorem 1 (Alfes–Bringmann–Schwagenscheidt, 2020)**.** *Certain linear combinations of the traces*
$$\sum_{Q \in \mathcal{Q}_D/\mathrm{SL}_2(\mathbb{Z})} \int_{\Gamma_Q \backslash c_Q} f_{k,A}(z) Q(z,1)^{k-1}\, dz$$
are rational.

A different motivation for our work is (that does not come from Green's functions):

- In his paper "Modular forms associated to real quadratic fields" Zagier defined holomorphic Hilbert modular forms that are the kernel functions of the Doi-Naganuma correspondence between elliptic modular forms of wt k for $\Gamma_0(D)$ with Nebentypus $\chi_D = \left(\frac{D}{\cdot}\right)$. and Hilbert modular forms of wt k for $\mathrm{SL}_2(\mathcal{O}_F)$.
- Kohnen–Zagier studied cycle integrals of diagonal restriction $f_{k\mathcal{A}}$ of these and showed that they are rational.
- Bengoechea was the first to define and investigate meromorphic analogues.

Today: Cycle integrals of the Hilbert analogues of the $f_{k,A}$

2 Setup

We define our setting.

- F a real quadratic field
- $D > 0$ its discriminant
- $\mathcal{O}_F$ its ring of integers
- ∂_F its different
- $$L = \left\{ \begin{pmatrix} a & \nu' \\ \nu & b \end{pmatrix} : \ a, b \in \mathbb{Z},\ \nu \in \mathcal{O}_F \right\} \qquad \text{with} \qquad Q(X) = -\det(X)$$
- $$L' = \left\{ \begin{pmatrix} a & \nu' \\ \nu & b \end{pmatrix} : \ a, b \in \mathbb{Z},\ \nu \in \partial_F^{-1} \right\}$$
- $$L_m = \left\{ X \in L' \,\middle|\, Q(X) = \frac{m}{D} \right\}$$
- $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$ acts on L by $\gamma \cdot X = \gamma X {\gamma'}^t$ and on L_m with finitely many orbits
- $z = (z_1, z_2) \in \mathbb{H} \times \mathbb{H}$, $z_i = x_i + iy_i$

Let $X \in L$. Define the polynomials
$$q_z(X) = -bz_1z_2 + \nu z_1 + \nu' z_2 - a$$
and
$$p_z(X) = \frac{1}{y_1} \left(-b\overline{z_1} z_2 + \nu \overline{z_1} + \nu' z_2 - a \right)$$

The *algebraic cycles* (also called Hirzebruch–Zagier divisors) are defined as follows: Let $X \in L'$ with $Q(X) = n < 0$ and let
$$T_n = \bigcup_{X \in L_n} T_X, \qquad T_X = \left\{ z \in \mathbb{H}^2 \,\middle|\, q_z(X) = 0 \right\}.$$

These give the analogues of CM points (/divisors).

We also define *real-analytic cycles*: Let $Y \in L'$ with $Q(y) > 0$. Then

$$C_y = \left\{ z \in \mathbb{H}^2 \,\middle|\, p_z(y) = 0 \right\}$$

defines a real-analytic cycle. It gives the analogues of geodesics.

3 Main result

Let $n < 0$ and define

$$\omega_n^{\text{mero}}(z) = \sum_{X \in L_n} q_z(X)^{-k}.$$

This function has the following properties:

- ω_n is a meromorphic Hilbert modular form of parallel weight (k, k) for $\mathrm{SL}_2(\mathcal{O}_F)$, i.e., a holomorphic function $f : \mathbb{H}^2 \longrightarrow \mathbb{C}$ such that

$$f(\gamma_1 z_1, \gamma_2 z_2) = (c_1 z_1 + d_1)^k (c_2 z_2 + d_2)^k f(z_1, z_2) \text{for all } (\gamma_1, \gamma_2) \in \mathrm{SL}_2(\mathcal{O}_F).$$

- ω_n has poles along T_n.
- ω_n decays like a cusp form at the cusps.

We make the following technical assumptions:

- T_n does not intersect any of the real-analytic cycles C_y for $y \in L_m$.
- $\operatorname{disc} D$ of F is an odd prime.
- We have $S_k^+\big(\Gamma_0(D), \chi_D\big) = 0$.

Under these assumptions the following theorem holds (otherwise, we also have to take certain linear combinations of the cycle integrals.)

Theorem 2 (Alfes–Depouilly–Kiefer–Schwagenscheidt, 2024)**.** *Let $k \geq 4$ be even, $m > 0$, $n < 0$. Then*

$$\sum_{y \in \Gamma \backslash L_m} \int_{\Gamma_y \backslash C_y} \omega_n^{\text{mero}}(z) q_z(y)^{k-2} dz_1 \, dz_2$$

is a rational multiple of πi.

4 A proof sketch

Step 1

$$0 = \langle \omega_n^{\text{mero}}, \omega_m^{\text{cusp}} \rangle .$$

Step 2

$$\Vert$$

$$\boxed{\text{traces of cycle integrals of } \omega_n^{\text{mero}}} + \boxed{\text{evaluation of } \omega_m^{\text{cusp}} \text{ at algebraic cycle}}$$

Here we prove a more general result on Petersson inner products of

1) a form with singularities along algebraic cycles,
2) develop a ξ-operator in this setting.
3) ξ of a form with singularities along the real-analytic cycle.

Step 3 Write

$$\omega_m^{\text{cusp}} = \xi \Omega_m^{\text{cusp}}.$$

Construct Ω_m^{cusp} such that Ω_m^{cusp} is a Hilbert analogue of a locally harmonic Maass form.

Interlude: What is a harmonic Maass form in the first place? A harmonic Maass form of weight k for a group Γ is a real-analytic function

$$f : \mathbb{H} \longrightarrow \mathbb{C}$$

that

- transforms like a modular form of wt k,

- $$\Delta_k f = 0, \qquad \Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$
- has at most linear exponential growth as $y \to \infty$.

Then the ξ-operator defined by Bruinier and Funke (when defining harmonic Maass forms) maps harmonic Maass forms of weight k to weakly holomorphic modular forms of weight $2-k$:

$$\xi_k := 2iv^k\overline{\frac{\partial f}{\partial \overline{\tau}}} : H_k \longrightarrow M^!_{2-k}.$$

In particular, Ω_m^{cusp} is constructed explicitly as a Borcherds' type theta lift of a certain harmonic Maass form.

Step 4

Use a technique of Kudla/Schofer to give a formula for the evaluation of of this theta lifting at the algebraic cycle.

Step 5

Use Stokes' theorem to obtain a formula for this quantity in terms of Fourier coefficients of harmonic weak Maass forms and unary theta functions. These Fourier coefficients are rational in this case.