

Stacks in Representation Theory.

What is a representation of an algebraic group ?

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0. Representations as geometric objects

In my talk I would like to introduce a new approach to (or rather a new language for) Representation Theory of groups.

Namely I propose to consider representation of a group as a sheaf on some geometric object.

This point of view implies that in case of an algebraic group G over a local or finite field the standard definition of a representation of the group G is in some sense "incorrect".

I would like to convince you that the category $Rep(G)$ of representations of G should be replaced by some larger category $\mathcal{M}(G)$.

1. What is a representation of an algebraic group ?

Standard approach.

F - p -adic field

\mathcal{G} algebraic group over F

$G = \mathcal{G}(F)$ topological group

$\text{Rep}(G)$ – appropriate category of representations of G

One of main goals of this lecture is to explain that

This approach is ideologically inconsistent.

First let me describe a striking example that illustrates this point.
It is based on the following "Geometric Ansatz".

2. Geometric Ansatz

Let $a : \mathcal{G} \times \mathcal{Z} \rightarrow \mathcal{Z}$ be a transitive action of an algebraic group on an algebraic variety.

Passing to F -points we get a continuous action $a : G \times Z \rightarrow Z$.

This action is usually not transitive and we can write Z as a union of open orbits $Z = \bigsqcup Z_i, i = 1, \dots, n$.

Ansatz

- 1 The space Z is "good", i.e. it is easy to describe
2. Every individual orbit Z_i is "bad" space, that means that it is difficult to describe.

3. Equivariant sheaves.

Consider a continuous action $a : G \times Z \rightarrow Z$. Central role in my approach plays the category $Sh_G(Z)$ of G -equivariant sheaves (of complex vector spaces) on Z .

Recall, that equivariant sheaf is a sheaf F on Z equipped with an isomorphism $\alpha : a^*(F) \rightarrow pr_Z^*(F)$ with some conditions.

Fact 1. The category $SH_G(pt)$ is equivalent to the category $Rep(G)$ of smooth representations of G .

Fact 2. Suppose that the action of G on Z is transitive. Fix a point $z \in Z$ and denote by H its stabilizer in G . Then $Sh_G(Z) \approx Sh_H(z) \approx Rep(H)$.

4. An example - representations of orthogonal groups.

V - n -dimensional space over F .

Group $G = GL(V)$ acts on the space Z of non-degenerate quadratic forms.

Fix a form $Q \in Z$, denote by H be the corresponding orthogonal group $O(Q)$ and by Z_0 the G -orbit of Q in Z . Then we have

$$\text{Rep}(H) \approx \text{Sh}_G(Z_0) \subset \text{Sh}_G(Z),$$

We see that according to the geometric Ansatz the category $\text{Rep}(H) \approx \text{Sh}_G(Z_0)$ is a bad category.

However this category can be naturally extended to a "good" category $\mathcal{M} := \text{Sh}_G(Z)$ of G -equivariant sheaves on Z

5. Representations and sheaves on Stacks.

This example suggests that in general we should extend the category $Rep(G)$ to some larger and more natural category $\mathcal{M}(G)$.

This agrees with observation by several mathematicians (e.g by D. Vogan) that when we classify irreducible representations it is better to work with the union of sets $Irr(G_i)$ for several forms of the group G than with one set $Irr(G)$.

In order to describe this category $\mathcal{M}(G)$ I propose to consider representations as sheaves on stacks. Let me discuss the notion of stack.

6. Stacks.

Informally stack is a "space" X such that every point $x \in X$ is endowed with a group G_x of automorphisms of inner degrees of freedom at this point.

We see that in order to consider stacks we should first fix a Geometric Environment, i.e. a category \mathcal{S} of spaces on which we model our stacks. In fact \mathcal{S} should be considered with some Grothendieck topology (standard term for such category \mathcal{S} is "site").

Usually one works with the following sites:

- (i) Category of schemes over a field F
- (ii) Category of smooth manifolds
- (iii) Category of topological spaces (e.g. totally discontinuous)
- (iv) Category *Sets* of sets.

7. Groupoids – stacks in category *Sets*.

Stacks in the category $\mathcal{S} = \text{Sets}$ are just **groupoids**. By definition groupoid is a category in which all morphisms are isomorphisms.

To every discrete group G we assign the **basic groupoid** $BG = pt/G$ as follows:

An object of the category BG is a G -torsor T and morphisms in this category are morphisms of G -sets.

More generally, given an action of the group G on a set Z we define the **action groupoid** $BG(Z) = Z/G$ as follows:

Object of $BG(Z)$ is a G -torsor T equipped with a G -morphism $\nu : T \rightarrow Z$. Morphisms are morphisms of G -sets over Z .

8. Representations as sheaves on groupoids.

Now consider an arbitrary groupoid \mathcal{X} . It is natural to think about \mathcal{X} as a geometric object (some kind of a space).

We define a **sheaf** R (of vector spaces) on a groupoid \mathcal{X} to be a functor $R : \mathcal{X} \rightarrow \text{Vect}$.

We denote by $Sh(\mathcal{X})$ the category of sheaves on \mathcal{X} .

Claim. (i) Category $Sh(BG)$ is naturally equivalent to the category $Rep(G)$.

(ii) Category $Sh(BG(Z))$ is naturally equivalent to the category $Sh_G(Z)$ of G -equivariant sheaves on Z .

This gives us a "geometric" description of categories $Rep(G)$ and $Sh_G(Z)$ as sheaves.

9. Topological groupoids.

If G is a totally discontinuous topological group one can consider the basic groupoid as a topological groupoid. One can then define sheaves on a topological groupoid \mathcal{X} taking this topology into account.

In this case the category $Sh(BG)$ is equivalent to the category of smooth representations of the group G .

10. Algebraic groups and stacks.

Let \mathcal{G} be an algebraic group defined over a field F .

By analogy with the discrete case one can define the **basic stack** $B\mathcal{G} = pt/\mathcal{G}$. This will be an algebraic stack modeled on the category \mathcal{S} of schemes over F (more details later).

For any algebraic stack \mathcal{X} over F its F -points $\mathcal{X}(F)$ form a groupoid. We define an F -**sheaf** on the stack \mathcal{X} to be a sheaf on the groupoid $\mathcal{X}(F)$.

Main object that I propose to study is the category $\mathcal{M} = \mathcal{M}(\mathcal{G}, F)$ of F -sheaves on the algebraic stack $B\mathcal{G}$.

This category \mathcal{M} should be considered as a "correct" category of representations of the algebraic group G . I call the objects of this category **stacky G -modules**.

11. Central subtle point.

Now we have two competing definitions of a representation of an algebraic group G .

1. A sheaf on the basic groupoid BG of the group $G = \mathcal{G}(F)$.
2. A sheaf on the groupoid $B\mathcal{G}(F)$ of F -points of the basic stack $B\mathcal{G}$.

We have a natural imbedding $BG \hookrightarrow (B\mathcal{G})(F)$ but it is not always an equivalence of categories. So the category $\mathcal{M}(\mathcal{G}, F)$ might be different from the category $Rep(G) = Sh(BG)$.

Usually people use definition 1. However in my opinion the definition 2 is much more appropriate.

In a sense my lecture is finished. Let me add several comments and technical remarks about groupoids and stacks.

12. Equivalence of groupoids

We know that if two objects of some category are isomorphic then we can consider them as two realizations of the same geometric structure.

Similarly, if two groupoids \mathcal{X} and \mathcal{Y} are equivalent (as categories) we can assume that they represent two realizations of the same geometric structure.

A subtle point here is that the equivalences between these groupoids form a groupoid. This means that if we fix an equivalence $Q : \mathcal{X} \rightarrow \mathcal{Y}$ then this equivalence itself has automorphisms, and it is not clear how we should think about them.

13. Matrix models of basic and quotient groupoids.

Example. Consider an action $a : G \times Z \rightarrow Z$.

Let us define a groupoid $BG_0(Z)$ as follows:

Objects of $BG_0(Z)$ are points $z \in Z$ and morphisms are defined by $Mor(z, z') := \{g \in G \mid gz = z'\}$.

We also denote the groupoid $BG_0(pt)$ by BG_0 .

Claim. The groupoid $BG_0(Z)$ is canonically equivalent to the groupoid $BG(Z)$.

The groupoid $BG_0(Z)$ might be considered as a "matrix" version of the groupoid $BG(Z)$. It is better suited for computations.

14. Theory of groups and theory of groupoids.

I would like to explain that the theories describing groups and groupoids are essentially equivalent.

Proposition. 1. Every groupoid \mathcal{X} is canonically decomposed as a disjoint union of connected groupoids.

2. A connected groupoid \mathcal{Y} is equivalent to the basic groupoid of some group G .

Thus we see that any question about groupoids can be reduced to the case of connected groupoids. Then this case can be reduced to a question in group theory.

In my opinion the relation between theory of groupoids and group theory is very similar to the relation between linear algebra and matrix calculus.

15. Equivalence between groups and connected groupoids.

Note that the group G and the equivalence of groupoids in part 2 of the proposition is not canonical. It depends on a choice of an object $Y \in \mathcal{Y}$.

If we choose an object Y then we get a canonical equivalence of categories $Q = Q_Y : \mathcal{Y} \rightarrow BG$, where $G := \text{Aut}(Y)$. If we pick another object Y' we get a different equivalence $Q' : \mathcal{Y} \rightarrow BG'$.

Note that any choice of an isomorphism between Y and Y' defines natural isomorphisms $G \simeq G'$ and $Q \simeq Q'$. However there is no natural choice for such an isomorphism.

Next three constructions show that usually in Mathematics we encounter groupoids and not groups.

16. Multiplicative groupoid of a category.

Construction I. Starting with any category C we construct the **multiplicative groupoid** $C^* = Iso(C)$ that has the same collection of objects as category C and isomorphisms of C as morphisms.

Example 1. $C = Finsets$ – the category of finite sets.

In this case the groupoid $Iso(C)$ is essentially the collection of all symmetric groups S_n .

Example 2. $C = Vect_k$ – the category of finite dimensional vector spaces over a field k .

In this case the groupoid $Iso(C)$ describes the collection of groups $GL(n, k)$ for all n .

17. Poincare groupoid.

Construction II. Poincare groupoid of a topological space X .

Objects of $Poin(X)$ are points of X . Morphisms – homotopy classes of paths.

If the space X is path connected then the groupoid $Poin(X)$ is connected. For any point $x \in X$ the group $Aut_{Poin(X)}(x)$ is the fundamental group $\pi_1(X, x)$.

This shows that the Poincare groupoid is more basic notion than the fundamental group.

18. Galois groupoid.

Construction III. Galois groupoid $Gal(F)$ of a field F .

Objects of the groupoid $Gal(F)$ are field extensions $F \rightarrow \Omega$ such that Ω is an algebraic closure of F . Morphisms are morphisms of field extensions.

The groupoid $Gal(F)$ is connected. If we fix an algebraic closure Ω then by definition the group $Aut_{Gal(F)}(\Omega)$ is the Galois group $Gal(\Omega/F)$.

Again we see that the notion of Galois groupoid is more basic than the notion of Galois group.

19. What is a stack ?

Let us fix some site \mathcal{S} . I would like to describe the notion of a stack \mathcal{X} modeled on \mathcal{S} . I assume two features of this notion.

1. For every two stacks \mathcal{X}, \mathcal{Y} the collection of morphisms from \mathcal{X} to \mathcal{Y} form a groupoid $Mor(\mathcal{X}, \mathcal{Y})$.
2. Every object $S \in \mathcal{S}$ is a stack.

The natural idea is to characterize a stack \mathcal{X} by collection of groupoids $\mathcal{X}(S) := Mor(S, \mathcal{X})$ for all objects $S \in \mathcal{S}$.

In fact usually it is enough to know the groupoids $\mathcal{X}(S)$ for objects S in some subcategory $\mathcal{B} \subset \mathcal{S}$ provided it is large enough. For example, if \mathcal{S} is the category of schemes we can restrict everything to the subcategory \mathcal{B} of affine schemes.

20. Technical definition of a stack.

Fix a large subcategory $\mathcal{B} \subset \mathcal{S}$. We define a stack \mathcal{X} over the site \mathcal{S} to be the following collection of data:

- (i) To every object $S \in \mathcal{B}$ we assign a groupoid $\mathcal{X}(S)$
- (ii) To every morphism $\nu : S \rightarrow S'$ in \mathcal{B} we assign a functor $\mathcal{X}(S') \rightarrow \mathcal{X}(S)$
- (iii) To every composition of morphisms in \mathcal{B} we assign an isomorphism of appropriate functors.

This data should satisfy a variety of compatibility conditions and some finiteness conditions. For details see for example a note by Barbara Fantechi "Stacks for everybody".

21. Basic stack and quotient stack.

Here are some examples of stacks over the category \mathcal{S} of schemes over the field F . Fix an algebraic group $G \in \mathcal{S}$.

Example 1. Basic stack BG .

For an affine F -scheme S an object of the groupoid $BG(S)$ is a principal G -bundle P over S .

Example 2. Quotient stack $BG(\mathcal{Z})$

Let G act on a scheme $\mathcal{Z} \in \mathcal{S}$. We define the quotient stack $\mathcal{X} = BG(\mathcal{Z}) = \mathcal{Z}/G$ as follows:

Object of $\mathcal{X}(S)$ is a principal G -bundle P over S equipped with a G -morphism $\nu : P \rightarrow \mathcal{Z}$. Morphisms are morphisms of G -bundles over \mathcal{Z} .

22. F -sheaves on an algebraic stacks.

Let F be a local non-Archimedean field (or finite field).

I will describe a technical definition of an F -sheaf on an algebraic stack \mathcal{X} that corresponds to the intuitive notion of a sheaf on F -points of \mathcal{X} .

For any F -scheme \mathcal{Z} we consider the topological space $Z = \mathcal{Z}(F)$ and define an F -sheaf R on \mathcal{Z} to be a sheaf on Z . The category of these sheaves we denote by $Sh_F(\mathcal{Z})$. Any morphism $\nu : \mathcal{Z} \rightarrow \mathcal{W}$ defines a functor $\nu^* : Sh_F(\mathcal{W}) \rightarrow Sh_F(\mathcal{Z})$

Question. How to extend these categories to stacks ? How to define the category $Sh_F(\mathcal{X})$?

23. Technical definition of F -sheaves on stacks.

Suppose we have some notion of F -sheaves on stacks.

Fix an F -sheaf R on a stack \mathcal{X} . Then for any affine scheme S and any point $p \in \mathcal{X}(S) = \text{Mor}(S, \mathcal{X})$ we get an F -sheaf $R_p = p^*(F)$ on S . We also get a family of isomorphisms connecting these sheaves. Now we want to use these sheaves and isomorphisms to characterize the F -sheaf R .

Definition. An F -sheaf R on the stack \mathcal{X} is a collection of F -sheaves R_p for all morphisms $p : S \rightarrow \mathcal{X}$ and a collection of isomorphisms satisfying correct compatibility relations.

It is not difficult to check that this definition is compatible with informal definitions discussed before.

24. How to describe F -sheaves on an algebraic stack.

Let \mathcal{X} be a stack over F . I would like to give a convenient descriptions of F -sheaves on the stack \mathcal{X} in terms of equivariant sheaves. I will do this for the case of a quotient stack $\mathcal{X} \approx \mathcal{Z}/\mathcal{G}$.

Construction I. Let T_1, \dots, T_i be representatives of isomorphisms classes of \mathcal{G} -torsors. They are described by elements in $H^1(\text{Gal}(F), G)$.

For every i set $G_i = \text{Aut}(T_i)$ – this is the collection of all pure inner forms of the group G . Consider also the topological G_i -space $Z_i = \text{Mor}(T_i, \mathcal{Z})$.

Claim. Category $Sh_F(\mathcal{X})$ of F -sheaves on \mathcal{X} is equivalent to the product of categories $\prod Sh_{G_i}(Z_i)$

25. Vogan's picture.

In particular we see that the collection of simple objects of the category $Sh_F(\mathcal{X})$ is a disjoint union of collections of simple G_i -equivariant sheaves on Z_i .

In case when $\mathcal{Z} = pt$ we see that $Irr(\mathcal{M}(\mathcal{G}, F)) = \coprod Irr(G_i)$.

If we postulate that the category $\mathcal{M}(\mathcal{G}, F)$ is the "correct" category describing representations of the algebraic group G then this bijection would explain the Vogan's picture.

26. Reduction to the case of group GL .

Let me present another description that is often convenient in computations.

Construction II. Suppose that \mathcal{G} is a linear algebraic group. Then we can imbed it into a group \mathcal{P} isomorphic to $GL(n)$.

Using this we can realize our quotient stack $\mathcal{X} = \mathcal{Z}/\mathcal{G}$ as the quotient \mathcal{W}/\mathcal{P} , where $\mathcal{W} = \mathcal{P} \times_{\mathcal{G}} \mathcal{Z}$.

Since the group \mathcal{P} has only one pure inner form we see that the category $Sh_F(\mathcal{X})$ can be realized as the category $Sh_P(W)$ of P -equivariant sheaves on W , where $P = \mathcal{P}(F)$ and $W = \mathcal{W}(F)$.