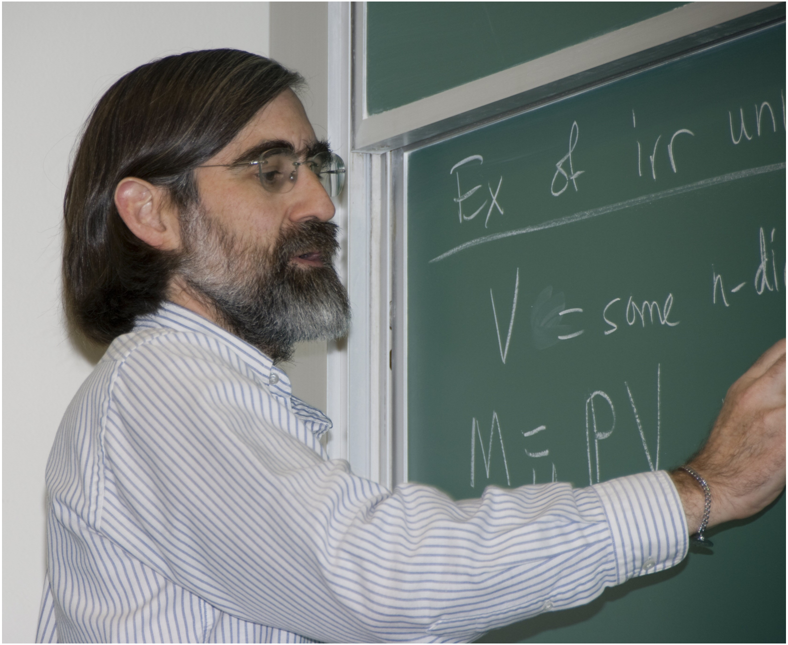
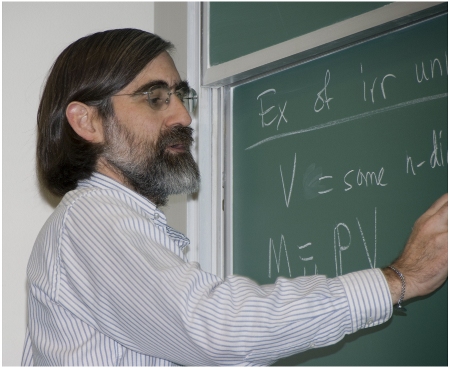
Cocenter-Representation duality for affine Hecke algebras

Xuhua He

HKUST/Univ. Maryland

May 21, 2014


Ex of irr
V = some


Ex of irr
V = some
M = PV

Overview

Finite dimensional representations of affine Hecke algebras have been studied by many people:

- Geometric method: Kazhdan, Lusztig, Ginzburg, Reeder, Kato, etc.
- Analytic method: Opdam, Solleveld

Overview

Finite dimensional representations of affine Hecke algebras have been studied by many people:

- Geometric method: Kazhdan, Lusztig, Ginzburg, Reeder, Kato, etc.
- Analytic method: Opdam, Solleveld
- Algebraic method: joint with Nie, joint with Ciubotaru

Overview

Finite dimensional representations of affine Hecke algebras have been studied by many people:

- Geometric method: Kazhdan, Lusztig, Ginzburg, Reeder, Kato, etc.
- Analytic method: Opdam, Solleveld
- Algebraic method: joint with Nie, joint with Ciubotaru

Affine Weyl groups

- $(X, R, Y, R^\vee, \Pi)$ a based root datum
- $W_0 = W(R)$ a finite Weyl group
- S the set of simple reflections in W_0.
- $W_a = \mathbb{Z}R \rtimes W_0$ an affine Weyl group
- $\tilde{S}$ the set of simple reflections in W_a
- $\tilde{W} = X \rtimes W_0$ an extended affine Weyl group.

Example

The group $\mathbb{Z}^n \rtimes S_n$ is the extended affine Weyl group of GL_n. It is a quasi-Coxeter group.

Iwahori-Matsumoto presentation of $\mathcal{H}$

Fix the parameter $\mathrm{q} = \{\mathrm{q}(s) \in \mathbb{C}^\times : s \in \tilde{S}\}$ such that $\mathrm{q}(s) = \mathrm{q}(t)$ if $s, t \in \tilde{S}$ are conjugate in $\tilde{W}$.

Definition (Iwahori-Matsumoto presentation)

The affine Hecke algebra $\mathcal{H} = \mathcal{H}_\mathrm{q}$ is the $\mathbb{C}$-algebra generated by $\{T_w : w \in \tilde{W}\}$ subject to the relations:

1. $T_w \cdot T_{w'} = T_{ww'}$, if $\ell(ww') = \ell(w) + \ell(w')$;
2. $(T_s + 1)(T_s - \mathrm{q}(s)^2) = 0$, if $s \in \tilde{S}$.

The Iwahori-Matsumoto presentation is related to the quasi-Coxeter structure of $\tilde{W}$.

Bernstein-Lusztig presentation of $\mathcal{H}$

Bernstein-Lusztig presentation: $\mathcal{H}$ is the $\mathbb{C}$-algebra generated by $\{\theta_x T_w; x \in X, w \in W_0\}$. This is related to the semi-direct product $\tilde{W} = X \rtimes W_0$.

For any $J \subset \Pi$, let $\mathcal{H}_J$ be the parabolic subalgebra of $\mathcal{H}$ generated by $\{\theta_x T_w; x \in X, w \in W_J\}$. This is the affine Hecke algebra for the parabolic subgroup $\tilde{W}_J = X \rtimes W_J$, but for (probably) different parameters.

Bernstein-Lusztig presentation of $\mathcal{H}$

Bernstein-Lusztig presentation: $\mathcal{H}$ is the $\mathbb{C}$-algebra generated by $\{\theta_x T_w; x \in X, w \in W_0\}$. This is related to the semi-direct product $\tilde{W} = X \rtimes W_0$.

For any $J \subset \Pi$, let $\mathcal{H}_J$ be the parabolic subalgebra of $\mathcal{H}$ generated by $\{\theta_x T_w; x \in X, w \in W_J\}$. This is the affine Hecke algebra for the parabolic subgroup $\tilde{W}_J = X \rtimes W_J$, but for (probably) different parameters.

Roughly speaking, $\mathcal{H}_J$ consists of two parts: the semisimple part $\mathcal{H}_J^{ss}$ and the central part.

For any representation σ of $\mathcal{H}_J^{ss}$ and any central character χ_t of $\mathcal{H}_J$, we have the parabolically induced module $i_J(\sigma \circ \chi_t)$.

Bernstein-Lusztig presentation of $\mathcal{H}$

Bernstein-Lusztig presentation: $\mathcal{H}$ is the $\mathbb{C}$-algebra generated by $\{\theta_x T_w; x \in X, w \in W_0\}$. This is related to the semi-direct product $\tilde{W} = X \rtimes W_0$.

For any $J \subset \Pi$, let $\mathcal{H}_J$ be the parabolic subalgebra of $\mathcal{H}$ generated by $\{\theta_x T_w; x \in X, w \in W_J\}$. This is the affine Hecke algebra for the parabolic subgroup $\tilde{W}_J = X \rtimes W_J$, but for (probably) different parameters.

Roughly speaking, $\mathcal{H}_J$ consists of two parts: the semisimple part $\mathcal{H}_J^{ss}$ and the central part.

For any representation σ of $\mathcal{H}_J^{ss}$ and any central character χ_t of $\mathcal{H}_J$, we have the parabolically induced module $i_J(\sigma \circ \chi_t)$.

Trace map

- $\bar{\mathcal{H}} = \mathcal{H}/[\mathcal{H}, \mathcal{H}]$—the cocenter of $\mathcal{H}$.
- $R(\mathcal{H})$—Grothendieck group of finite dim representations of $\mathcal{H}$.
- $Tr : \bar{\mathcal{H}} \to R(\mathcal{H})^*$, $h \mapsto (V \mapsto Tr_V(h))$, the trace map.

Definition

A function $f \in R(\mathcal{H})^*$ is good if for any $J \subset \Pi$ and $\sigma \in R(\mathcal{H}_J^{ss})$, the function $t \mapsto f(i_J(\sigma \circ \chi_t))$ is a regular function on t.

Trace map

- $\bar{\mathcal{H}} = \mathcal{H}/[\mathcal{H}, \mathcal{H}]$—the cocenter of $\mathcal{H}$.
- $R(\mathcal{H})$—Grothendieck group of finite dim representations of $\mathcal{H}$.
- $Tr : \bar{\mathcal{H}} \to R(\mathcal{H})^*$, $h \mapsto (V \mapsto Tr_V(h))$, the trace map.

Definition

A function $f \in R(\mathcal{H})^*$ is good if for any $J \subset \Pi$ and $\sigma \in R(\mathcal{H}_J^{ss})$, the function $t \mapsto f(i_J(\sigma \circ \chi_t))$ is a regular function on t.

Theorem (Bernstein, Deligne, Kazhdan)

Let H_q be the Hecke algebra of a p-adic group (algebra of locally constant, compact support measures). Then

1. The image of Tr is $R(H_q)^*_{good}$ (trace Paley-Wiener Theorem).
2. The map Tr is injective (density Theorem).

Trace map

- $\bar{\mathcal{H}} = \mathcal{H}/[\mathcal{H}, \mathcal{H}]$—the cocenter of $\mathcal{H}$.
- $R(\mathcal{H})$—Grothendieck group of finite dim representations of $\mathcal{H}$.
- $Tr : \bar{\mathcal{H}} \to R(\mathcal{H})^*$, $h \mapsto (V \mapsto Tr_V(h))$, the trace map.

Definition

A function $f \in R(\mathcal{H})^*$ is good if for any $J \subset \Pi$ and $\sigma \in R(\mathcal{H}_J^{ss})$, the function $t \mapsto f(i_J(\sigma \circ \chi_t))$ is a regular function on t.

Theorem (Bernstein, Deligne, Kazhdan)

Let H_{q} be the Hecke algebra of a p-adic group (algebra of locally constant, compact support measures). Then

1. The image of Tr is $R(H_{\mathrm{q}})^*_{good}$ (trace Paley-Wiener Theorem).
2. The map Tr is injective (density Theorem).

Cocenter of group algebra

Naive example: Consider $\mathcal{H}_1 = \mathbb{Z}[W]$. Then

1. For any conjugacy class $\mathcal{O}$ of W, and $w, w' \in \mathcal{O}$. The image of w and w' in $\bar{\mathcal{H}}_1$ are the same.
2. $\bar{\mathcal{H}}_1$ has a standard basis $\{w_{\mathcal{O}}\}$. Here $\mathcal{O}$ runs over all the conjugacy classes of W and $w_{\mathcal{O}}$ is a representative of $\mathcal{O}$.

Question

What happens for Hecke algebra $\mathcal{H}$?

Notice here that in general, for $w, w' \in \mathcal{O}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are different.

Cocenter of group algebra

Naive example: Consider $\mathcal{H}_1 = \mathbb{Z}[W]$. Then

1. For any conjugacy class $\mathcal{O}$ of W, and $w, w' \in \mathcal{O}$. The image of w and w' in $\bar{\mathcal{H}}_1$ are the same.
2. $\bar{\mathcal{H}}_1$ has a standard basis $\{w_{\mathcal{O}}\}$. Here $\mathcal{O}$ runs over all the conjugacy classes of W and $w_{\mathcal{O}}$ is a representative of $\mathcal{O}$.

Question

What happens for Hecke algebra $\mathcal{H}$?

Notice here that in general, for $w, w' \in \mathcal{O}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are different.

Finite Coxeter groups

Theorem (Geck-Pfeiffer, Geck-Kim-Pfeiffer, H., H.-Nie)

Let W_0 be a finite Coxeter group and $\mathcal{O}$ be a conjugacy class. Let $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$. Then

Finite Coxeter groups

Theorem (Geck-Pfeiffer, Geck-Kim-Pfeiffer, H., H.-Nie)

Let W_0 be a finite Coxeter group and $\mathcal{O}$ be a conjugacy class. Let $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$. Then

1. Each element w of $\mathcal{O}$ can be brought to a minimal length element w' by conjugation by simple reflections which reduce or keep constant the length. We write $w \to w'$.

Finite Coxeter groups

Theorem (Geck-Pfeiffer, Geck-Kim-Pfeiffer, H., H.-Nie)

Let W_0 be a finite Coxeter group and $\mathcal{O}$ be a conjugacy class. Let $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$. Then

1. Each element w of $\mathcal{O}$ can be brought to a minimal length element w' by conjugation by simple reflections which reduce or keep constant the length. We write $w \to w'$.
2. Any two elements in $\mathcal{O}_{\min}$ are "strongly conjugate", i.e., conjugate in the associated Braid group.

Finite Coxeter groups

Theorem (Geck-Pfeiffer, Geck-Kim-Pfeiffer, H., H.-Nie)

Let W_0 be a finite Coxeter group and $\mathcal{O}$ be a conjugacy class. Let $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$. Then

1. Each element w of $\mathcal{O}$ can be brought to a minimal length element w' by conjugation by simple reflections which reduce or keep constant the length. We write $w \rightarrow w'$.
2. Any two elements in $\mathcal{O}_{\min}$ are "strongly conjugate", i.e., conjugate in the associated Braid group.

Elliptic conjugacy classes of type E_8

C	$\lvert C \rvert$	$\lvert C_{\min} \rvert$	d	$l(w)$	$w \in$	w^d
E_8	$2^{13}.3^4.5.7$	128	30	8	12345678	Δ^2
$E_8(a_1)$	$2^{11}.3^4.5^2.7$	320	24	10	1234254678	Δ^2
$E_8(a_2)$	$2^{12}.3^5.5.7$	624	20	12	123425465478	Δ^2
$E_8(a_4)$	$2^{13}.3^2.5^2.7$	732	18	14	12342542345678	$\Delta^2\Delta_{24}^4$
$E_8(a_5)$	$2^{13}.3^4.5.7$	1516	15	16	1231423454657658	Δ^2
$E_7 + A_1$	$2^{12}.3^2.5^2.7$	192	18	16	1234234542345678	$\Delta^2\Delta_{24}^{16}$
D_8	$2^{12}.3^5.5^2$	852	14	18	123423454234565768	$\Delta^2\Delta_2^{12}$
$E_8(a_3)$	$2^9.3^3.5^2.7$	2696	12	20	12314234542365476548	Δ^2
$D_8(a_1)$	$2^{11}.3^3.5^2.7$	2040	12	22	1234234542345654765876	$\Delta^2\Delta_{4578}^4$
$E_8(a_7)$	$2^9.3^3.5^2.7$	2360	12	22	1231423454316542345678	$\Delta^2\Delta_{2345}^2$
$E_8(a_6)$	$2^{11}.3^4.7$	3370	10	24	123142314542345654765876	Δ^2
$E_7(a_2) + A_1$	$2^{10}.3^3.5^2.7$	1758	12	24	123142314354316542345678	$\Delta^2\Delta_{2345}^2\Delta_{24}^8$
$E_6 + A_2$	$2^9.3^3.5^2.7$	840	12	26	12314231454231456542345678	$\Delta^2\Delta_{2345}^6$
$D_8(a_2)$	$2^{12}.3^4.5.7$	4996	30	26	12342354234654276542345678	$\Delta^6\Delta_{2456}^4\Delta_5^{20}$
A_8	$2^{13}.3^2.5^2.7$	2816	9	28	1231423145423145654234567687	$\Delta^2\Delta_{34}^4$
$D_8(a_3)$	$2^8.3^4.5^2.7$	7748	8	30	123142314354231465423476548765	Δ^2
$D_6 + 2A_1$	$2^{10}.3^4.7$	256	10	32	12342345423456542345676542345678	$\Delta^2\Delta_{2456}^8$
$A_7 + A_1$	$2^7.3^5.5^2.7$	2080	8	34	1231423454231456542345676542345678	$\Delta^2\Delta_{2345}^2\Delta_{25}^4$
$E_8(a_8)$	$2^7.5.7$	4480	6	40	1 1 143542314565423145676542345678765	Δ^2
$E_7(a_4) + A_1$	$2^9.5^2.7$	11592	6	42	1 1 14354231435426542314567654234567876	$\Delta^2\Delta_{34}^4$
$2D_4$	$2^9.3.5^2.7$	4070	6	44	1 1 1435423456542314567654231435465768765	$\Delta^2\Delta_{1367}^4$
$E_6(a_2) + A_2$	$2^8.3^2.5^2.7$	16374	6	44	1 1 1454231436542314354265431765423456787	$\Delta^2\Delta_{2345}^2$
$A_5 + A_2 + A_1$	$2^9.3^2.5^2.7$	3752	6	46	1231423145423145654234567654234567876542345678	$\Delta^2\Delta_{234578}^2\Delta_{78}^2$
$D_5(a_1) + A_3$	$2^8.3^3.5^2.7$	15134	12	46	1231423154231654317654231435426543176542345678	$\Delta^4\Delta_{123456}^2$
$2A_4$	$2^{11}.3^4.7$	7952	5	48	123142314542314565423145676542314567876542345678	Δ^2
$2D_4(a_1)$	$2^4.3^3.5.7$	15120	4	60	1231423143· ·54231435426542345765423143548765423143542654765876	Δ^2
$D_4 + 4A_1$	$2^6.5.7$	56	6	64	1231423143· ·542314354265423143542654317654231435426543176542345678	$\Delta^2\Delta_{123456}^4$
$2A_3 + 2A_1$	$2^3.3^3.5^2.7$	1260	4	66	1231423143· ·54231435426542314354265431765423143542654317876542345678	$\Delta^2\Delta_{2345}^2$
$4A_2$	$2^7.5.7$	4480	3	80	123142314354231435· ·65423143542676542314354265431787654231435426543176542345678765	Δ^2
$8A_1$	1	1	2	120	w_0	Δ^2

Figure: Geck-Pfeiffer, Characters of finite Coxeter groups and Iwahori-Hecke algebras, table B.6.

Finite Hecke algebras

Theorem (Geck-Pfeiffer)

Let $\mathcal{H}$ be the Hecke algebra for the finite Weyl group W_0. Then

- For any conjugacy class $\mathcal{O}$ of W_0 and $w, w' \in \mathcal{O}_{\min}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are the same. We denote it by $T_{\mathcal{O}}$.

Finite Hecke algebras

Theorem (Geck-Pfeiffer)

Let $\mathcal{H}$ be the Hecke algebra for the finite Weyl group W_0. Then

- For any conjugacy class $\mathcal{O}$ of W_0 and $w, w' \in \mathcal{O}_{\min}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are the same. We denote it by $T_{\mathcal{O}}$.
- $\{T_{\mathcal{O}}\}$ is a basis of $\bar{\mathcal{H}}$.

Finite Hecke algebras

Theorem (Geck-Pfeiffer)

Let $\mathcal{H}$ be the Hecke algebra for the finite Weyl group W_0. Then

- For any conjugacy class $\mathcal{O}$ of W_0 and $w, w' \in \mathcal{O}_{\min}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are the same. We denote it by $T_{\mathcal{O}}$.
- $\{T_{\mathcal{O}}\}$ is a basis of $\bar{\mathcal{H}}$.

Here minimal length representatives give the same image in the cocenter follows from the fact that these elements are "strongly conjugate".

Finite Hecke algebras

Theorem (Geck-Pfeiffer)

Let $\mathcal{H}$ be the Hecke algebra for the finite Weyl group W_0. Then

- For any conjugacy class $\mathcal{O}$ of W_0 and $w, w' \in \mathcal{O}_{\min}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are the same. We denote it by $T_{\mathcal{O}}$.
- $\{T_{\mathcal{O}}\}$ is a basis of $\bar{\mathcal{H}}$.

Here minimal length representatives give the same image in the cocenter follows from the fact that these elements are "strongly conjugate".

$\{T_{\mathcal{O}}\}$ spans $\bar{\mathcal{H}}$ follows from the fact that each element w can be reduced to a minimal length element w' via $w \to w'$.

Finite Hecke algebras

Theorem (Geck-Pfeiffer)

Let $\mathcal{H}$ be the Hecke algebra for the finite Weyl group W_0. Then

- For any conjugacy class $\mathcal{O}$ of W_0 and $w, w' \in \mathcal{O}_{\min}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are the same. We denote it by $T_{\mathcal{O}}$.
- $\{T_{\mathcal{O}}\}$ is a basis of $\bar{\mathcal{H}}$.

Here minimal length representatives give the same image in the cocenter follows from the fact that these elements are "strongly conjugate".

$\{T_{\mathcal{O}}\}$ spans $\bar{\mathcal{H}}$ follows from the fact that each element w can be reduced to a minimal length element w' via $w \to w'$.

And $\{T_{\mathcal{O}}\}$ is linearly independent follows from the fact that

$$\sharp \text{ irreducible representations of } W_0 = \sharp \text{ conjugacy classes of } W_0.$$

Finite Hecke algebras

Theorem (Geck-Pfeiffer)

Let $\mathcal{H}$ be the Hecke algebra for the finite Weyl group W_0. Then

- For any conjugacy class $\mathcal{O}$ of W_0 and $w, w' \in \mathcal{O}_{\min}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are the same. We denote it by $T_{\mathcal{O}}$.
- $\{T_{\mathcal{O}}\}$ is a basis of $\bar{\mathcal{H}}$.

Here minimal length representatives give the same image in the cocenter follows from the fact that these elements are "strongly conjugate".

$\{T_{\mathcal{O}}\}$ spans $\bar{\mathcal{H}}$ follows from the fact that each element w can be reduced to a minimal length element w' via $w \to w'$.

And $\{T_{\mathcal{O}}\}$ is linearly independent follows from the fact that

$$\sharp \text{ irreducible representations of } W_0 = \sharp \text{ conjugacy classes of } W_0.$$

From finite Weyl groups to affine Weyl groups

We'd like to understand the behaviour of elements in a given conjugacy class $\mathcal{O}$ of an (extended) affine Weyl group and to see if

- Any element $w \in \mathcal{O}$ can be reduced to an element $w' \in \mathcal{O}_{\min}$.
- Any two elements in $\mathcal{O}_{\min}$ are "strongly conjugate".

The main difficulty is that in general, $\mathcal{O}$ contains infinite elements.

This make it impossible to check by computer via brute force even for a simple affine Weyl group such as $\tilde{A}_2$.

From finite Weyl groups to affine Weyl groups

We'd like to understand the behaviour of elements in a given conjugacy class $\mathcal{O}$ of an (extended) affine Weyl group and to see if

- Any element $w \in \mathcal{O}$ can be reduced to an element $w' \in \mathcal{O}_{\min}$.
- Any two elements in $\mathcal{O}_{\min}$ are "strongly conjugate".

The main difficulty is that in general, $\mathcal{O}$ contains infinite elements.

This make it impossible to check by computer via brute force even for a simple affine Weyl group such as $\tilde{A}_2$.

How can we see the infinity?

From finite Weyl groups to affine Weyl groups

We'd like to understand the behaviour of elements in a given conjugacy class $\mathcal{O}$ of an (extended) affine Weyl group and to see if

- Any element $w \in \mathcal{O}$ can be reduced to an element $w' \in \mathcal{O}_{\min}$.
- Any two elements in $\mathcal{O}_{\min}$ are "strongly conjugate".

The main difficulty is that in general, $\mathcal{O}$ contains infinite elements.

This make it impossible to check by computer via brute force even for a simple affine Weyl group such as $\tilde{A}_2$.

How can we see the infinity?

MIThenge

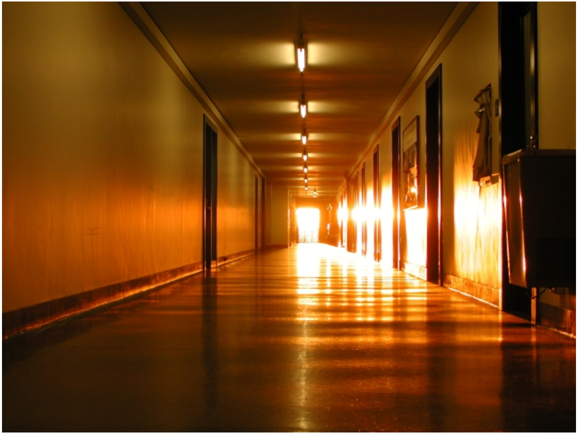


Figure: MIT Infinite Corridor, January 2001. Photograph by Matt Yourst.

Predictions: 11/11/2014-11/14/2014, Time: 4:18:52 to 4:22:38.

MIThenge

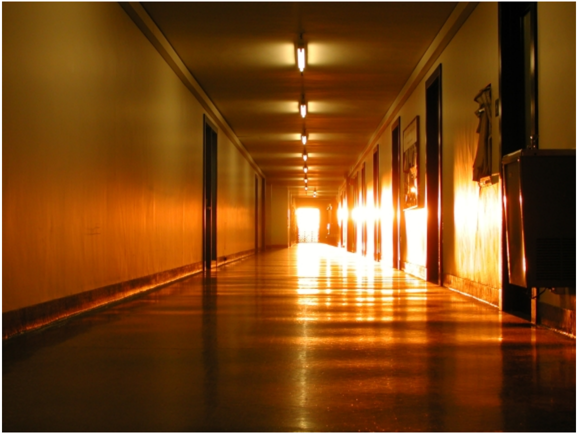


Figure: MIT Infinite Corridor, January 2001. Photograph by Matt Yourst.

Predictions: 11/11/2014-11/14/2014, Time: 4:18:52 to 4:22:38.

Arithmetic invariants of conjugacy classes in $\tilde{W}$

Definition

For $w \in \tilde{W}$, $w^{|W_0|} = t^\lambda$ for some $\lambda \in X$. Set $\nu_w = \lambda/|W_0| \in X_{\mathbb{Q}}$, the *Newton point* of w. Let $\bar{\nu}_w$ be the unique dominant element in $W_0 \cdot \nu_w$.

The map $f : \tilde{W} \to \tilde{W}/W_a \times X_{\mathbb{Q}}, \quad w \mapsto (wW_a, \bar{\nu}_w)$ is constant on each conjugacy class of $\tilde{W}$.

Theorem (Kottwitz, H.)

Let G be a split p-adic group with Iwahori-Weyl group $\tilde{W}$ and σ be the standard Frobenius map. Then

1. The map f factors as $\tilde{W}/conj \xrightarrow{\Phi} G/\sigma - conj \xrightarrow{1-1} Im(f)$.
2. The map Φ is surjective and has a natural cross-section given by the set of straight conjugacy classes of $\tilde{W}$.

Arithmetic invariants of conjugacy classes in $\tilde{W}$

Definition

For $w \in \tilde{W}$, $w^{|W_0|} = t^\lambda$ for some $\lambda \in X$. Set $\nu_w = \lambda/|W_0| \in X_{\mathbb{Q}}$, the *Newton point* of w. Let $\bar{\nu}_w$ be the unique dominant element in $W_0 \cdot \nu_w$.

The map $f : \tilde{W} \to \tilde{W}/W_a \times X_{\mathbb{Q}}, \quad w \mapsto (wW_a, \bar{\nu}_w)$ is constant on each conjugacy class of $\tilde{W}$.

Theorem (Kottwitz, H.)

Let G be a split p-adic group with Iwahori-Weyl group $\tilde{W}$ and σ be the standard Frobenius map. Then

1. The map f factors as $\tilde{W}/conj \xrightarrow{\Phi} G/\sigma - conj \xrightarrow{1-1} Im(f)$.
2. The map Φ is surjective and has a natural cross-section given by the set of straight conjugacy classes of $\tilde{W}$.

Affine Weyl group

In the study of minimal length elements in a given conjugacy class of $\tilde{W}$, the Newton point provide a crucial invariant. It allows us to reduce from affine Weyl groups to finite Weyl groups (associated to the Newton point).

Theorem (H., H.-Nie)

Let W be an extended affine Weyl group and $\mathcal{O}$ be a conjugacy class. Then

Affine Weyl group

In the study of minimal length elements in a given conjugacy class of $\tilde{W}$, the Newton point provide a crucial invariant. It allows us to reduce from affine Weyl groups to finite Weyl groups (associated to the Newton point).

Theorem (H., H.-Nie)

Let W be an extended affine Weyl group and $\mathcal{O}$ be a conjugacy class. Then

Affine Weyl group

In the study of minimal length elements in a given conjugacy class of $\tilde{W}$, the Newton point provide a crucial invariant. It allows us to reduce from affine Weyl groups to finite Weyl groups (associated to the Newton point).

Theorem (H., H.-Nie)

Let W be an extended affine Weyl group and $\mathcal{O}$ be a conjugacy class. Then

1. For any $w \in \mathcal{O}$, there exists $w' \in \mathcal{O}_{\min}$ such that $w \rightarrow w'$.

Affine Weyl group

In the study of minimal length elements in a given conjugacy class of $\tilde{W}$, the Newton point provide a crucial invariant. It allows us to reduce from affine Weyl groups to finite Weyl groups (associated to the Newton point).

Theorem (H., H.-Nie)

Let W be an extended affine Weyl group and $\mathcal{O}$ be a conjugacy class. Then

1. For any $w \in \mathcal{O}$, there exists $w' \in \mathcal{O}_{\min}$ such that $w \to w'$.
2. Any two elements in $\mathcal{O}_{min}$ are "strongly conjugate".

Affine Weyl group

In the study of minimal length elements in a given conjugacy class of $\tilde{W}$, the Newton point provide a crucial invariant. It allows us to reduce from affine Weyl groups to finite Weyl groups (associated to the Newton point).

Theorem (H., H.-Nie)

Let W be an extended affine Weyl group and $\mathcal{O}$ be a conjugacy class. Then

1. For any $w \in \mathcal{O}$, there exists $w' \in \mathcal{O}_{\min}$ such that $w \to w'$.
2. Any two elements in $\mathcal{O}_{min}$ are "strongly conjugate".

Iwahori-Matsumoto presentation of $\bar{\mathcal{H}}$

Theorem (H.-Nie)

Let $\mathcal{H}$ be an extended affine Hecke algebra. Then

- For any conjugacy class $\mathcal{O}$ of $\tilde{W}$ and $w, w' \in \mathcal{O}_{\min}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are the same. We denote it by $T_{\mathcal{O}}$.

Iwahori-Matsumoto presentation of $\bar{\mathcal{H}}$

Theorem (H.-Nie)

Let $\mathcal{H}$ be an extended affine Hecke algebra. Then

- For any conjugacy class $\mathcal{O}$ of $\tilde{W}$ and $w, w' \in \mathcal{O}_{\min}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are the same. We denote it by $T_{\mathcal{O}}$.
- $\{T_{\mathcal{O}}\}$ spans of $\bar{\mathcal{H}}$.

Iwahori-Matsumoto presentation of $\bar{\mathcal{H}}$

Theorem (H.-Nie)

Let $\mathcal{H}$ be an extended affine Hecke algebra. Then

- For any conjugacy class $\mathcal{O}$ of $\tilde{W}$ and $w, w' \in \mathcal{O}_{\min}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are the same. We denote it by $T_{\mathcal{O}}$.
- $\{T_{\mathcal{O}}\}$ spans of $\bar{\mathcal{H}}$.

We'll see later that $\{T_{\mathcal{O}}\}$ is always a basis of $\bar{\mathcal{H}}$. This is the Iwahori-Matsumoto presentation of the cocenter.

Iwahori-Matsumoto presentation of $\bar{\mathcal{H}}$

Theorem (H.-Nie)

Let $\mathcal{H}$ be an extended affine Hecke algebra. Then

- For any conjugacy class $\mathcal{O}$ of $\tilde{W}$ and $w, w' \in \mathcal{O}_{\min}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are the same. We denote it by $T_{\mathcal{O}}$.
- $\{T_{\mathcal{O}}\}$ spans of $\bar{\mathcal{H}}$.

We'll see later that $\{T_{\mathcal{O}}\}$ is always a basis of $\bar{\mathcal{H}}$. This is the Iwahori-Matsumoto presentation of the cocenter.

However, the relation between Iwahori-Matsumoto presentation and representation theory (the induction functor and restriction functors, etc.) is hard to understand.

Iwahori-Matsumoto presentation of $\bar{\mathcal{H}}$

Theorem (H.-Nie)

Let $\mathcal{H}$ be an extended affine Hecke algebra. Then

- For any conjugacy class $\mathcal{O}$ of $\tilde{W}$ and $w, w' \in \mathcal{O}_{\min}$, the image of T_w and $T_{w'}$ in $\bar{\mathcal{H}}$ are the same. We denote it by $T_{\mathcal{O}}$.
- $\{T_{\mathcal{O}}\}$ spans of $\bar{\mathcal{H}}$.

We'll see later that $\{T_{\mathcal{O}}\}$ is always a basis of $\bar{\mathcal{H}}$. This is the Iwahori-Matsumoto presentation of the cocenter.

However, the relation between Iwahori-Matsumoto presentation and representation theory (the induction functor and restriction functors, etc.) is hard to understand.

Bernstein-Lusztig presentation of $\bar{\mathcal{H}}$

In order to give the Bernstein-Lusztig presentation of the cocenter, one needs to have a nice paramitrization of conjugacy classes of $\tilde{W}$.

Proposition (H.-Nie)

Let $\mathcal{A}$ be the set of all pairs (J, C), where C is an elliptic conjugacy class of $\tilde{W}_J$ and ν_w is dominant for all $w \in C$. Then

$$\pi : \mathcal{A}/\sim \xrightarrow{1-1} \text{conjugacy classes of } \tilde{W}.$$

Theorem (H.-Nie)

Let $i_J : \mathcal{H}_J \to \mathcal{H}$ be the inclusion and $\bar{i}_J : \bar{\mathcal{H}}_J \to \bar{\mathcal{H}}$ the induced map. For any $(J, C) \in \mathcal{A}$,

$$T_{\mathcal{O}} = \bar{i}_J(T^J_C).$$

Bernstein-Lusztig presentation of $\bar{\mathcal{H}}$

In order to give the Bernstein-Lusztig presentation of the cocenter, one needs to have a nice paramitrization of conjugacy classes of $\tilde{W}$.

Proposition (H.-Nie)

Let $\mathcal{A}$ be the set of all pairs (J, C), where C is an elliptic conjugacy class of $\tilde{W}_J$ and ν_w is dominant for all $w \in C$. Then

$$\pi : \mathcal{A}/\sim \xrightarrow{1-1} \text{conjugacy classes of } \tilde{W}.$$

Theorem (H.-Nie)

Let $i_J : \mathcal{H}_J \to \mathcal{H}$ be the inclusion and $\bar{i}_J : \bar{\mathcal{H}}_J \to \bar{\mathcal{H}}$ the induced map. For any $(J, C) \in \mathcal{A}$,

$$T_{\mathcal{O}} = \bar{i}_J(T_C^J).$$

Cocenter-Representation duality, I

Theorem (Ciubotaru-H.)

Cocenter-Representation duality holds for affine Hecke algebras for generic parameters.

Cocenter-Representation duality, I

Theorem (Ciubotaru-H.)

Cocenter-Representation duality holds for affine Hecke algebras for generic parameters.

Theorem (Ciubotaru-H.)

- $\{T_{\mathcal{O}}\}$ is a basis of $\bar{\mathcal{H}}$ (basis Theorem).

Cocenter-Representation duality, I

Theorem (Ciubotaru-H.)

Cocenter-Representation duality holds for affine Hecke algebras for generic parameters.

Theorem (Ciubotaru-H.)

- $\{T_{\mathcal{O}}\}$ is a basis of $\bar{\mathcal{H}}$ (basis Theorem).

Cocenter-Representation duality, I

Theorem (Ciubotaru-H.)

Cocenter-Representation duality holds for affine Hecke algebras for generic parameters.

Theorem (Ciubotaru-H.)

- $\{T_{\mathcal{O}}\}$ is a basis of $\bar{\mathcal{H}}$ (basis Theorem).
- For any parameters, the image of Tr is $R(\mathcal{H}_{\mathrm{q}})^*_{good}$ (trace Paley-Wiener Theorem).

Cocenter-Representation duality, I

Theorem (Ciubotaru-H.)

Cocenter-Representation duality holds for affine Hecke algebras for generic parameters.

Theorem (Ciubotaru-H.)

- $\{T_{\mathcal{O}}\}$ is a basis of $\bar{\mathcal{H}}$ (basis Theorem).
- For any parameters, the image of Tr is $R(\mathcal{H}_{\mathrm{q}})^*_{good}$ (trace Paley-Wiener Theorem).
- For generic parameter, the map Tr is injective (density Theorem).

Cocenter-Representation duality, I

Theorem (Ciubotaru-H.)

Cocenter-Representation duality holds for affine Hecke algebras for generic parameters.

Theorem (Ciubotaru-H.)

- $\{T_{\mathcal{O}}\}$ is a basis of $\bar{\mathcal{H}}$ (basis Theorem).
- For any parameters, the image of Tr is $R(\mathcal{H}_{\mathrm{q}})^*_{good}$ (trace Paley-Wiener Theorem).
- For generic parameter, the map Tr is injective (density Theorem).

Cocenter-Representation duality, II

To explain the basic idea of cocenter-representation duality, we start with the group algebra for a finite group and its representation over $\mathbb{C}$.

Then $\sharp$ irreducible representations=$\sharp$ conjugacy classes. In other words,

$$\text{rank of Grothendieck group}=\text{rank of cocenter.}$$

And the trace map leads to an invertible square matrix.

Cocenter-Representation duality, II

To explain the basic idea of cocenter-representation duality, we start with the group algebra for a finite group and its representation over $\mathbb{C}$.

Then $\sharp$ irreducible representations$=\sharp$ conjugacy classes. In other words,

$$\text{rank of Grothendieck group} = \text{rank of cocenter}.$$

And the trace map leads to an invertible square matrix.

Difficulty: for affine Hecke algebras, both the rank of $R(H)$ and the rank of $\bar{H}$ are infinite.

Cocenter-Representation duality, II

To explain the basic idea of cocenter-representation duality, we start with the group algebra for a finite group and its representation over $\mathbb{C}$.

Then $\sharp$ irreducible representations$=\sharp$ conjugacy classes. In other words,

$$\text{rank of Grothendieck group}=\text{rank of cocenter}.$$

And the trace map leads to an invertible square matrix.

Difficulty: for affine Hecke algebras, both the rank of $R(H)$ and the rank of $\bar{H}$ are infinite.

Cocenter-Representation duality, III

The key idea is to construct filtrations on $\bar{\mathcal{H}}$ and on $R(\mathcal{H})$ so that the corresponding subquotient "matches".

Question

What are the filtrations?

Elliptic Theory (Arthur): There is a standard filtration on $R(\mathcal{H})$ obtained from induced modules: $R(\mathcal{H}) \supset R(\mathcal{H})^1 \supset \cdots$, where $R(\mathcal{H})^1$ is spanned by all the induced modules. The elliptic quotient $\bar{R}(\mathcal{H})_{ell} = R(\mathcal{H})/R(\mathcal{H})^1$.

Cocenter-Representation duality, III

The key idea is to construct filtrations on $\bar{\mathcal{H}}$ and on $R(\mathcal{H})$ so that the corresponding subquotient "matches".

Question

What are the filtrations?

Elliptic Theory (Arthur): There is a standard filtration on $R(\mathcal{H})$ obtained from induced modules: $R(\mathcal{H}) \supset R(\mathcal{H})^1 \supset \cdots$, where $R(\mathcal{H})^1$ is spanned by all the induced modules. The elliptic quotient $\bar{R}(\mathcal{H})_{ell} = R(\mathcal{H})/R(\mathcal{H})^1$.

However, the dual filtration on $\bar{\mathcal{H}}$ is very complicated to understand.

Cocenter-Representation duality, III

The key idea is to construct filtrations on $\bar{\mathcal{H}}$ and on $R(\mathcal{H})$ so that the corresponding subquotient "matches".

Question

What are the filtrations?

Elliptic Theory (Arthur): There is a standard filtration on $R(\mathcal{H})$ obtained from induced modules: $R(\mathcal{H}) \supset R(\mathcal{H})^1 \supset \cdots$, where $R(\mathcal{H})^1$ is spanned by all the induced modules. The elliptic quotient $\bar{R}(\mathcal{H})_{ell} = R(\mathcal{H})/R(\mathcal{H})^1$.

However, the dual filtration on $\bar{\mathcal{H}}$ is very complicated to understand.

Rigid quotient, I

Here we consider different filtrations. For simplicity, we only consider semisimple root datum here.

On the $\bar{\mathcal{H}}$ side, we consider:

$$\{0\} \subset \bar{\mathcal{H}}_1 \subset \cdots \subset \bar{\mathcal{H}},$$

where $\bar{\mathcal{H}}_1$ is spanned by $T_{\mathcal{O}}$ with $\nu_{\mathcal{O}} = 0$.

Rigid quotient, I

Here we consider different filtrations. For simplicity, we only consider semisimple root datum here.

On the $\bar{\mathcal{H}}$ side, we consider:

$$\{0\} \subset \bar{\mathcal{H}}_1 \subset \cdots \subset \bar{\mathcal{H}},$$

where $\bar{\mathcal{H}}_1$ is spanned by $T_{\mathcal{O}}$ with $\nu_{\mathcal{O}} = 0$.

On the $R(H)$ side, we consider the dual filtration:

$$R(\mathcal{H}) \supset R(\mathcal{H})^1_{rigid} \supset \cdots,$$

where $R(\mathcal{H})^1_{rigid}$ is spanned by $i_J(\sigma) - i_J(\sigma \circ \chi_t)$ for all J, σ, t. The quotient $\bar{R}(H)_{rigid} = R(\mathcal{H})/R(\mathcal{H})^1_{rigid}$ is called the rigid quotient.

Rigid quotient, I

Here we consider different filtrations. For simplicity, we only consider semisimple root datum here.

On the $\bar{\mathcal{H}}$ side, we consider:

$$\{0\} \subset \bar{\mathcal{H}}_1 \subset \cdots \subset \bar{\mathcal{H}},$$

where $\bar{\mathcal{H}}_1$ is spanned by $T_{\mathcal{O}}$ with $\nu_{\mathcal{O}} = 0$.

On the $R(H)$ side, we consider the dual filtration:

$$R(\mathcal{H}) \supset R(\mathcal{H})^1_{rigid} \supset \cdots ,$$

where $R(\mathcal{H})^1_{rigid}$ is spanned by $i_J(\sigma) - i_J(\sigma \circ \chi_t)$ for all J, σ, t. The quotient $\bar{R}(H)_{rigid} = R(\mathcal{H})/R(\mathcal{H})^1_{rigid}$ is called the rigid quotient.

Rigid quotient, II

Theorem (Ciubotaru-H.)

For generic parameters, the trace map $Tr : \bar{\mathcal{H}} \times R(\mathcal{H}) \to \mathbb{C}$ induces a perfect pairing

$$Tr : \bar{\mathcal{H}}_1 \times \bar{R}(\mathcal{H})_{rigid} \to \mathbb{C}.$$

Rigid quotient, II

Theorem (Ciubotaru-H.)

For generic parameters, the trace map $Tr : \bar{\mathcal{H}} \times R(\mathcal{H}) \to \mathbb{C}$ induces a perfect pairing

$$Tr : \bar{\mathcal{H}}_1 \times \bar{R}(\mathcal{H})_{rigid} \to \mathbb{C}.$$

Corollary (Ciubotaru-H.)

1. For generic parameters, the rank of $\bar{R}(\mathcal{H}_q)_{rigid}$ equals the number of conjugacy classes of $\tilde{W}$ of finite order.
2. For generic parameters, the rank of $\bar{R}(\mathcal{H}_q)_{ell}$ equals the number of elliptic conjugacy classes of $\tilde{W}$.

Rigid quotient, II

Theorem (Ciubotaru-H.)

For generic parameters, the trace map $Tr : \bar{\mathcal{H}} \times R(\mathcal{H}) \to \mathbb{C}$ induces a perfect pairing

$$Tr : \bar{\mathcal{H}}_1 \times \bar{R}(\mathcal{H})_{rigid} \to \mathbb{C}.$$

Corollary (Ciubotaru-H.)

1. For generic parameters, the rank of $\bar{R}(\mathcal{H}_{\mathrm{q}})_{rigid}$ equals the number of conjugacy classes of $\tilde{W}$ of finite order.
2. For generic parameters, the rank of $\bar{R}(\mathcal{H}_{\mathrm{q}})_{ell}$ equals the number of elliptic conjugacy classes of $\tilde{W}$.

My wild guess for modular case

Let $\mathbf{k}$ be an algebraically closed field and $q(s) \in \mathbf{k}^\times$. Let $R(\mathcal{H}_{q,\mathbf{k}})$ be the Grothendieck group of finite dimensional representation of $\mathcal{H}_q$ over $\mathbf{k}$.

For any $K \subset \tilde{S}$ with W_K is finite, let Ω_K be the subset of Ω that stabilizes K and $W_K^\sharp = W_K \rtimes \Omega_K$. Let $\mathcal{I}$ be the set of all K with maximal $W_K^\sharp$.

Question

Is the rank of $\bar{R}(\mathcal{H}_{q,\mathbf{k}})_{ell}$ equal to the $\sum_{K \in \mathcal{I}/conj} \text{rank}\, \bar{R}(\mathcal{H}_{K,q,\mathbf{k}} \rtimes \Omega_K)_{ell}$?

Here the left side is for affine Hecke algebras and the right side is for finite Hecke algebras (with twist).

Example

We list the expected rank of $\bar{R}(\mathcal{H}_{q,\mathbf{k}})_{ell}$ for SL_3 and PGL_3. Here $\Phi_3(q) = q^2 + q + 1$.

SL_3	$char(\mathbf{k}) \neq 3$	$char(\mathbf{k}) = 3$
$\Phi_3(q) \neq 0$	3	3
$\Phi_3(q) = 0$	0	0

PGL_3	$char(\mathbf{k}) \neq 3$	$char(\mathbf{k}) = 3$
$\Phi_3(q) \neq 0$	3	1
$\Phi_3(q) = 0$	2	0

One more thing: on the real group

Thank you, David:
for being a great teacher, guide and friend of my wife and I.

Happy birthday!

Thank you, David:
for being a great teacher, guide and friend of my wife and I.

Happy birthday!