LANGLANDS CORRESPONDENCE AND HITCHIN SYSTEMS ON NODAL CURVES

Henrick Rabinovitz, Joseph Vulakh

Abstract. In this paper, we continue the study of Hitchin systems, introduced by Hitchin [9], and their quantizations, introduced by Beilinson and Drinfeld [2], in the context of the analytic Langlands correspondence as described by Etingof, Frenkel, and Kazhdan [4]. We develop the theory needed for the analytic Langlands correspondence for nodal curves, performing specific calculations for what we call "bubble curves", curves formed by gluing together copies of the projective line over a field. We determine the classical and quantum Hitchin hamiltonians for group bundles on these curves, and compute the eigenfunctions of the Hitchin hamiltonians.

Contents

1. Introduction

For a smooth projective curve X and connected reductive algebraic group G over field $\mathbf{k}$, the stack of G-bundles $\mathrm{Bun}_G(X)$ over X is a central object in the Langlands program. Roughly speaking, the Langlands program studies a correspondence between spectra of Hecke functors on modules of differential operators on $\mathrm{Bun}_G(X)$ and flat ${}^LG$-bundles on X, where ${}^LG$ is the Langlands dual of G. The original formulation, arithmetic Langlands correspondence, studies finite fields, and the related geometric Langlands correspondence studies the complex case. The more recently introduced *analytic* Langlands program, introduced by Etingof, Frenkel, and Kazhdan [4], lets $\mathbf{k}$ be an arbitrary local field, allowing, for example, $\mathbf{k} = \mathbb{R}$. The program has since been refined (see, for example, [6]), and has been studied extensively for smooth curves X (see, for example, [7]).

In this paper, we study $\mathrm{Bun}_G(X)$ for nodal curves X, developing the necessary theory for one side of the Langlands correspondence, namely the "spectral theory" on $\mathrm{Bun}_G(X)$. We define $\mathrm{Bun}_G(X)$ for nodal curves X and affine algebraic groups G, and set up much of the needed apparatus for studying analytic Langlads in this setting. While many of our definitions and results work for general nodal curves, we focus in particular on nodal curves formed by gluing together copies of $\mathbb{P}^1$, which we call *bubble curves*. These are relatively simple cases to study, and the techniques developed should apply to the more general case of G-bundles on arbitrary nodal curves. We perform explicit calculations on the special case of *trinionic curves*, in which each copy of $\mathbb{P}^1$ is identified with other *bubbles* at exactly three points. This special case is particularly nice, allowing for rather simple calculations, while still shedding light on the general theory.

We begin by outlining the necessary components. We define $\mathrm{Bun}_G(X)$ for nodal X in Section 2. We also give formal definitions of bubble curves and trinionic curves and describe why these are particularly easy cases to work with.

The next important notion for analytic Langlands is the *Hitchin integrable system*. For a smooth projective curve X and a reductive group G, the Hitchin integrable system is an integrable system of elements of the cotangent space of a certain substructure of $\mathrm{Bun}_G(X)$, namely the moduli space $\mathrm{Bun}_G^\circ(X)$ of stable principal G-bundles on X, first defined by Hitchin [9]. As Beilinson and Drinfeld [2] showed, the Hitchin system admits a natural quantization, which consists of twisted differential operators on the square root of the canonical line bundle on $\mathrm{Bun}_G(X)$. We extend the classical definitions to nodal X in Section 3 and Section 4, and determine the quantized Hitchin system in Section 5. Note that we do not prove the necessary independence conditions for Hitchin systems in this case to be integrable systems, but we do prove that they Poisson commute.

The last component we study is the most immediately relevant to the spectral problem. Etingof, Frenkel, and Kazhdan [4] define the Hilbert space $\mathcal{H} = L^2(\mathrm{Bun}_G(X))$ on $\mathrm{Bun}_G(X)$ using the quantized Hitchin hamiltonians. The quantum Hitchin hamiltonians act on $\mathcal{H}$, and a central question in the analytic langlands program is to understand the spectral decomposition of this action, that is, the eigenfunctions of the Hitchin hamiltonians. We develop the theory needed for this Hilbert space over nodal curves in Section 6. In Section 7, we study wobbly divisors on $\mathrm{Bun}_G(X)$, which are expected to determine the singularities of the eigenfunctions [5]. We also develop theory which is helpful for studying the real case in Section 8; while we do not build on this further in this paper, this is important for future work on studying real bundles on trinion curves. Finally, we tackle the spectral problem in full in Section 9, computing an integral formula the eigenfunctions of the Hitchin hamiltonians for trinion curves and proving that the integrals converge for generic parameters.

1.1. **Acknowledgements.** The authors would like to thank their mentor, Ivan Motorin of the MIT Department of Mathematics, for his guidance and support in the research process and in the preparation of this paper. We are also grateful to Prof. Pavel Etingof of the MIT Department of Mathematics for suggesting the project and for providing valuable suggestions and pointing out further directions for research. We also thank Prof. Roman Bezrukavnikov and Dr. Jonathan Bloom of the MIT Department of Mathematics for their advice on organization and presentation throughout the duration of the research. Finally, we would like to thank the SPUR program for making this research opportunity possible.

2. Bundles on nodal curves

We study local geometry on the stack $\mathrm{Bun}_G C$ for certain nodal curves C which we call *bubble curves*. In our conventions, by genus of a projective scheme X we mean the arithmetic genus $(-1)^{\dim X}(\chi(X, \mathcal{O}_X) - 1)$. For (not necessarily smooth) projective curves C with $n = h^0(C, \mathcal{O}_C)$ connected components, a formula for genus is then $h^1(C, \mathcal{O}_C) - (n - 1)$, and if C has dualizing sheaf ω_C, Serre duality allows us to write genus as $h^0(C, \omega_C) - (n - 1)$.

2.1. **Nodal curves.** Let $\tilde{C}$ be a smooth, projective curve over a field $\mathbf{k}$ with distinct points $z_1, \ldots, z_n, z'_1, \ldots, z'_n$. Define the curve C by identifying z_i with z'_i for each i. Curves obtained this way are called *nodal curves*, and the images p_i of z_i and z'_i are called *nodal points*. Up to relabeling of the z_i, the curve $\tilde{C}$ is uniquely determined by C, as the quotient map $\varphi : \tilde{C} \to C$ is also the normalization of C.

Like smooth curves, nodal curves have a line bundle as their dualizing sheaf:

Proposition 2.1. *The dualizing sheaf ω_C of the nodal curve C is a line bundle and is given by the kernel of the map*

$$\varphi_* \left(\Omega_{\tilde{C}}(z_1, \ldots, z_n, z'_1, \ldots, z'_n)\right) \to k(p_1) \oplus \cdots \oplus k(p_n)$$

where $k(p_i)$ denotes the skyscraper sheaf at p_i and the map is given by, for each direct summand $k(p_i)$, taking the image to be the sum of the residues of the preimage (when thought of as a differential) at z_i and z'_i.

Proof. See proposition 13.2.9 of [12]. (Note that the reference erroneously states that the residue map is given by differences of residues rather than sums of residues.) □

2.2. **Moduli stack of bundles.** If X is a $\mathbf{k}$-scheme and G is an affine algebraic group over $\mathbf{k}$, there is a moduli stack of principal G-bundles over X, $\mathrm{Bun}_G(X)$, with functor of points $\mathrm{Bun}_G(X)(A)$, A ranging over commutative $\mathbf{k}$-algebras, defined to be the set of principal G-bundles on X defined over scalars in A [7].

Recall that any principal G-bundle may be presented as transition maps between trivial bundles on a cover $\{U_i\}$ of X. Furthermore, this cover may always be chosen to be étale. The transition maps may be written as $g_{ij} : U_i \cap U_j \to G$. The imposed conditions are then $g_{ii} = \mathrm{id}$ over $U_i \cap U_i$, $g_{ij} g_{ji} = \mathrm{id}$ over $U_i \cap U_j$, and $g_{ij} g_{jk} = g_{ik}$ over $U_i \cap U_j \cap U_k$ (here we take the convention that the "intersection" of two étale opens is given by their pullback). An isomorphism between two bundles presented in this way, say g_{ij} and g'_{ij}, is given by maps $h_i : U_i \to G$ such that $h_j g_{ij} = g'_{ij} h_i$. This describes isomorphism classes of principal G-bundles as homology classes of Čech 1-cocycles. We thus get that $\mathrm{Bun}_G(X)$ may be written as $H^1_{\text{ét}}(X, \mathcal{O}_{X,G})$, the first étale cohomology of $\mathcal{O}_{X,G}$.

Consider the G-bundle given by transition maps $\{g_{ij}\}$ on open cover $\{U_i\}$. A trivialization at a point $x \in X$ may be defined formally by maps $t_i^x : U_i \cap \{x\} \to G$ for each U_i satisfying $t_i^x g_{ij} = t_j^x$, with the image of a trivialization $\{t_i^x\}$ under an isomorphism $\{h_i\}$ from $\{g_{ij}\}$ to $\{g'_{ij}\}$ given by $t_i'^x := t_i^x h_i^{-1}$. It is not hard to see that a trivialization at a point x is not dependent on the open cover $\{U_i\}$, as to specify a trivialization at x it is necessary and sufficient to specify it on a subcover, and so for any two open covers a trivialization at x with respect to either is equivalent to one with respect to their union.

If we fix marked points $x_1, \ldots, x_n \in X$, we may equip a principal G-bundle on X with trivializations at each of the marked points. While there exist such trivializations no matter the choice of principal G-bundle, the trivializations add extra structure (isomorphisms must then be compatible with the equipped the trivializations at each marked point), resulting in a larger moduli space $\mathrm{Bun}_G(X, x_1, \ldots, x_n)$, with a surjective map back to $\mathrm{Bun}_G(X)$ corresponding to quotienting out by the action of $G \times \cdots \times G$ with each copy of G acting on the trivialization at a different point x_i.

Now let C be a nodal curve with normalization $\tilde{C}$ such that the pairs z_i, z_i' for $1 \le i \le n$ are identified in the quotient $\tilde{C} \to C$.

Proposition 2.2. *We have*

$$\mathrm{Bun}_G(C) \cong \mathrm{Bun}_G(\tilde{C}, z_1, \ldots, z_n, z_1', \ldots, z_n')/G^n,$$

where the ith component of G^n acts by left multiplication on the trivializations at z_i and z_i'.

Proof. Given a bundle on $\tilde{C}$ equipped with trivializations on the marked points, we first choose an étale open cover $\{U_j\}$ that trivializes the bundle. For each marked point z_i (resp. z_i'), add a punctured copy of some U_j, denoted by V_i (resp. V_i'), such that the new open contains exactly one preimage of the marked point in question and no preimages of any other marked point. Then redefine each U_j to be its previous definition except punctured at all preimages of all marked points. The result of this process is a new open cover of $\tilde{C}$, given by $\{V_i, V_i', U_j\}$, which trivializes our bundle. Then let transition maps for our bundle be given by $\{g_{kl}\}$, where subscripts range over the disjoint union of subscripts i, i', and j, with i corresponding to V_i, i' corresponding to V_i', and j corresponding to U_j. Similarly, let the trivialization at z_i (resp. z_i') be given by t_k^i (resp. $t_k^{i'}$).

For each pair of marked points (z_i, z_i'), define W_i to be the étale open of C given by identifying the preimages of z_i and z_i' in the disjoint union of V_i and V_i'; the étale map is then induced by the composition of the étale map from $V_i \sqcup V_i'$ to $\tilde{C}$ and the normalization map $\tilde{C} \to C$. Then $\{W_i, U_j\}$ is an étale open cover of C. We may then define a bundle on G with transition maps g_{kl}' defined by

$$\begin{cases} g'_{j_1 j_2} & := g_{j_1 j_2} \\ g'_{ij} & := t_i^i g_{ij} \sqcup t_{i'}^{i'} g_{i'j} \\ g'_{ji} & := g_{ji}(t_i^i)^{-1} \sqcup g_{ji'}(t_{i'}^{i'})^{-1} \\ g'_{i_1 i_2} & := t_{i_1}^{i_1} g_{i_1 i_2}(t_{i_2}^{i_2})^{-1} \sqcup t_{i_1'}^{i_1'} g_{i_1' i_2}(t_{i_2}^{i_2})^{-1} \sqcup t_{i_1}^{i_1} g_{i_1 i_2'}(t_{i_2'}^{i_2'})^{-1} \sqcup t_{i_1'}^{i_1'} g_{i_1' i_2'}(t_{i_2'}^{i_2'})^{-1} \qquad (i_1 \neq i_2) \\ g'_{ii} & := \mathrm{id} \end{cases}$$

where we extend each trivialization map t_k^k to a constant map on the corresponding étale open (of C).

It is then easy to show that the map we have defined from $\mathrm{Bun}_G(\tilde{C}, z_1, \ldots, z_n, z_1', \ldots, z_n')$ to Bun_G is the desired quotient map, as this is equivalent to the assertion that the map is essentially surjective and an isomorphism on bundles over C is equivalent to an isomorphism on bundles over $\tilde{C}$. Essentially surjective is easy to see as starting with a bundle on C, we can generate an open cover trivializing that is in the same form we had before, and then it is easy to our constructed quotient map backwards (we may arbitrarily assume that for each x_i' we have $t_{i'}^{i'} = \mathrm{id}$). The correspondence between isomorphisms is given by $\{h_i, h_i', h_j, h_i''\}$ over $\tilde{C}$ (where $h_i'' \in G$ corresponds to the action of G^n) corresponds to $\{(h_i'' t_i^i h_i^{-1})|^{V_i} h_i (t_i^i|^{V_i})^{-1} \cup (h_i'' t_{i'}^{i'} (h_i')^{-1})|^{V_i'} h_i' (t_{i'}^{i'}|^{V_i'})^{-1}, h_j\}$ over C (where our superscript notation indicates extending the domain of a constant map). □

Corollary 2.3. $\mathrm{Bun}_G(C)$ *is given by bundles on $\tilde{C}$ equipped with isomorphisms from the stalks at z_i to the stalks at z_i'. Representing C using an open cover and transition maps, these isomorphisms may be represented by maps $s_{j_1 j_2}^i : (U_{j_1} \cap \{x_i'\}) \times_{\mathbf{k}} (U_{j_2} \cap \{x_i\}) \to G$ satisfying $g_{j_1 j_2} s_{j_2 j_3}^i g_{j_3 j_4} = s_{j_1 j_4}^i$, and the image of $s_{j_1 h_2}^i$ under an isomorphism of bundles h_j is given by $s'^i_{j_1 j_2} := h_{j_1} s_{j_1 j_2}^i h_{j_2}^{-1}$.*

Proof. Using a representation by trivializations, let $s^i_{j_1j_2} := t^{i'}_{j_1}(t^i_{j_2})^{-1}$. □

2.3. **Principal bundles for special groups.** For some special groups, principal G-bundles have special bijective relations to vector bundles, due to their transition maps both being given by matrices. In particular, principal GL_n-bundles are in bijection with vector bundles of rank n, principal SL_n-bundles are in bijection with vector bundles of rank n of trivial determinant, and principal PGL_n-bundles are in bijection with vector bundles of rank n up to twisting by line bundles [7].

2.4. **Bubble curves.** Define a *bubble curve* to be a nodal curve C, whose normalization $\tilde{C}$ is a disjoint union of copies of $\mathbb{P}^1$, which we refer to as "bubbles." We can associate to C a graph Γ (self-loops and parallel edges allowed), whose vertices are the copies of $\mathbb{P}^1$ and whose edges are pairs of identified points (or equivalently, nodal points), connecting to the bubbles containing the two identified points in $\tilde{C}$. Sometimes it will also be convenient to arbitrarily assign a direction to the edges of the graph, in which case we get a quiver Q associated to C. We then take the convention that an edge points from z_i to z_i' in line with our notation from previous sections (which technically carries more information exactly when z_i and z_i' are on the same bubble).

We may also go in the reverse direction: if $\Gamma = (V(\Gamma), E(\Gamma))$ is a graph, with self-loops and repeated edges allowed, we may associate a bubble curve $C = C_\Gamma$ to our graph given by assigning a copy of $\mathbb{P}^1$ to each vertex in $V(\Gamma)$ and for each edge in $E(\Gamma)$ identifying points of the corresponding copies of $\mathbb{P}^1$. For graphs with a vertex of degree at least 4, C will not be uniquely defined up to isomorphism by Γ as there will be a choice of which four or more points to pick on a bubble that has graph-theoretic degree at least 4. However, many properties of C will only be dependent on Γ. In particular, C will have the same number of connected components as Γ, and the genus of C will be equal to the graph-theoretic genus of Γ given by $e(\Gamma) - v(\Gamma) + 1$ (this follows from our description of ω_C).

In light of Corollary 2.3 vertex-trivial GL_n-bundles on C (that is, bundles whose restriction to each bubble is trivial) are in correspondence with quiver representations of a quiver Q associated to C whose vector spaces and maps are all of rank n. A similar description works more generally for any group G by replacing the vector space at each vertex with a G-torsor. More generally, all G-bundles admit a similar description to this; fixing the G-bundle on each vertex, it is then equivalent to assign an element of G to each edge, but the isomorphisms of such bundles will be more complicated than simply assigning an element of G to each vertex and will depend on more than just the Γ.

We note here that vertex-trivial bundles may also be described nicely in terms of the graph Γ without mention of the quiver Q; in particular, it is equivalent to assign elements of G to each direction of each edge (where direction is defined in terms of identified points) such that the two elements assigned to the same edge in different directions are inverses of one another. This description is essentially the same as the quiver description, but will sometimes be more convenient. In particular, we can interpret this as a representation of the fundamental groupoid of Γ on n-dimensional vector spaces for $G = \mathrm{GL}_n$ (or, to a weaker extent, when $G = \mathrm{SL}_n$ or $G = \mathrm{PGL}_n$) and more generally on G-torsors.

Let the substack of $\mathrm{Bun}_G(C)$ of vertex-trivial bundles be denoted by $\mathrm{Bun}^\circ_G(C)$. Then, by our quiver description, we may write this stack as a quotient stack:

$$\mathrm{Bun}^\circ_G(C) \cong \prod_{e\in Q_1} G / \prod_{v\in Q_0} G$$

where the action is defined by $(g_v)_{v\in Q_0}(g_e)_{e\in Q_1} := (g_{t(e)}g_e(g_{s(e)})^{-1})_{e\in Q_1}$, where $t(e)$ and $s(e)$ are the target and source of e respectively. The groups we are interested in are GL_n,

PGL_n, and SL_n. For these groups, $\mathrm{Bun}_G^\circ(C)$ is an open substack of $\mathrm{Bun}_G(C)$ that is dense in the connected component of bundles that are degree 0 on each bubble. The reason for this is that it is the inverse image of trivial bundles under the pullback map $\mathrm{Bun}_G(C) \to \mathrm{Bun}_G(\tilde{C})$, and so it follows from the fact that for these groups G, $\mathrm{Bun}_G(\mathbb{P}^1)$ has a connected component of degree 0 bundles and this component has a dense open substack consisting only of the trivial bundle [7, Remark 3.3]. (Note here that what we mean by "degree" depends on G; for $G = \mathrm{GL}_n$, it is degree of the associated vector bundle, for $G = \mathrm{SL}_n$ it is the same but due to trivial determinant it is always 0, and for $G = \mathrm{PGL}_n$ it is degree of a corresponding vector bundle modulo n.) It follows from the quotient definition that the dimension of the stack of vertex-trivial GL_n-bundles is $n^2(e(\Gamma) - v(\Gamma)) + 1 = n^2(g-1) + 1$. Quotienting further gives the dimension of the moduli stack of vertex-trivial SL_n-bundles to be $n^2(g-1) + 1 - g = (n^2-1)(g-1)$.

2.5. **Trinion curves.** We are particularly interested in *trivalent graphs*, that is, those with every vertex having degree three. We call a bubble curve whose graph is trivalent a *trinion curve.* In this case, for any trivalent graph Γ, there is a unique (up to isomorphism) bubble curve C_Γ which has Γ as its graph. Furthermore, for any graph isomorphism $\Gamma \to \Gamma'$, there is a unique isomorphism of schemes $C_\Gamma \to C_{\Gamma'}$. Hence, $\Gamma \mapsto C_\Gamma$ gives an equivalence of the groupoids of trivalent graphs and trinion curves.

For a trinion curve C with graph Γ, we have $e(\Gamma) = \frac{3}{2}v(\Gamma)$, so the genus of both C and Γ is given by $\frac{1}{2}v(\Gamma) + 1 = \frac{1}{3}e(\Gamma) + 1$. Equivalently, we have $e(\Gamma) = 3g - 3$ and $v(\Gamma) = 2g - 2$, where g is the genus of Γ. Thus the dimension of the stack of SL_n-bundles on trinion curves can be easily expressed as a multiple of the number of edges or vertices.

Fig. 1 shows two examples of trivalent graphs, which we call, from top to bottom, the *pair of lips* and the *pair of glasses* respectively.

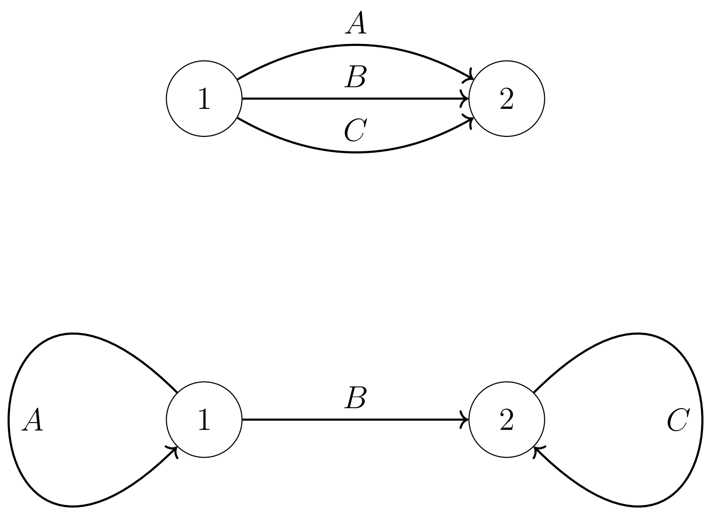

Fig. 1. The *pair of lips* graph of three edges, labeled with transition matrices A, B, and C, connecting two vertices is above. The *pair of glasses* gaph of two self-loops connected by an edge, again labeled with transition matrices, is below.

3. Higgs fields

Let G be an algebraic group over $\mathbf{k}$, let X be a Cohen-Macaulay $\mathbf{k}$-variety of dimension 1 with dualizing sheaf ω_X, let E be a principal G-bundle on X, let $\mathfrak{g}$ be the Lie algebra of G, and assume $\mathfrak{g}$ is equipped with an invariant nondegenerate bilinear form $B(-,-)$. For G equal to any of GL_n, SL_n, or PGL_n, we define this pairing by $B(X,Y) = \mathrm{tr}(XY)$. Define the *adjoint bundle* $\mathrm{ad}\, E$ of E to be the associated bundle to E from the adjoint representation of E on $\mathfrak{g}$; that is, the fiber bundle with fiber $\mathfrak{g}$ and transition maps the same as those of E acting on $\mathfrak{g}$ via the adjoint representation. Then, since $\mathfrak{g}$ is a vector space, $\mathrm{ad}\, E$ is a vector bundle, and it is self-dual by duality with respect to $B(-,-)$. In the case that G is equal to any of GL_n, SL_n, or PGL_n, then if we think of E as a vector bundle, $\mathrm{ad}\, E$ becomes $E \otimes E^*$.

Recall that $\mathrm{Bun}_G(X) \cong H^1_{\text{ét}}(X, \mathcal{O}_{X,G})$. We may differentiate both the cocycle condition and the group action (that we quotient out by) to get that

$$T_E \,\mathrm{Bun}_G(X) \cong H^1(X, \mathrm{ad}\, E).$$

Dualizing both sides, applying Serre duality to the right side, and applying the isomorphism from $(\mathrm{ad}\, E)^*$ to $\mathrm{ad}\, E$, we get

$$T_E^* \,\mathrm{Bun}_G(X) \cong H^0(X, \omega_X \otimes \mathrm{ad}\, E)$$

Elements of the right side are called *Higgs fields* on E.

Let G be GL_r, SL_r, or PGL_r, let $X = C$ for some bubble curve C with quiver Q, and assume that E is vertex-trivial so that we may restrict $\mathrm{Bun}_G(C)$ to its open substack $\mathrm{Bun}_G^\circ(C)$. In that case, a Higgs field restricted to a bubble of graph-theoretic degree d with marked points at $z = \alpha_1, \alpha_2, \ldots, \alpha_d$ (where we choose a local coordinate z such that none of these points are $z = \infty$) looks like

$$\left(\frac{Q_1}{\alpha_1 - z} + \frac{Q_2}{\alpha_2 - z} + \cdots + \frac{Q_d}{\alpha_d - z}\right) \mathrm{d}z$$

such that $Q_1 + Q_2 + \cdots + Q_d = 0$, where $Q_i \in \mathfrak{g}$ is an $r \times r$ square matrix required to have determinant 0 in the case that $G = \mathrm{SL}_r$ or $G = \mathrm{PGL}_r$. The conditions to glue these sections on each bubble to a global section look like $Q_i + AQ_j' A^{-1} = 0$, where A is a matrix corresponding to an edge of the quiver Q that points from the marked point of Q_j' to the marked point of Q_i.

Let $G = GL_r$. We may present another way to parametrize

$$T^* \,\mathrm{Bun}_G^\circ(C) \cong T^*\big(\prod_{e \in Q_1} G / \prod_{v \in Q_0} G\big).$$

In particular, we have matrices of coordinate functions for $\prod_{e\in Q_1} G$ given by A_e for each edge $e \in Q_1$. We can extend this system of coordinates to $T^*(\prod_{e \in Q_1} G)$ by introducing new matrices of coordinate functions

$$A_e^* := \left(\frac{\partial}{\partial A_e}\right)^{\intercal}$$

where by $\frac{\partial}{\partial A}$ we mean the matrix whose entries are partial derivatives with respect to the corresponding entries of A (these partial derivatives are defined here, as our coordinates on $\prod_{e\in Q_1} G$ form standard coordinates on affine space if we extend from invertible matrices to all matrices).

Define $\bar{Q}$, the *dual quiver* to Q, have the same set of vertices as Q and a set of edges in one-to-one correspondence with edges of Q such that if $e^* \in \bar{Q}_1$ corresponds to $e \in Q_1$, then $s(e^*) = t(e)$ and $t(e^*) = s(e)$. We say e^* is *dual* to e, and define $(e^*)^* := e$. We define $Q \cup \bar{Q}$ to be the quiver that has the same vertices as Q and $\bar{Q}$ and whose edges are the disjoint union of the edges of Q and $\bar{Q}$. Then, based on our coordinates $\{A_e, A_e^*\}_{e\in Q_1}$ for $T^*(\prod_{e\in Q_1} G)$, we can interpret elements of $T^*(\prod_{e\in Q_1} G)$ as quiver representations $\{A_e\}_{e \in (Q\cup\bar{Q})_1}$ of $Q \cup \bar{Q}$, where we define $A_e := A_{e^*}^*$ for $e \in \bar{Q}_1$ (for convenience, we also define the right side to be the left side when $e \in Q_1$). While different elements of $T^*(\prod_{e\in Q_1} G)$ may map to isomorphic quiver representations in this way, this is no longer the case once we quotient out by $\prod_{v\in Q_0} G$, and we really do get a correspondence of $T^*(\prod_{e\in Q_1} G)/\prod_{v\in Q_0} G$ with quiver representations of $Q \cup \bar{Q}$ that are rank r at every vertex and invertible at every edge of Q.

Recall that we want to get at $T^*(\prod_{e\in Q_1} G/\prod_{v\in Q_0} G)$ rather than $T^*(\prod_{e\in Q_1} G)/\prod_{v\in Q_0} G$. For that, we use hamiltonian reduction to say that for a moment map

$$\mu : T^*(\prod_{e\in Q_1} G) \to \mathrm{Lie}(\prod_{v\in Q_0} G)^*$$

we have

$$T^*(\prod_{e\in Q_1} G/\prod_{v\in Q_0} G) \cong \mu^{-1}(0)/\prod_{v\in Q_0} G \subset T^*(\prod_{e\in Q_1} G)/\prod_{v\in Q_0} G.$$

Hence, we are able to realize $T^*(\prod_{e\in Q_1} G/\prod_{v\in Q_0} G)$ as a quiver variety. In particular, the moment map equal to 0 condition is equivalent to

$$\sum_{e\in(Q\cup\bar{Q})_1, t(e)=v} \epsilon(e) A_e A_{e^*} = 0$$

for all vertices $v \in Q_0$ where

$$\epsilon(e) := \begin{cases} 1 & e \in Q_1 \\ -1 & e \in \bar{Q}_1 \end{cases}.$$

We may go between our two different ways of presenting $T^*(\mathrm{Bun}_{\mathrm{GL}_r}(C))$; in particular, in the first way we used Serre duality on vector fields presented in coordinates of the Lie algebra of GL_r which consists of vector fields invariant under translation by a multiplicative action of GL_r, whereas in the second way we instead used vector fields invariant under translation by an additive action of $r \times r$ matrices, and so we arrive at the formulas

$$Q_i = \epsilon(e_i) A_{e_i} A^*_{e_i}$$

where e_i is the edge pointing towards the marked point $z = \alpha_i$. When we focus on a fixed bubble, we may write A_{e_i} as A_i and $A^*_{e_i}$ as A^*_i.

4. Hitchin map

Assume $\mathbf{k} = \mathbb{C}$. Let G be a reductive algebraic group over $\mathbf{k}$, let X be a Cohen-Macaulay $\mathbf{k}$-variety of dimension 1 with dualizing sheaf ω_X, let $\mathfrak{g}$ be the Lie algebra of G, and assume $\mathfrak{g}$ is equipped with an invariant nondegenerate bilinear form $B(-,-)$ so that we may apply the previous section.

First, recall Chevalley's Theorem [7]:

Theorem 4.1. *Polynomials on $\mathfrak{g}$ invariant under the adjoint action of G form a free algebra and may be generated by homogeneous polynomials $R_1, R_2, \ldots, R_r$ of degrees $d_1 \leq d_2 \leq \cdots \leq d_r$ called the* degrees *of G, and r is the rank of G.*

The *Hitchin base* is defined as

$$\mathcal{B} = \mathcal{B}_{X,G} := \bigoplus_{i=1}^{r} H^0(X, \omega_X^{\otimes d_i})$$

and the *Hitchin map* is defined as a map

$$p : T^* \mathrm{Bun}_G \to \mathcal{B}_{X,G}$$

where

$$p : (E, Q) \mapsto (R_1(Q), R_2(Q), \ldots, R_r(Q))$$

where Q is a Higgs field over a principal G-bundle E. It is a theorem that this map defines an integrable system if X is a smooth projective curve and we restrict the domain to a dense open subset.

Define *Hitchin hamiltonians* to be the components (or coefficients) of the Hitchin map. There may be multiple ways of defining such components depending on choices of bases for each direct summand of the Hitchin base, and any of these give valid Hitchin hamiltonians. In particular, whether or not these hamiltonians form an integrable system is independent of how we choose linear coordinates on the Hitchin base, as the Hamiltonians will generate the same subalgebra regardless. This also means we may use overdetermined linear coordinates, and we will do this.

Of interest here is the Hitchin map when G is either GL_r, SL_r, or PGL_r, and $X = C$ where C is a bubble curve, and we restrict the domain to vertex-trivial bundles. In that case, we first note that our polynomials $R_1, R_2, \ldots, R_r$ are given by $R_i(Q) = \mathrm{tr}(Q^i)$ if $G = \mathrm{GL}_r$, and in the other two cases the rank is actually $r-1$ and our polynomials $R_1, R_2, \ldots, R_{r-1}$ are given by $R_i(Q) = \mathrm{tr}(Q^{i+1})$. In particular, for general bubble curves, we assume $G = \mathrm{GL}_r$, and we prove that all Hitchin hamiltonians (components of the Hitchin map) Poisson commute. In fact, we prove the stronger claim that they have *quantizations* (defined in the next section) that commute at the quantum level.

Explicitly, in our case of a bubble curve C and $G = \mathrm{GL}_r$, hamiltonians may be computed by the restriction of the higher traces to each bubble, as knowing the value on each bubble determines the global section of $\omega_C^{\otimes i}$ (these give rise to possibly overdetermined linear coordiantes on the Hitchin base). In the notation of the previous section, our Hamiltonians will then be given by coefficients of higher traces of

$$\left(\frac{\epsilon_1 A_1 A_1^*}{\alpha_1 - z} + \frac{\epsilon_2 A_2 A_2^*}{\alpha_2 - z} + \cdots + \frac{\epsilon_d A_d A_d^*}{\alpha_d - z}\right) \mathrm{d}z.$$

5. Quantum Hitchin Hamiltonians

5.1. **Quantization of Commutative Poisson Algebras.** Let $\hbar$ be a formal symbol so that $\mathbf{k}[\hbar]$ is a polynomial ring. Let $\mathcal{O}$ be a commutative Poisson algebra over $\mathbf{k}$ with bracket $\{-,-\}$. Define a $\mathbf{k}[\hbar]$-algebra structure on it by $\hbar\mathcal{O} := 0$.

Definition 5.1. A *quantization* of $\mathcal{O}$ is an associative algebra A over $\mathbf{k}[\hbar]$, free as a $\mathbf{k}[\hbar]$-module, equipped with a surjective $\mathbf{k}[\hbar]$-algebra homomorphism $\kappa : A \to \mathcal{O}$ with kernel $\hbar A$ such that

$$\kappa\left(\frac{fg - gf}{\hbar}\right) = \{\kappa(f), \kappa(g)\}$$

holds for all $f, g \in A$.

Note that quantization is more commonly done over $\mathbf{k}[[\hbar]]$, with a definition analogous to that given above. For our purposes, it will be more convenient to use the definition over $\mathbf{k}[\hbar]$ that we have given. One way of presenting a quantization is as follows:

Definition 5.2. A *filtered quantization* of $\mathcal{O}$ is a $\mathbb{Z}_{\geq 0}$-filtered associative $\mathbf{k}$-algebra $A = \bigcup_{i\geq 0} F_i A$ equipped with an identification of its associated graded algebra $\mathrm{gr}(A) := \bigoplus_{i\geq 0} F_i A/F_{i-1}A$ with $\mathcal{O}$ that respects associative $\mathbf{k}$-algebra structure and is such that for any $f \in F_i A$ and $g \in F_j A$, if $[f]$ and $[g]$ are the respective images in $F_i A/F_{i-1}A$ and $F_j A/F_{j-1}A$, the Poisson bracket $\{[f], [g]\}$ is given by the image of $[f, g] = fg - gf \in F_{i+j-1}A$ in $F_{i+j-1}A/F_{i+j-2}A$.

Note that for any filtered algebra A as above with commutative associated graded algebra, $\mathrm{gr}(A)$ will have the structure of a Poisson algebra as given above, and A will be a filtered quantization of $\mathrm{gr}(A)$.

If A is a filtered quantization of $\mathcal{O}$, then the Rees algebra of A, defined to be the $\mathbf{k}[\hbar]$-subalgebra $\sum_{i\geq 0}\hbar^i F_i A$ of the $\mathbf{k}[\hbar]$-algebra $A[\hbar]$, is a quantization of $\mathcal{O}$.

Let $\mathcal{O} = \mathcal{O}(T^*Y)$ for the symplectic manifold T^*Y, where Y is a smooth affine $\mathbf{k}$-scheme. An example of a filtered quantization of $\mathcal{O}$ is given by the *algebra of differential operators* $D(Y)$. This is defined to be a $\mathbf{k}$-subalgebra of the algebra of linear operators on $\mathcal{O}(Y)$, namely the subalgebra generated by the operators given by multiplication by an element of $\mathcal{O}(Y)$ and the operators that are derivations (i.e. given by vector fields). The filtration $D_{\leq n}(Y)$ on $D(Y)$ is defined by letting $D_{\leq n}(Y)$ to be the space of differential operators of order at most n, meaning that they can be written as the sum of monomials in the aforementioned generators (derivations and multiplication by scalar fields) such that each monomial has at most n factors that are derivations (and the rest of the factors are multiplication by scalar fields).

5.2. **Quantum Hamiltonian Reduction.** Given a symplectic $\mathbf{k}$-scheme X and an algebraic group H with a Hamiltonian action on X, recall that the classical Hamiltonian reduction of X by H is given by $\mu^{-1}(0)/H$, where $\mu : X \to \mathfrak{h}^*$ is a classical moment map. The classical Hamiltonian reduction is another symplectic manifold. Now, suppose that we have a quantization A of $\mathcal{O}(X)$, and a quantization of the action of H so that H acts on A so that $\kappa : A \to \mathcal{O}(X)$ is H-equivariant. Then a quantum moment map $\mu : U(\mathfrak{h}) \to A$ should quantize the classical comoment map $\mu^* : U(\mathfrak{h}) \to \mathcal{O}(X)$, meaning that it should return the classical comoment map upon composition with the map $\kappa : A \to \mathcal{O}(X)$. It should further satisfy the quantum moment map conditions of being H-equivariant and satisfying $z \cdot a = [\mu(z), a]$ for all $z \in \mathfrak{h}$ and $a \in A$. Then the quantum Hamiltonian reduction of A by H quantizes $\mathcal{O}$ of the classical Hamiltonian reduction, and is given by $A^H/(A\mu(\mathfrak{h}))^H$.

Note that we may also use a filtered quantization for quantum Hamiltonian reduction; the quantum moment map conditions are the same and the expression for Hamiltonian reduction is still $A^H/(A\mu(\mathfrak{h}))^H$. We may ignore the condition of returning the classical comoment map if we do not fix a classical comoment map; any quantum moment map quantizes some classical comoment map.

Now, suppose $X = T^*Y$, and the action of H on X is induced by an action of H on Y. Recall that the classical Hamiltonian reduction of T^*Y by H is isomorphic to $T^*(X/H)$, and so a quantum Hamiltonian reduction would quantize $\mathcal{O}(T^*(X/H))$. We may use the filtered quantization of T^*Y given by the algebra of differential operators, $D(Y)$. In this case, there is a canonical choice of quantum moment map μ, generated from the map $\mathfrak{h} \to D(Y)$ given by mapping an element of the Lie algebra $\mathfrak{h}$ to the derivation corresponding to its vector field on Y. The quantum moment map conditions are then tautological, so we get a quantum Hamiltonian reduction in this case.

5.3. **Quantizations of Hitchin Hamiltonians over** $\mathrm{Bun}^\circ_{\mathrm{GL}_r}$ **of Bubble Curves.** Let C be a bubble curve, with quiver Q. Recall that $\mathrm{Bun}^\circ_{\mathrm{GL}_r} C$ can be written as $\prod_{e\in Q_1} \mathrm{GL}_r / \prod_{v\in Q_0} \mathrm{GL}_r$. We use the quantization of $\mathcal{O}(T^* \mathrm{Bun}^\circ_G C)$ given by the quantum Hamiltonian reduction of $D(\prod_{e\in Q_1} \mathrm{GL}_r)$.

Recall that over our bubble curve C, Hitchin hamiltonians on a given bubble with nodal points at $z = \alpha_1$,$z = \alpha_2$,...,$z = \alpha_d$ (assume we have chosen a local coordinate z such that none of the α_i are 0 or ∞) are given by coefficients of higher traces of

$$\left(\frac{\epsilon_1 A_1 A_1^*}{\alpha_1 - z} + \frac{\epsilon_2 A_2 A_2^*}{\alpha_2 - z} + \cdots + \frac{\epsilon_d A_d A_d^*}{\alpha_d - z}\right) \mathrm{d}z$$

where A_i and A_i^* are the coordinate functions of the presentation of $T^* \mathrm{Bun}^\circ_{\mathrm{GL}_r} C$ as a quiver variety prior to quotienting out by the GL_r-action at each vertex, and (A_i, A_i^*)

being the pair of dual edges corresponding to the nodal point at α_i such that A_i is the edge pointing towards the vertex of the quiver corresponding to the selected bubble.

We desire preimages of the Hitchin hamiltonians in the quantization that generate a commutative subalgebra of the quantization. This extends the property of the classical Hitchin hamiltonians Poisson commuting with one another. Since Hitchin hamiltonians are homogeneous, it makes sense to consider preimages in the filtered quantization instead of the Rees algebra, so this is what we do.

We work in the algebra of differential operators $D(\prod_{e\in Q_1}\mathrm{GL}_r)$ prior to quotienting and taking invariants (hence our quantizations must be invariant in order to be valid). For a transition matrix A corresponding to some edge of Q (and, by abuse of notation and terminology, we may refer to the matrix and the edge as the same), define

$$[A] := A\left(\frac{\partial}{\partial A}\right)^{\intercal}$$

and

$$[A^*] := -A^{\intercal}\frac{\partial}{\partial A}$$

where A^* is the edge of $\bar{Q}$ dual to A. Then $[A]$ is a matrix of right-invariant vector fields over the copy of GL_r at A, while $[A^*]$ is a matrix of left-invariant vector fields over the same copy of GL_r; hence, the entries of $[A]$ commute with the entries of $[A^*]$. Furthermore, the entries of $[A]$ and $[B]$ where A and B are vector fields on different copies of GL_r obviously commute; hence, the entries of $[A]$ commute with those of $[B]$ so long as A and B are not the same edge in $Q\cup\bar{Q}$. Furthermore, the entries of $[A]$ generate the a copy of $\mathfrak{gl}_r$ as the elementary matrices.

Quantum Hitchin hamiltonians are given by coefficients of the following traces:

$$\begin{aligned}\mathrm{tr}&\sum_{i=1}^{k}\sum_{j_1+j_2+\cdots+j_i=k}\left(\prod_{l=1}^{i}\binom{l+\sum_{m=0}^{l}j_m}{j_l}\right)\left(\overrightarrow{\prod}_{l=1}^{i}\sum_{m=1}^{d}\frac{j_l![A_m]}{(\alpha_m-z)^{j_l+1}}\right)\\ =&\sum_{i=1}^{k}\sum_{j_1+j_2+\cdots+j_i=k}\left(\prod_{l=1}^{i}\binom{l+\sum_{m=0}^{l}j_m}{j_l}\right)\mathrm{tr}\left(\overrightarrow{\prod}_{l=1}^{i}\sum_{m=1}^{d}\frac{j_l![A_m]}{(\alpha_m-z)^{j_l+1}}\right).\end{aligned}$$

Here, k corresponds to taking trace of the kth power in the classical case. These expressions are invariant under the $\prod_{v\in Q_0}\mathrm{GL}_r$-action, they map to the classical expressions in the associated graded ring, and, nontrivially, their coefficients commute. Note that the latter is only nontrivial as we are looking at coefficients of these expressions for a fixed bubble; if we instead took different bubbles, this statement would trivialize, as entries in the matrices $[A]$ of one expression would commute with matrices $[B]$ of the other, as A and B, pointing to different vertices, must be different edges of $Q\cup\bar{Q}$. Hence, to prove commutativity, we shall fix a bubble. Our traces are constant coefficients (with respect to ∂_z) of the following traces:

$$\mathrm{tr}\left(\partial_z+\sum_{m=1}^{d}\frac{[A_m]}{\alpha_m-z}\right)^{k}.$$

We make the stronger claim that in fact all coefficients of these expressions commute, where by coefficients here we mean coefficients of the expressions in z that are coefficients of these expressions with respect to ∂_z. Letting our trace be $F^k(z,\partial_z)$, an equivalent formulation of this commutativity is the equation $[F^k(z,\partial_z),F^l(w,\partial_w)]=0$ where z and w are independent indeterminates and k and l are variables ranging over positive integers independently.

Consider the algebra $U(t^{-1}\mathfrak{gl}_r[t^{-1}])$. Let $E_{ij}[n]$ be the image in $U(t^{-1}\mathfrak{gl}_r[t^{-1}])$ of $E_{ij}t^n \in t^{-1}\mathfrak{gl}_r[t^{-1}]$, where $n \leq -1$. These form a basis for the algebra. Let their generating function be $E_{ij}(u) \coloneqq \sum_{n\geq 0} E_{ij}[-n-1]u^n$. Define matrix $E(u)$ to be that with entries $E_{ij}(u)$. It is known that the coefficients of $\operatorname{tr}(\partial_u + E(u))^k$, elements of $U(t^{-1}\mathfrak{gl}_r[t^{-1}])$, commute, where k ranges over positive integers [11]. Hence, $[\operatorname{tr}(\partial_u + E(u))^k, \operatorname{tr}(\partial_v + E(v))^l] = 0$ for independent indeterminates u and v and any positive integers k and l.

Given our fixed bubble, define a map $\psi : U(t^{-1}\mathfrak{gl}_r[t^{-1}]) \to D(\prod_{e\in Q_1} \mathrm{GL}_r)$ by

$$E_{ij}[n] \mapsto (\alpha_1^n[A_1] + \alpha_2^n[A_2] + \cdots + \alpha_d^n[A_n])_{ij} = \alpha_1^n[A_1]_{ij} + \alpha_2^n[A_2]_{ij} + \cdots + \alpha_d^n[A_n]_{ij}.$$

In order for this definition to work in defining an associative algebra map, it suffices for it to work in defining a Lie algebra map from $t^{-1}\mathfrak{gl}_r[t^{-1}]$ to the algebra of differential operators. This latter property follows from direct calculation of brackets of preimages and images, noting that that the different terms of the sum defining the images of this map commute with one another, and each term $[A_k]_{ij}$ has the same bracket laws as E_{ij}. Map u to z and v to w. Applying our map to both sides of the equality $[\operatorname{tr}(\partial_u + E(u))^k, \operatorname{tr}(\partial_v + E(v))^l] = 0$ gives $[F^k(z, \partial_z), F^l(w, \partial_w)] = 0$, as desired.

6. L^2 functions

Let C be a nodal curve with normalization $\tilde{C}$ having marked points $z_1, \ldots, z_n, z_1', \ldots, z_n'$. It is useful to calculate the eigenvalues and eigenfunctions of the quantum Hitchin hamiltonians and Hecke operators. In this paper, we focus on Hitchin hamiltonians, performing these calculations in Section 9. To build up to this, we must define the space on which quantum Hitchin hamiltonians act.

For $1 \leq i \leq n$, let π_i be an irreducible unitary representation of G occurring in the Plancherel formula

$$L^2(G) = \int_{V \text{ irred. unitary rep}} V \otimes V^* \mathrm{d}\mu(V)$$

for the integral decomposition of the regular representations of G on $L^2(G)$ into irreducible representations. Consider the action of G^n on $\mathrm{Bun}_G(\tilde{C}, z_1, \ldots, z_n, z_1', \ldots, z_n')$ in which the jth component of G^n acts on z_j and z_j'. Thus we may identify a representation of G with each j. We define the Hilbert space $\mathcal{H} = L^2(\mathrm{Bun}_G(C))$ by its sections at $\pi_1, \ldots, \pi_n$.

Definition 6.1. The L^2-sections of $\mathcal{H}$ are defined by quotients

$$L^2(\mathrm{Bun}_G(C))_{\pi_1,\ldots,\pi_n} = (\mathrm{Bun}_G(\tilde{C}, z_1, \ldots, z_n, z_1', \ldots, z_n') \times \pi_1 \otimes \cdots \otimes \pi_n)/G^n,$$

where the π_i are fixed unitary irreducible representations of G at z_i, z_i'. Then

$$L^2(\mathrm{Bun}_G(C)) = \int L^2(\mathrm{Bun}_G(C))_{\pi_1,\ldots,\pi_n} \mathrm{d}\mu(\pi_1) \cdots \mathrm{d}\mu(\pi_n).$$

This is motivated by the case of a compact group G, where Plancherel's formula is simply a completed direct sum of irreducible unitary representations and thus it is natural to consider L^2-sections of $\mathcal{H}$ with respect to these summands. The set of real points, that is, those fixed by the real structure, is the union of the copy of the real numbers in each bubble.

7. Wobbly Divisors

Let X be a $\mathbf{k}$-variety of dimension 1, and let G be GL_n, SL_n, or PGL_n.

Definition 7.1. The *wobbly divisor* of X is defined to be the set of principal G-bundles that admit a nontrivial nilpotent Higgs field (that is, whose value at each point of X is nilpotent).

The wobbly divisor is related to Hitchin hamiltonians, as a nilpotent Higgs field is equivalently one that is mapped to 0 by the Hitchin map. We calculate the wobbly divisor in the case of trinion curves, restricted to only vertex-trivial bundles. We then show how this is related to singularities of eigenfunctions of Laplacians at each edge of the trinion curve's graph (in this case these Laplacians quantize our Hitchin hamiltonians).

7.1. **Wobbly divisor of trinion curves.** Let $X = C$ be a trinion curve over $\mathbf{k}$ with graph Γ.

Theorem 7.2. *Considering the presentation of* $\mathrm{Bun}_G^\circ(C)$ *as the space of representations of the fundamental groupoid of* Γ*, a vertex-trivial bundle* E *is in the wobbly divisor if and only if at least one of the following holds:*

(1) *There exists a simple loop mapped to a matrix that has repeated eigenvalue (picking an arbitrarily base point on the simple loop and direction around the loop).*
(2) *There exists two disjoint simple loops and a simple path from one to the other such that, picking a base point on the path, the loops are mapped to matrices that share an eigenvector.*
(3) *There exists two distinct simple loops with nonempty connected intersection such that, picking a base point in the intersection, the loops are mapped to matrices that share an eigenvector.*

Remark 7.3. The conditions "share an eigenvector" and "has a repeated eigenvalue" are well-defined for G equal to any of SL_r, GL_r, or PGL_r.

We prove some useful lemmas.

Lemma 7.4. *The stabilizer in* G *of a nonzero nilpotent matrix* $X \in \mathfrak{g}$ *under the adjoint representation is given by the product of scalar matrices and a unipotent subgroup (of dimension* 1*) of* G*, and the set of elements of* G *that act on* X *as multiplication by some scalar is a Borel subgroup of* G*.*

Proof. Without loss of generality, let $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If we let $A \in G$ have $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then direct calculation shows A commutes with X when $a = d$ and $c = 0$, and A commutes with X up to some scalar when $c = 0$. □

Lemma 7.5. *If* X*,* Y*, and* Z *are nilpotent* 2×2 *matrices,* β_1*,* β_2*, and* β_3 *are nonzero scalars, and* $\beta_1 X + \beta_2 Y + \beta_3 Z = 0$*, then the* $\dim(\mathrm{span}(X, Y, Z)) \leq 1$.

Proof. Suppose not. Without loss of generality, let $\beta_1 = \beta_2 = -\beta_3 = 1$. Then $X + Y = Z$. Then X and Y must span a space of rank at least 2, so they have to both be nonzero and have distinct kernels (or equivalently, distinct images). Then let x generate the kernel of X and y generate the kernel of Y. Then $(X+Y)^2 = Z^2 = 0$, so $0 = (X+Y)^2 x = (X+Y)Yx$. Yx is then a nonzero multiple of y, so $0 = (X + Y)y = Xy$, which implies y is in the kernel of X, and hence the kernels of X and Y are the same, contradiction. □

Lemma 7.6. *Every nonempty trivalent graph has a the pair of lips or the pair of glasses as a topological minor*

Proof. First, since the graph is trivalent, it is not a tree, so it has at least one loop, and therefore has at least one simple loop (meaning a loop that does not intersect itself; a single self-loop counts here). This is the starting point for our subgraph representing the toplogical minor. Then, starting from a vertex of the simple loop followed by the edge from that vertex that is not part of the simple loop, add to the subgraph by going on a path until the first time a vertex has been reached that was already in the subgraph. As this vertex must have been reached by an edge not traversed before, it could not have been the vertex on the simple loop at which we started our path, as that vertex had all three (counted with multiplicity) of its edges traversed. Hence, either we have reached the simple loop and the resulting subgraph is homeomorphic to a pair of lips, or we have reached an earlier part of the path and the resulting graph is homeomorphic to a pair of glasses. □

From lemma Lemma 7.5, it follows that the residues of Q on a given bubble, as they add to 0 and are nilpotent, must all be scalar multiples of some nilpotent matrix. Furthermore, given that that is the case, it follows that the Higgs field will be nilpotent at any point in that bubble, as it will always be a multiple of that nilpotent matrix throughout the bubble. Also, the condition of being nontrivial is equivalent to being nonzero at one of the residues, as if zero at all residues then a Higgs field will then be 0 on all bubbles. Hence, the question of being in the wobbly divisor is completely formulated in terms of the graph Γ, so from now on we only consider E to be a representation of the fundamental group of this graph. In particular, a Higgs field Q then consists of the information of assigning a matrix Q_{ei} to each side i (say, 0 or 1) of each edge e of Γ, satisfying that the sum of such Q-matrices at each vertex is 0 and for any edge conjugating the Q-matrix on one side of the edge by the E-matrix attached to the edge (in the appropriate direction) gives the negative of the Q-matrix on the other side of the edge. We are interested in which representations E admit a nontrivial Q satisfying these properties such that additionally all Q-matrices are nilpotent, but not all of them are 0.

Now we prove the main theorem:

Proof of Theorem 7.2, forward implication. The following are equivalent:

- There exists a nontrivial nilpotent Higgs field Q of E.
- There exists a nilpotent Higgs field Q on E restricted to some subgraph Γ_0 of Γ with at least one edge such that all Q-matrices are nonzero (as we can restrict nontrivial Higgs fields to the subgraph containing the edges at which they are nonzero).
- There exists a nilpotent Higgs field Q on E restricted to some *connected* subgraph Γ_0 of Γ with at least one edge such that all Q-matrices are nonzero (as considering disconnected subgraphs is redundant since we may restrict a nontrivial nilpotent Q to a connected component).
- There exists a nilpotent Higgs field Q on E restricted to some connected subgraph Γ_0 of Γ with at least one edge but no vertices of degree 1 such that all Q-matrices are nonzero (as the condition of Q-matrices adding to 0 at a vertex of degree 1 is contradictory to the condition that the Q-matrix at the edge from that vertex must be nonzero).
- There exists a nilpotent Higgs field Q on E restricted to some topological minor Γ_0 of Γ that is either nonempty connected trivalent or a self-loop on one vertex such that all Q-matrices are nonzero (as it is equivalent to remove a vertex of degree 2 and combine the two edges connecting it; it can be checked that the resulting conditions on Q will be equivalent; then any subgraph with no leaves becomes a toplogical minor that is trivalent or a self-loop in this way).

If Γ_0 is a self-loop, then there are two nonzero nilpotent matrices Q_0 and Q_1 on each side of the edge such that $Q_0 + Q_1 = 0$ and $Q_0 + E_{01}Q_1(E_{01})^{-1} = 0$ where E_{01} is the E-matrix at the edge from side 1 to side 0. Hence, the existence of such a Q is equivalent to the existence of a Q_0 commuting with E_{01}, which, by Lemma 7.4, is equivalent to E_{01} having a repeated eigenvalue.

If Γ_0 is nonempty connected trivalent, pick a base point (that is a vertex of Γ_0), and then the condition of existence of such a Q implies that all loops of the fundamental group at this base point get mapped to matrices whose conjugation action on the nilpotent Q-matrices at the base point is by a scalar. This is because when we start at one end of the loop with a Q-matrix of the base point, we then may go around the loop. When we pass through an edge, we conjugate by the E-matrix attached to that edge and negate to get the Q-matrix on the other side of the edge. When we pass through a vertex from one edge to another, we know from before that we simply multiply our Q-matrix by some scalar. Hence, once we have reached the end of the loop, we will be back at the base point and hence be a scalar multiple of the Q-matrix at the start, and we will have multiplied by some scalars along the way and we will have conjugated by the E-matrix of the loop. Now, apply Lemma 7.4 to say that the existence of such a Q is equivalent to the matrices in the image of the E-representation of the fundamental group lying in a common Borel subgroup, or equivalently, sharing some eigenvector. It is then implied that this condition holds for any topological minor of Γ_0. By Lemma 7.6, there exists a topological minor of Γ_0 that is isomorphic to the pair of lips or to the pair of glasses. Hence, it is implied that some topological minor of Γ isomorphic to the pair of lips or the pair of glasses satisfies the shared eigenvector property of the representation E restricted to the fundamental group.

We now have enough to prove the forward direction of Theorem 7.2. From what we have said, if E is in the wobbly divisor, then one of the following holds:

- One case is that Γ has some topological minor that is a loop such that E restricted to that minor sends the loop to a matrix with repeated eigenvalue. Then that minor corresponds to a simple loop of Γ and (1) holds.
- Another case is that Γ has some topological minor isomorphic to the pair of glasses such that all loops in the fundamental group of this minor are mapped by E to matrices sharing an eigenvector. As the loops are generated by the two simple loops of this graph, we can then lift to the original graph Γ, where this minor corresponds to two disjoint simple loops connected by a path, and we get (2).
- The third an final case is that Γ has some topological minor isomorphic to the pair of lips such that all loop sin the fundamental group of this minor are mapped by E to matrices sharing an eigenvector. As the loops are generated by the two "lips" of this graph, we can lift to the original graph Γ where this minor corresponds to a pair of distinct simple loops with nonempty connected intersection, and we get (3).

We conclude the forward direction.

□

Proof of Theorem 7.2, backward implication. Note that the conditions related to topological minors that we proved were equivalent to E being in the wobbly divisor can still be used. We split up into the three cases:

(1) holds: The simple loop gives a topological minor Γ_0 that is one vertex with one loop, and nilpotent Higgs field Q with no zero Q-matrices exists on E restricted to it as, as previously seen, the conditions on Q are $Q_0 + Q_1 = 0$ and $Q_0 + E_{01}Q_1(E_{01})^{-1} = 0$, and the existence of such a Q is then equivalent to the existence of a nonzero nilpotent commuting with E of the loop, which is in turn equivalent by Lemma **??** to E of the loop having repeated eigenvalue, which is the statement of (1).

(2) holds: The two simple loops and the path connecting them give rise to a pair of glasses minor. Hence, it suffices to construct a nilpotent Higgs field Q on E restricted to this minor such that none of the Q-matrices are 0. Label the loops of the

glasses graph e and f and label the third edge p. Then we want matrices nilpotent Q_e, Q_f, and Q_p satisfying certain conditions, where $-E_eQ_eE_e^{-1}$, $-E_fQ_fE_f^{-1}$, and $-E_pQ_pE_p^{-1}$ are resulting Q-matrices on the other sides of the respective edges. Assuming E_p goes along p in the direction from e to f, the conditions are then

$$Q_e - E_eQ_eE_e^{-1} + Q_p = 0$$
$$Q_f - E_fQ_fE_f^{-1} - E_pQ_pE_p^{-1} = 0.$$

Now, we can conjugate our matrices in our groupoid representation to get an equivalent one in which $E_p = \mathrm{id}$. Then it suffices to find nonzero nilpotents Q_e and Q_f such that they are multiples of one another and $E_eQ_eE^{-1}$ and $E_fQ_fE_f^{-1}$ are also multiples of them, then either $E_eQ_eE_e^{-1} - Q_e = 0$ in which case by Lemma 7.4 E_e has repeated eigenvalue and we are back in the previous case of (1) holding, $Q_f - E_fQ_fE_f^{-1} = 0$ and again (1) holds, or we can scale Q_f so that $E_eQ_eE_e^{-1} - Q_e$ and $Q_f - E_fQ_fE_f^{-1}$ are the same nonzero nilpotent which we define to be Q_p.

(3) holds: This is analogous to the case that (2) holds except for the pair of lips instead of glasses. In particular, if the edges are e, f, and g, after we assume that one of the edges, say g, gets mapped to the identity by E, the conditions we get are

$$Q_e + Q_f + Q_g = 0$$
$$E_eQ_eE_e^{-1} + E_fQ_fE_f^{-1} + Q_g = 0.$$

Now, again by (3) and Lemma 7.4, we know we can choose Q_e and Q_f to be nonzero nilpotent multiples of one another such that also multiples of them are $E_eQ_eE_e^{-1}$ and $E_fQ_fE_f^{-1}$. Now, if either $E_eQ_eE_e^{-1} = Q_e$ or $E_fQ_fE_f^{-1} = Q_f$, similarly to the case of (2) holds, we degenerate to the case of (1) holds on simple loop eg or fg respectively. Additionally, if $E_f^{-1}E_eQ_fE_f^{-1}E_e = \mathrm{id}$, we degenerate in the same way via simple loop ef. Finally, we are left with the nondegenerate case, in which $E_eQ_eE_e^{-1} = \alpha Q_e$ and $E_fQ_fE_F^{-1} = \beta Q_f$ for some nonzero scalars $\alpha \neq \beta$. Then, if $Q_f = \gamma Q_e$ where we choose γ (but it must be nonzero and not), in order for Q_g to be defined such that both conditions hold, we want $1 + \gamma = \alpha + \beta\gamma$ to be nonzero. Solving for γ, we see that our nondegeracy conditions imply that γ will exist, will not be 0, and $0 \neq 1 + \gamma$ as desired. □

7.2. **Computations of singularities for system of differential equations.** For the pair of lips graph from Section 2 of two vertices connected by three edges, we describe an equivalent approach to computing the wobbly divisor and show that it produces results equivalent to our more general methods. For the bubble curve C obtained from the lips graph with edges labeled A, B, and C, we have coordinates

$$u = \mathrm{tr}\, BC^{-1}$$
$$v = \mathrm{tr}\, CA^{-1}$$
$$w = \mathrm{tr}\, AB^{-1},$$

which are, of course, invariant under the stabilizer group action at the two vertices. As in Section 6, we consider the action of quantum Hitchin hamiltonians on sections of $L^2(\mathrm{Bun}_G(C))$ at unitary representations V_A, V_B, V_C of $G = \mathrm{SL}_2$. There are $3g - 3 = 3$ Hitchin hamiltonians in this case (three independent hamiltonians are given by $\mathrm{tr}(AA^*)$, $\mathrm{tr}(BB^*)$, and $\mathrm{tr}(AA^*BB^*)$), and they correspond to the Casimir elements Δ_A, Δ_B, Δ_C of SL_2 acting as differential operators on V_A, V_B, V_C, respectively. Now since the Casimir elements are central, their action should preserve invariance, so their action should be expressible as a differential operator in the coordinates u, v, w. Then if there is a common eigenfunction f of Δ_A, Δ_B, Δ_C in $V_A \otimes V_B \otimes C_V$ (we will actually compute f in Section 9), then expressing $\Delta_A f$, $\Delta_B f$, and $\Delta_C f$ in terms of derivatives of f with respect to u, v, and w gives three differential equations for f. Now consider the vector

$$F = (f, \partial_u f, \partial_v f, \partial_w f, \partial_u\partial_v f, \partial_u\partial_w f, \partial_v\partial_w f, \partial_u\partial_v\partial_w f)$$

of partials of f. For generic u, v, and w, the differential equations f satisfies can be used to express $\partial_u F$, $\partial_v F$, and ∂_w F in terms of F:

$$\begin{aligned}\partial_u F &= M_u F\\ \partial_v F &= M_v F\\ \partial_w F &= M_w F,\end{aligned}$$

where M_u, M_v, and M_w are matrices with polynomial entries in u, v, w. We wish to find when these matrices are not well-defined, that is, when they have singularities.

This is equivalent to the picture of wobbly divisors as elements of $\mathrm{Bun}_G(C)$ with nilpotent preimage in $T^*\mathrm{Bun}_G(C)$. Indeed, the nilpotency condition gives that the principal symbol of the differential operators Δ_A, Δ_B, and Δ_C are zero. Since for an eigenfunction f of Δ_A, Δ_B, and Δ_C, we have $\Delta_A f = \lambda_A f$, $\Delta_B f = \lambda_B f$, and $\Delta_C f = \lambda_C f$ for eigenvalues λ_A, λ_B, λ_C, this gives a relation among lower derivatives of f, giving that $(1, \partial_u, \partial_v, \partial_w, \partial_u\partial_v, \partial_u\partial_w, \partial_v\partial_w, \partial_u\partial_v\partial_w)$ do not form a complete set of differentials. This is precisely the case when one of the computed matrices has a singularity.

Computation gives the form to be

$$\Delta_A = \frac{3}{2}v\partial_v + \frac{3}{2}w\partial_w + (vw - 2u)\partial_v\partial_w + \frac{1}{2}(v^2-4)(\partial_v)^2 + \frac{1}{2}(w^2-4)(\partial_w)^2$$

and cyclic permutations for Δ_B and Δ_C, where ∂_x denotes $\frac{\partial}{\partial x}$ for a variable x. This can be obtained by assuming $C = \mathrm{id}$, expressing u, v, and w in terms of components of A and B, and computing Δ_A with a computer algebra system.

Applying to the eigenfunction f of Δ_A, Δ_B, Δ_C with eigenvalues λ_A, λ_B, λ_C, respectively, we get

$$\begin{aligned}\lambda_A f &= \tfrac{3}{2}v\partial_v f + \tfrac{3}{2}w\partial_w f + (vw-2u)\partial_v\partial_w f + \tfrac{1}{2}(v^2-4)\partial_v^2 f + \tfrac{1}{2}(w^2-4)\partial_w^2 f\\ \lambda_B f &= \tfrac{3}{2}u\partial_u f + \tfrac{3}{2}w\partial_w f + (uw-2v)\partial_u\partial_w f + \tfrac{1}{2}(u^2-4)\partial_u^2 f + \tfrac{1}{2}(w^2-4)\partial_w^2 f\\ \lambda_C f &= \tfrac{3}{2}u\partial_u f + \tfrac{3}{2}v\partial_v f + (uv-2w)\partial_u\partial_v f + \tfrac{1}{2}(u^2-4)\partial_u^2 f + \tfrac{1}{2}(v^2-4)\partial_v^2 f.\end{aligned}$$

Solving for $\partial_u^2 f$ gives

$$\begin{aligned}\partial_u^2 f = \frac{1}{u^2-4}\big(&(\lambda_b + \lambda_c - \lambda_a)f + 3u\partial_u f\\ &- (vw-2u)\partial_v\partial_w f + (uv-2w)\partial_u\partial_v f + (uw-2v)\partial_u\partial_w f\big),\end{aligned}$$

which has a singularity for $U(u,v,w)$ at $u = \pm 2$. By symmetry there are singularities at $v = \pm 2$ for $V(u,v,w)$ and at $w = \pm 2$ for $W(u,v,w)$.

Also

$$\begin{aligned}\lambda_A\partial_u f &= \tfrac{3}{2}v\partial_u\partial_v f + \tfrac{3}{2}w\partial_u\partial_w f - 2\partial_v\partial_w f + (vw-2u)\partial_u\partial_v\partial_w f\\ &\quad + \tfrac{1}{2}(v^2-4)\partial_u\partial_v^2 f + \tfrac{1}{2}(w^2-4)\partial_u\partial_w^2 f\\ \lambda_A\partial_v f &= \tfrac{3}{2}\partial_v f + \tfrac{5}{2}v\partial_v^2 f + \tfrac{5}{2}w\partial_v\partial_w f\\ &\quad + (vw-2u)\partial_v^2\partial_w f + \tfrac{1}{2}(v^2-4)\partial_v^3 f + \tfrac{1}{2}(w^2-4)\partial_v\partial_w^2 f\\ \lambda_A\partial_w f &= \tfrac{3}{2}\partial_w f + \tfrac{5}{2}w\partial_w^2 f + \tfrac{5}{2}v\partial_v\partial_w f\\ &\quad + (vw-2u)\partial_v\partial_w^2 f + \tfrac{1}{2}(w^2-4)\partial_w^3 f + \tfrac{1}{2}(v^2-4)\partial_v^2\partial_w f\\ \lambda_B\partial_v f &= \tfrac{3}{2}w\partial_v\partial_w f + \tfrac{3}{2}u\partial_v\partial_u f - 2\partial_w\partial_u f + (uw-2v)\partial_u\partial_v\partial_w f\end{aligned}$$

$$
\begin{aligned}
&+ \tfrac{1}{2}(w^2-4)\partial_v\partial_w^2 f + \tfrac{1}{2}(u^2-4)\partial_u^2\partial_v f \\
\lambda_B \partial_w f &= \tfrac{3}{2}\partial_w f + \tfrac{5}{2}w\partial_w^2 f + \tfrac{5}{2}u\partial_w\partial_u f \\
&+ (uw-2v)\partial_u\partial_w^2 f + \tfrac{1}{2}(w^2-4)\partial_w^3 f + \tfrac{1}{2}(u^2-4)\partial_u^2\partial_w f \\
\lambda_B \partial_u f &= \tfrac{3}{2}\partial_u f + \tfrac{5}{2}u\partial_u^2 f + \tfrac{5}{2}w\partial_u\partial_w f \\
&+ (uw-2v)\partial_u^2\partial_w f + \tfrac{1}{2}(u^2-4)\partial_u^3 f + \tfrac{1}{2}(w^2-4)\partial_u\partial_w^2 f \\
\lambda_C \partial_w f &= \tfrac{3}{2}u\partial_u\partial_w f + \tfrac{3}{2}v\partial_v\partial_w f - 2\partial_u\partial_v f + (uv-2w)\partial_u\partial_v\partial_w f \\
&+ \tfrac{1}{2}(u^2-4)\partial_u^2\partial_w f + \tfrac{1}{2}(v^2-4)\partial_v^2\partial_w f \\
\lambda_C \partial_u f &= \tfrac{3}{2}\partial_u f + \tfrac{5}{2}u\partial_u^2 f + \tfrac{5}{2}v\partial_u\partial_v f \\
&+ (uv-2w)\partial_u^2\partial_v f + \tfrac{1}{2}(u^2-4)\partial_u^3 f + \tfrac{1}{2}(v^2-4)\partial_u\partial_v^2 f \\
\lambda_C \partial_v f &= \tfrac{3}{2}\partial_v f + \tfrac{5}{2}v\partial_v^2 f + \tfrac{5}{2}u\partial_u\partial_v f \\
&+ (uv-2w)\partial_u\partial_v^2 f + \tfrac{1}{2}(v^2-4)\partial_v^3 f + \tfrac{1}{2}(u^2-4)\partial_u^2\partial_v f.
\end{aligned}
$$

This system in nine equations can be solved for the nine quantities $\partial_u^2\partial_v f$, $\partial_u\partial_v^2 f$, $\partial_u^2\partial_w f$, $\partial_u\partial_w^2 f$, $\partial_v^2\partial_w f$, $\partial_v\partial_w^2 f$, ∂_u^3, ∂_v^3, ∂_w^3 in terms of the other differentials. This will have singularities at the zeroes of the determinant of the matrix

$$
\begin{bmatrix}
0 & 0 & 0 & \frac{1}{2}(u^2-4) & 0 & 0 & 0 & (uv-2w) & \frac{1}{2}(u^2-4) \\
\frac{1}{2}(v^2-4) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(v^2-4) & (uv-2w) \\
0 & 0 & 0 & 0 & \frac{1}{2}(u^2-4) & (uw-2v) & \frac{1}{2}(u^2-4) & 0 & 0 \\
\frac{1}{2}(w^2-4) & 0 & 0 & 0 & (uw-2v) & \frac{1}{2}(w^2-4) & 0 & 0 & 0 \\
0 & (vw-2u) & \frac{1}{2}(v^2-4) & 0 & 0 & 0 & \frac{1}{2}(v^2-4) & 0 & 0 \\
0 & \frac{1}{2}(w^2-4) & (vw-2u) & \frac{1}{2}(w^2-4) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}(u^2-4) & 0 & \frac{1}{2}(u^2-4) & 0 \\
0 & \frac{1}{2}(v^2-4) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(v^2-4) \\
0 & 0 & \frac{1}{2}(w^2-4) & 0 & \frac{1}{2}(w^2-4) & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

The bottom three rows once again give singularities when one of u, v, or w is ± 2. Factoring those out, we wish to find the determinant of

$$
\begin{bmatrix}
0 & 0 & 0 & \frac{1}{2}(u^2-4) & 0 & 0 & 0 & (uv-2w) & \frac{1}{2}(u^2-4) \\
\frac{1}{2}(v^2-4) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(v^2-4) & (uv-2w) \\
0 & 0 & 0 & 0 & \frac{1}{2}(u^2-4) & (uw-2v) & \frac{1}{2}(u^2-4) & 0 & 0 \\
\frac{1}{2}(w^2-4) & 0 & 0 & 0 & (uw-2v) & \frac{1}{2}(w^2-4) & 0 & 0 & 0 \\
0 & (vw-2u) & \frac{1}{2}(v^2-4) & 0 & 0 & 0 & \frac{1}{2}(v^2-4) & 0 & 0 \\
0 & \frac{1}{2}(w^2-4) & (vw-2u) & \frac{1}{2}(w^2-4) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

A computer algebra system gives

$$\frac{1}{2}(u^2-4)(v^2-4)(w^2-4)(u^2+v^2+w^2-uvw-4)^2,$$

which, in addition to the old singularities, gives a possible singularity at $u^2+v^2+w^2-uvw-4$.

Finally, we need to obtain equations for $\partial_u^2\partial_v\partial_w f$ and cyclic permutations. We will actually need twelve equations in twelve unknowns: $\partial_u^2\partial_v\partial_w f$, $\partial_u\partial_v^2\partial_w f$, $\partial_u\partial_v\partial_w^2 f$, as well as $\partial_u^2\partial_v^2 f$, $\partial_u^2\partial_w^2 f$, $\partial_v^2\partial_w^2 f$, $\partial_u^3\partial_v f$, $\partial_u\partial_v^3 f$, $\partial_u^3\partial_w f$, $\partial_u\partial_w^3 f$, $\partial_v^3\partial_w f$, $\partial_v\partial_w^3 f$. All we will care about are poles, so we neglect the "constants" (differentials besides those in the list of twelve unknowns) in the equations

$$
\begin{aligned}
0 &\approx (vw-2u)\partial_u^2\partial_v\partial_w f + \tfrac{1}{2}(v^2-4)\partial_u^2\partial_v^2 f + \tfrac{1}{2}(w^2-4)\partial_u^2\partial_w^2 f \\
0 &\approx (vw-2u)\partial_v^2\partial_w^2 f + \tfrac{1}{2}(v^2-4)\partial_v^3\partial_w f + \tfrac{1}{2}(w^2-4)\partial_v\partial_w^3 f \\
0 &\approx (vw-2u)\partial_u\partial_v^2\partial_w f + \tfrac{1}{2}(v^2-4)\partial_u\partial_v^3 f + \tfrac{1}{2}(w^2-4)\partial_u\partial_v\partial_w^2 f \\
0 &\approx (vw-2u)\partial_u\partial_v\partial_w^2 f + \tfrac{1}{2}(v^2-4)\partial_u\partial_v^2\partial_w f + \tfrac{1}{2}(w^2-4)\partial_u\partial_w^3 f,
\end{aligned}
$$

where we differentiate the equality for $(-\frac{1}{4}-a^2)f$ by ∂_u^2, $\partial_u\partial_w$, $\partial_u\partial_v$, and $\partial_u\partial_w$, respectively. Symmetrically we have

$$\begin{aligned}
0 &\approx (uw-2v)\partial_u\partial_v^2\partial_w f + \tfrac{1}{2}(w^2-4)\partial_v^2\partial_w^2 f + \tfrac{1}{2}(u^2-4)\partial_u^2\partial_v^2 f\\
0 &\approx (uw-2v)\partial_u^2\partial_w^2 f + \tfrac{1}{2}(w^2-4)\partial_u\partial_w^3 f + \tfrac{1}{2}(u^2-4)\partial_u^3\partial_w f\\
0 &\approx (uw-2v)\partial_u\partial_v\partial_w^2 f + \tfrac{1}{2}(w^2-4)\partial_v\partial_w^3 f + \tfrac{1}{2}(u^2-4)\partial_u^2\partial_v\partial_w f\\
0 &\approx (uw-2v)\partial_u^2\partial_v\partial_w f + \tfrac{1}{2}(w^2-4)\partial_u\partial_v\partial_w^2 f + \tfrac{1}{2}(u^2-4)\partial_u^3\partial_v f\\
0 &\approx (uv-2w)\partial_u\partial_v\partial_w^2 f + \tfrac{1}{2}(u^2-4)\partial_u^2\partial_w^2 f + \tfrac{1}{2}(v^2-4)\partial_v^2\partial_w^2 f\\
0 &\approx (uv-2w)\partial_u^2\partial_v^2 f + \tfrac{1}{2}(u^2-4)\partial_u^3\partial_v f + \tfrac{1}{2}(v^2-4)\partial_u\partial_v^3 f\\
0 &\approx (uv-2w)\partial_u^2\partial_v\partial_w f + \tfrac{1}{2}(u^2-4)\partial_u^3\partial_w f + \tfrac{1}{2}(v^2-4)\partial_u\partial_v^2\partial_w f\\
0 &\approx (uv-2w)\partial_u\partial_v^2\partial_w f + \tfrac{1}{2}(u^2-4)\partial_u^2\partial_v\partial_w f + \tfrac{1}{2}(v^2-4)\partial_v^3\partial_w f.
\end{aligned}$$

The matrix for the denominator of Cramer's rule is then

$$\begin{bmatrix}
(vw-2u) & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(u^2-4) & (uw-2v) & 0 & 0 & (uv-2w) & \frac{1}{2}(u^2-4)\\
0 & 0 & (vw-2u) & \frac{1}{2}(v^2-4) & (uw-2v) & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(v^2-4) & (uv-2w)\\
0 & 0 & \frac{1}{2}(w^2-4) & (vw-2u) & 0 & 0 & (uw-2v) & \frac{1}{2}(w^2-4) & (uv-2w) & 0 & 0 & 0\\
\frac{1}{2}(v^2-4) & 0 & 0 & 0 & \frac{1}{2}(u^2-4) & 0 & 0 & 0 & 0 & (uv-2w) & 0 & 0\\
\frac{1}{2}(w^2-4) & 0 & 0 & 0 & 0 & (uw-2v) & 0 & 0 & \frac{1}{2}(u^2-4) & 0 & 0 & 0\\
0 & (vw-2u) & 0 & 0 & \frac{1}{2}(w^2-4) & 0 & 0 & 0 & \frac{1}{2}(v^2-4) & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(u^2-4) & 0 & \frac{1}{2}(u^2-4) & 0 & 0\\
0 & 0 & \frac{1}{2}(v^2-4) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(v^2-4) & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}(u^2-4) & 0 & 0 & 0 & 0 & \frac{1}{2}(u^2-4) & 0\\
0 & 0 & 0 & \frac{1}{2}(w^2-4) & 0 & \frac{1}{2}(w^2-4) & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{2}(v^2-4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(v^2-4)\\
0 & \frac{1}{2}(w^2-4) & 0 & 0 & 0 & 0 & \frac{1}{2}(w^2-4) & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

and factoring out poles we already obtained from the last several rows gives

$$\begin{bmatrix}
(vw-2u) & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(u^2-4) & (uw-2v) & 0 & 0 & (uv-2w) & \frac{1}{2}(u^2-4)\\
0 & 0 & (vw-2u) & \frac{1}{2}(v^2-4) & (uw-2v) & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(v^2-4) & (uv-2w)\\
0 & 0 & \frac{1}{2}(w^2-4) & (vw-2u) & 0 & 0 & (uw-2v) & \frac{1}{2}(w^2-4) & (uv-2w) & 0 & 0 & 0\\
\frac{1}{2}(v^2-4) & 0 & 0 & 0 & \frac{1}{2}(u^2-4) & 0 & 0 & 0 & 0 & (uv-2w) & 0 & 0\\
\frac{1}{2}(w^2-4) & 0 & 0 & 0 & 0 & (uw-2v) & 0 & 0 & \frac{1}{2}(u^2-4) & 0 & 0 & 0\\
0 & (vw-2u) & 0 & 0 & \frac{1}{2}(w^2-4) & 0 & 0 & 0 & \frac{1}{2}(v^2-4) & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

which has determinant

$$4(u^2+v^2+w^2-uvw-4)^4,$$

giving no new poles.

Thus this explicit calculation of the wobbly divisor in terms of differential operators gives the same result as more our more abstract reasoning.

8. Real structures on nodal curves

A real structure on a nodal curve C over the field $\mathbb{C}$ is an anti-holomorphic involution on C. In this section, we classify real structures on trinionic curves up to conjugation by holomorphic functions. This is important for computation of the Hilbert space in the real case; while we do not actually perform these calculations in this paper, this is useful motivation for Section 9 and sets the stage for continued study of the Langlands correspondence for trinionic curves.

We begin with a single bubble. Let $\varphi\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ be an anti-holomorphic involution. Then $\bar{\varphi}$ is a holomorphic bijection on $\mathbb{CP}^1 = \mathbb{C} \sqcup \{\infty\}$, and therefore must be meromorphic with a single simple pole. It is well-known that the meromorphic functions on $\mathbb{CP}^1$ are just the rational functions, so φ must be of the form

$$z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d}$$

for some complex numbers a, b, c, d. If this is an involution then

$$\begin{aligned} z &= \frac{a\left(\frac{\bar{a}z+\bar{b}}{\bar{c}z+\bar{d}}\right) + b}{c\left(\frac{\bar{a}z+\bar{b}}{\bar{c}z+\bar{d}}\right) + d} \\ &= \frac{a\bar{a}z + a\bar{b} + b\bar{c}z + b\bar{d}}{c\bar{a}z + c\bar{b} + d\bar{c}z + d\bar{d}}, \end{aligned}$$

which gives

$$\begin{aligned} a\bar{b} + b\bar{d} &= 0 \\ c\bar{a} + d\bar{c} &= 0 \\ a\bar{a} + b\bar{c} &= c\bar{b} + d\bar{d}. \end{aligned}$$

Taking the conjugate of the second equality, and then taking a linear combination with the first, gives

$$b\bar{c}a = \bar{b}ca.$$

Now there are two cases.

- $a \neq 0$. Then $\bar{b}c = b\bar{c}$, and so we obtain $a\bar{a} = d\bar{d}$. We may assume WLOG that $a = 1$, so $|d| = 1$. Let $d = -\omega^2$ (where ω is of magnitude 1). Then $\bar{b} = -b\bar{d} = \omega^{-2}b$ and similarly $\bar{c} = -d^{-1}c = \omega^{-2}c$. So there are reals β and γ such that $b = \beta\omega$ and $c = \gamma\omega$. So the map φ is of the form

  $$z \mapsto \frac{\bar{z} + \beta\omega}{\gamma\omega\bar{z} - \omega^2}.$$

  Conjugating by $z \mapsto \omega z$ gives

  $$z \mapsto \omega \cdot \frac{\omega\bar{z} + \beta\omega}{\gamma\omega^2\bar{z} - \omega^2} = \frac{\bar{z} + \beta}{\gamma\bar{z} - 1}.$$

  This has real coefficients so its conjugate corresponds to the action of the real matrix

  $$\begin{bmatrix} 1 & \beta \\ \gamma & -1 \end{bmatrix}$$

  on $\mathbb{CP}^1$. We must have $\beta\gamma \neq -1$, since otherwise the map is not bijective. Then the matrix is diagonalizable by a real matrix, so the map φ is conjugate to $z \mapsto r\bar{z}$ for some real r. Since φ is an involution, we have $r = \pm 1$, and if $r = -1$, conjugating by $z \mapsto iz$ gives $z \mapsto z$. Thus in this case φ is conjugate to the conjugation map.
- $a = 0$. Then $b\bar{d} = 0$ and $d\bar{c} = 0$. If $d \neq 0$, then $b = c = 0$, clearly not giving an involution. So $d = 0$, which leaves functions of the form $\frac{r}{\bar{z}}$ for real r as possible involutions. Up to conjugacy, this gives $\pm\frac{1}{\bar{z}}$. Conjugating the map $z \mapsto \frac{1}{\bar{z}}$ by $z \mapsto \frac{z-i}{iz-1}$ gives $z \mapsto \bar{z}$, while $z \mapsto -\frac{1}{\bar{z}}$ is not conjugate to $z \mapsto \bar{z}$ because the former has no fixed points, while the latter has a curve of them.

Thus up to conjugation there are two real systems: $z \mapsto \bar{z}$ and $z \mapsto \frac{1}{\bar{z}}$.

Now consider a bubble with labeled points 0, 1, and ∞. The conjugacy class $z \mapsto \bar{z}$ corresponds to the identity permutation on these three points, while $z \mapsto \frac{1}{\bar{z}}$ corresponds to a transposition. Now for a general trinionic curve, real systems are given by a permutation of the bubbles together with a real system on each of the bubbles, with constraints that the real systems must agree along edges. This is encapsulated by considering graph automorphisms, viewed as acting on edges; we additionally want a flip of a self-loop to be a non-trivial automorphism, since this corresponds to a non-trivial action on a bubble. The following theorem states this formally.

Theorem 8.1. *Up to conjugation by holomorphic functions, real systems on a trinionic curve C are given by conjugacy classes of elements of order 2 in the group of graph automorphisms of the trivalent graph Γ corresponding to C, where automorphisms are viewed as acting on edges and a flip of a self-loop is a non-trivial automorphism.*

A canonical real structure on C is given by the conjugacy class of the identity in the automorphism group of Γ. This corresponds to simply conjugating each bubble.

9. Eigenfunctions

Let $G = \mathrm{SL}_2(\mathbf{k})$. Let C be a trinion curve corresponding to quiver Q. The quantum Hitchin system acts on Hilbert space of $\mathrm{Bun}_G(C)$. In this section, we find eigenfunctions of this action.

Recall that

$$L^2(\mathrm{Bun}_G(C)) = \int L^2(\mathrm{Bun}_G(C))_{\pi_1,\dots,\pi_n} \mathrm{d}\mu(\pi_1)\cdots \mathrm{d}\mu(\pi_n),$$

where the integration is over irreducible unitary representations of G, so we are interested in the eigenfunctions in parts obtained by assigned irreducible unitary representations to each edge of Q. For $\mathbf{k} = \mathbb{C}$ the online unitary irreducible representations are principal series representations, so we may fix real principal series weights s_j so we only need to consider $\pi_j = V_{s_j}$, the space of $(\frac{1}{2} + is_j)$-densities on $\mathbb{P}^1$ with action given by

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} f\right)(z) = f\left(\frac{az+b}{cz+d}\right) |cz+d|^{-m(1+2is)} \psi_{abcd}(z),$$

where m is 1 when $\mathbf{k} = \mathbb{R}$ and 2 when $\mathbf{k} = \mathbb{C}$ and $\psi_{abcd}(z)$ is an expression of magnitude 1 depending on discrete parameters of the representation, which will not matter for our purposes. Specifically, for the complex case

$$\psi_{abcd}(z) = \left(\frac{cz+d}{|cz+d|}\right)^k$$

for descrete parameter $k \in \mathbb{Z}$ and for the complex case

$$\psi_{abcd}(z) = \mathrm{sgn}(cz+d)^k,$$

where $k \in \{0, 1\}$.

When C is a trinion curve and $G = \mathrm{PGL}_2$, there are $3g-3$ quantized Hitchin hamiltonians, corresponding to Casimir elements acting on the irreducible unitary representations at each edge. The Casimir element $ef + fe + \frac{1}{2}h^2$ should act on $V_{s_j} \otimes \overline{V_{s_j}} = V_{s_j} \otimes V_{-s_j}$ by acting by the Casimir element on V_{s_j} and by the identity on V_{-s_j}.

We first compute the eigenvalue of the action of the Casimir element on V_s. A curve having tangent h at $t = 0$ is given by

$$\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Curves tangent to e and f are given by

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix},$$

respectively. We have

$$\begin{aligned}
\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} F\right)(z) &= \left(\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}^{-1} F\right)(z) \\
&= F(z-t) \\
\left(\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} F\right)(z) &= \left(\begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}^{-1} F\right)(z) \\
&= \psi(z)|-tz+1|^{-m(1+2is)} F\left(\frac{z}{-tz+1}\right) \\
\left(\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} F\right)(z) &= \left(\begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix}^{-1} F\right)(z) \\
&= e^{-m(1+2is)t} F(e^{-2t} z).
\end{aligned}$$

In the real case, taking small enough t guarantees that $\psi(z) = 1$ and $-tz+1 > 0$, so differentiating at $t = 0$ gives

$$\begin{aligned}
(eF)(z) &= -F'(z) \\
(fF)(z) &= m(1+2is)zF(z) + z^2F'(z) \\
(hF)(z) &= -m(1+2is)F(z) - 2zF'(z),
\end{aligned}$$

so

$$\begin{aligned}
\left(\left(fe+ef+\frac{1}{2}h^2\right)F\right)(z) = &-m(1+2is)zF'(z) - z^2F''(z) \\
&- m(1+2is)F(z) - m(1+2is)zF'(z) \\
&- 2zF'(z) - z^2F''(z) \\
&+ \frac{m^2(1+2is)^2}{2}F(z) + m(1+2is)zF'(z) \\
&+ m(1+2is)zF'(z) + 2z^2F''(z) + 2zF'(z) \\
= &\frac{m^2(1+2is)^2 - 2m(1+2is)}{2} \cdot F(z).
\end{aligned}$$

Setting $m = 1$ this gives $-\frac{1}{2} - 2s^2$. A similar, but more technical, calculation can be used to find the eigenvalue for the general complex case. If the discrete parameter of the representation is disregarded, then plugging $m = 2$ into the results from the real case give gives $4is - 8s^2$.

For principal series weights s_1, s_2, and s_3, we wish to find a vector in $V_{s_1} \otimes V_{s_2} \otimes V_{s_3}$ invariant under the action of G. Elements of $V_{s_1} \otimes V_{s_2} \otimes V_{s_3}$ are densities on $(\mathbb{P}^1)^3$ of the form $F(x_1, x_2, x_3)(\mathrm{d}x_1)^{\frac{1}{2}+is_1}(\mathrm{d}x_2)^{\frac{1}{2}+is_2}(\mathrm{d}x_3)^{\frac{1}{2}+is_3}$. Fix a positive real number $t \neq 1$, and assume $f(0,1,t) = 1$. Using invariance, we determine F. Let

$$A = \begin{bmatrix} 1 & -x_1 \\ \frac{(x_1-x_3)-t(x_1-x_2)}{t(x_2-x_3)} & \frac{tx_3(x_1-x_2)-x_2(x_1-x_3)}{t(x_2-x_3)} \end{bmatrix}.$$

It can be verified that the automorphism on $\mathbb{P}^1$ associated to A maps x_1, x_2, and x_3 to 0, 1, and t, respectively. Also,

$$\det A = \frac{t-1}{t} \cdot \frac{(x_1 - x_3)(x_1 - x_2)}{x_2 - x_3}.$$

So acting by $\frac{1}{\sqrt{\det A}}A$ on x_1, x_2, x_3, gives 0, 1, and t, respectively. Thus

$$\begin{aligned} F(x_1, x_2, x_3) &= \left(\left(\frac{A}{\sqrt{\det A}} \right)^{-1} f \right) (x_1, x_2, x_3) \\ &= \left| \frac{(x_1 - x_2)(x_1 - x_3) + t(x_1 - x_3)(x_1 - x_2)}{t(x_2 - x_3)} \right|^{-1-2is_1} |x_1 - x_2|^{-1-2is_2} \left| \frac{x_1 - x_3}{t} \right|^{-1-2is_3} \\ &\quad \left| \frac{t-1}{t} \cdot \frac{(x_1 - x_3)(x_1 - x_2)}{x_2 - x_3} \right|^{\frac{3}{2}+is_1+is_2+is_3} \psi(x_1, x_2, x_3) \end{aligned}$$

which up to a constant factor depending only on the parameter t and an expression of magnitude 1 depending on discrete parameters is

$$F_{abc}(x_1, x_2, x_3) = |x_1 - x_2|^{-\frac{1}{2}-is_1-is_2+is_3} |x_1 - x_3|^{-\frac{1}{2}-is_1+is_2-is_3} |x_2 - x_3|^{-\frac{1}{2}+is_1-is_2-is_3}.$$

Piecing together these invariant vectors over all triples of principal series representations occurring at vertices of Q will give the desired eigenvector.

Introduce a variable x_e for each edge e. Let the matrix acting at each edge be A_e, and the eigenvalue of the Laplacian Δ_{A_e} be $\frac{1}{2} + is_e$. Denote by $\bar{e}$ the reverse of edge e, and define $A_{\bar{e}}$ to be the identity matrix and $s_{\bar{e}} = -s_e$, while keeping $x_{\bar{e}} = x_e$. For each vertex v, let the three incoming edges be e_v^1, e_v^2, e_v^3, keeping track of direction (so an edge is counted backwards if it is outgoing). Then, disregarding the function ψ depending on discrete parameters, the integral is

$$\int \left(\prod_{v \in V} \prod_{\text{cyc}} |A_{e_v^1} x_{e_v^1} - A_{e_v^2} x_{e_v^2}|^{m(-\frac{1}{2}-is_{e_v^1}-is_{e_v^2}+is_{e_v^3})} \right) \left(\prod_{e \in E} |(A_e)_{21} x_e + (A_e)_{22}|^{m(-1-2is_e)} \right) \prod_{e \in E} \mathrm{d}x_e^m, \tag{9.1}$$

where $\mathrm{d}x_e^1 = \mathrm{d}x_e$ and $\mathrm{d}x_e^2 = \mathrm{d}x_e \overline{\mathrm{d}x_e}$ is the measure of integration.

Remark 9.1. For compact groups G in the real case, things are simpler. As discussed in Section 8, there is a canonical real structure τ over trinion curves C. For $G = \mathrm{GL}_2(\mathbb{C})$, there is a canonical anti-holomorphic involution σ given by complex conjugation. Consider real structures on $\mathrm{Bun}_G(C)$, that is, isomorphisms $A\colon (\sigma, \tau)(\mathrm{Bun}_G(C)) \to \mathrm{Bun}_G(C)$ such that $A \circ (\sigma, \tau)(A) = \mathrm{id}$. The goal is to study the Hilbert space of the real bundle, that is, the part of $\mathrm{Bun}_G(C)$ invariant under the canonical real structure. Now the subgroup of real points of $\mathrm{GL}_2(\mathbb{C})$ is $\mathrm{U}(\mathbb{R})$, which is compact group, and any unitary representation of a compact group is fintite-dimensional. Plancherel decompositions of a compact group are given by completed direct sums. So the Hilbert space of the real bundle is a completed direct sum over irreducible finite-dimensional representations. Since representations are finite-dimensional, invariant vectors can be described very explicitly. Then the pairing is just a finite sum, rather than the integral, so the eigenfunction can be written explicitly in polynomial form. Thus the existence of eigenfunctions is immediate, and does not depend on the convergence of an integral.

Example 9.2. *For the graph of two vertices connected by three edges, all directed the same way, the integral is*

$$\int |x_1 - x_2|^{m(-\frac{1}{2}+is_1+is_2-is_3)} |x_1 - x_3|^{m(-\frac{1}{2}+is_1-is_2+is_3)} |x_2 - x_3|^{m(-\frac{1}{2}-is_1+is_2+is_3)}$$

$$|Ax_1 - Bx_2|^{m(-\frac{1}{2}-is_1-is_2+is_3)}|Ax_1 - Cx_3|^{m(-\frac{1}{2}-is_1+is_2-is_3)}|Bx_2 - Cx_3|^{m(-\frac{1}{2}+is_1-is_2-is_3)}\mathrm{d}x_1^m\mathrm{d}x_2^m\mathrm{d}x_3^m$$

Example 9.3. *For the graph of two vertices, each with a self-loop and with an edge between them, the integral is*

$$\int |Ax_1 - x_1|^{m(-\frac{1}{2}-is_2)}|Ax_1 - x_3|^{m(-\frac{1}{2}-2is_1+is_2)}|x_1 - x_3|^{m(-\frac{1}{2}+2is_1+is_2)}$$
$$|Cx_2 - x_2|^{m(-\frac{1}{2}+is_2)}|Cx_2 - Bx_3|^{m(-\frac{1}{2}+2ix-is_2)}|x_2 - Bx_3|^{(-\frac{1}{2}-2is_3-is_2)}$$
$$|A_{21}x_1 + A_{22}|^{m(-1-2is_1)}|C_{21}x_2 + C_{22}|^{m(-1-2is_2)}|B_{21}x_3 + B_{22}|^{m(-1-2is_3)}\mathrm{d}x_1^m\mathrm{d}x_2^m\mathrm{d}x_3^m.$$

Theorem 9.4. *For generic values of matrices A_e, the integral (9.1) converges absolutely.*

Proof. As before, we denote $\mathrm{d}x_e\overline{\mathrm{d}x_e}$ by simply $\mathrm{d}x_e^2$ in the complex case, allowing for more consistent notation. The integral of the absolute value is

$$\int \left(\prod_{v\in V}\prod_{\text{cyc}} |A_{e_v^1}x_{e_v^1} - A_{e_v^2}x_{e_v^2}|^{-\frac{1}{2}m}\right)\left(\prod_{e\in E}|(A_e)_{21}x_e + (A_e)_{22}|^{-m}\right)\prod_{e\in E}\mathrm{d}x_e^m.$$

Expanding the factors of the first product in terms of matrix elements and canceling out with factors in the second product gives

$$\int \left(\prod_{e,f}|((A_e)_{11}x_e + (A_e)_{12})((A_f)_{21}x_f + (A_f)_{22}) - ((A_f)_{11}x_f + (A_f)_{12})((A_e)_{21}x_e + (A_e)_{22})|^{-\frac{1}{2}m}\right)$$
$$\prod_{e\in E}\mathrm{d}x_e^m$$
$$= \int \Bigg(\prod_{e,f}|((A_e)_{11}(A_f)_{21} - (A_e)_{21}(A_f)_{11})x_ex_f + ((A_e)_{11}(A_f)_{22} - (A_e)_{21}(A_f)_{12})x_e$$
$$- ((A_e)_{22}(A_f)_{11} - (A_e)_{12}(A_f)_{21})x_f + (A_e)_{12}(A_f)_{22} - (A_e)_{22}(A_f)_{12}|^{-\frac{1}{2}m}\Bigg)$$
$$\prod_{e\in E}\mathrm{d}x_e^m.$$

where the first product is over edges e, f incoming into a single vertex (where edges are directed and we count both edges of the original quiver and their dual edges). Let F_{ef} be the polynomial in x_e, x_f in the product, so that the integral becomes

$$\int \prod_{e,f}|F_{ef}|^{-\frac{1}{2}m}\prod_{e\in E}\mathrm{d}x_e^m. \tag{9.2}$$

We prove that this converges locally with respect to every affine chart, which will prove that it converges over all $\mathbb{C}^n$.

Consider the change of coordinates in which x_e is replaced by $x'_e = \frac{1}{x_e}$ and x_f for $f \neq e$ is replaced by $x'_f = \frac{x_f}{x_e}$. This corresponds to taking a different affine chart of $(\mathbb{CP})^n$. Letting g be the genus of the quiver Q and recalling that there are $3g-3$ edges, and thus $3g-3$ variables, the Jacobian has determinant $\frac{1}{x_e^{3g-3+1}}$, so the integrand is multiplied by $x_e^{m(-3g+3-1)}$. Now to make $F_{fg}(x'_f, x'_g)$ a polynomial, we must multiply by x_e^2. Doing this for each factor multiplies the integrand by $x_e^{\frac{1}{2}m\cdot 2(6g-6)}$, because we have $6g-6$ equations coming from the $2g-2$ vertices of Q. This leaves a factor of $x_e^{m(3g-3-1)}$ in the integrand. We are interested in the behavior near $x_e = 0$. At the origin, poles generically come only from factors of the form $F_{fg} = x_f - x_g$, which become of the form $x_ex_f - x_ex_g$ after the coordinate change. There are at most $3g-3$ such factors because they correspond to pairs of edges directed away from a single vertex of Q. In x_e, this leaves a factor of at worst $x_e^{m(3g-3-1-\frac{1}{2}(3g-3)} = x_e^{m(\frac{1}{2}(3g-3)-1}$, and since $g \geq 2$, this is at worst $x_e^{-\frac{1}{2}}$, which converges. In other variables, we possible have independent factors of the form $|x_f - x_g|^{-\frac{1}{2}m}|x_g - x_h|^{-\frac{1}{2}m}$, which also converge (this can be seen by a change of coordinates).

Now consider poles not at the origin. Thus some x_f is not equal to zero. Setting x_f to be a constant, the transformed Fs once again are polynomials. Integrating locally in x_f then reduces the problem to proving that the integral (9.2) converges locally under the assumption that each F_{ef} is a polynomial in x_e and x_f, with coefficients depending on parameters A_e and A_f (we may absorb the choice of nonzero x_f into these) in a way dictated by the graph structure of Q. We proceed to prove that (9.2) does indeed converge.

We consider the integral around hypersurfaces where some of the F are equal to zero. Namely, for $k \geq 1$ consider a set of unordered pairs $\{(e_j, f_j)\}_{j=1}^k$ such that there is a nonempty set of points with all $F_{e_j f_j}$ equal to zero, and assume that $\{(e_j, f_j)\}_{j=1}^k$ is maximal with this property for the given set of common zeroes. Consider an open set containing the common zeroes of the $F_{e_j f_j}$, which form a hypersurface, and remove neighborhoods of points on this curve at which other Fs are zero. Let $\{g_j\}_{j=1}^m = \{e_j\}_{j=1}^k \cup \{f_j\}_{j=1}^k$ as a set. We will prove that $\int \prod_{j=1}^k F_{e_j f_j} \prod_{j=1}^m \mathrm{d}x_{g_j}$ and the corresponding complex integral converge near the origin; it will then follow that the integral (9.2) converges locally.

Consider a graph Γ with vertices $\{x_{g_j}\}_{j=1}^m$ and undirected edges corresponding to $F_{e_j f_j}$ (thus self-loops and repeated edges are allowed). We may assume without loss of generality that the resulting graph is connected (otherwise the integral factors, reducing to a smaller problem). Each edge determines one variable x in terms of another, up to a set of finite points (self-loops determine a single x up to finitely many options). We wish to use the nonemptiness of the solution set to constrain the structure of the graph in the generic case. Thus in particular we wish to constrain which F_{ef} can be independent of the matrices; we call edges correspond to such F_{ef} *bad edges*.

First, we note that F_{ef} is independent of the A only if it is of the form $\pm(x_e - x_f)$, corresponding to when e and f are outgoing from a single vertex in the trivalent quiver Q. Thus if there are two bad edges sharing a vertex, we may assume they form a triangle, since the constraints $x_e - x_f = 0 = x_f - x_g$ imply $x_e - x_g = 0$, and are not adjacent to any other bad edges. Now we contract our graph along all bad edges, giving a graph Γ'. Now no bad edges of Γ' can share a vertex, since then they would correspond to bad edges of Γ sharing a vertex or adjacent to a triangle. We claim that Γ' has genus at most 1. Suppose otherwise. Then there are two edges which can be removed so that Γ' remains connected, in which case the values of the x_{g_j} can be parametrized in terms of a single one, giving an algebraic curve of solutions. Adding in one of the removed edges restricts this down to a set of finite points. Then if the remaining edge is not bad, a generic choice will yield that no solutions exist, a contradiction. So it must be the case that both removed edges are bad. Thus any non-bridge of Γ' is bad. But since Γ' has a cycle and by the constraint on adjacency of bad edges some edge of the cycle is not bad, this is a contradiction.

We now go back to the original graph Γ, which is obtained by replacing some vertices of the genus ≤ 1 graph Γ' with triangles. We introduce a change of coordinates. Fix a vertex e of Γ such that, if Γ' is of genus 1, the image of e in Γ' is in a cycle. For each edge (x_f, x_g) of Γ which maps to an edge of Γ', introduce the local coordinate $F_{fg}(x_f, x_g)$. For triangles at x_f, x_g, x_h, introduce coordinates $x_f - x_g, x_g - x_h$. Finally, if Γ has genus 0, keep the coordinate x_e. Up to a factor depending on the Jacobian, this reduces the integral to a product of independent factors of the form $|uv(u+v)^{-\frac{1}{2}m}|$ and $y^{-\frac{1}{2}m}$, which converge.

Thus it remains to check that the determinant of the Jacobian is nonzero.

We begin with the case when Γ' has genus 0, and describe the coordinates more explicitly. Arbitrarily delete an edge of each bad triangle, so that Γ becomes a tree. For each variable x_f with $f \neq e$, let y_f be the coordinate given by the value of F_{fg}, where x_g corresponds to the edge "above" x_f in the spanning tree. The coordinate change maps $x_e \mapsto x_e$, $x_f \mapsto y_f$ near the origin (note that this is equivalent to what we described earlier). Up to a sign, the Jacobian has determinant $\prod_{f \neq e} \frac{\partial}{\partial x_g} F_{fg}$, where implicitly g depends on f. The set of zeroes is a curve, and taking the A generically, the sets of zeroes of $\frac{\partial}{\partial x_g} F_{fg}$ and $\frac{\partial}{\partial x_f} F_{fg}$ will not overlap with each other or with the same sets for other choices of f and g. Thus changing the root of the tree, which we can do because e was unconstrained in the genus 0 case, gives a valid change of coordinates locally at each point on the curve.

Now if Γ' has genus 1, the set of singularities is a finite set of isolated points, not a curve. Then a generic choice of the A will guarantee that determinant of the Jacobian is nonzero at this finite set of points, so the change of coordinates is again valid. Thus in both cases, the integral converges. □

References

[1] Roman Bezrukavnikov, Roman Travkin, Tsao-Hsien Chen, Xinwen Zhu, *Quantization of Hitchin integrable system via positive characteristic* arXiv:1603.01327v4 [math.RT] 4 Oct 2023

[2] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, Preprint http://www.math.uchicago.edu/~drinfeld/langlands/QuantizationHitchin.pdf

[3] Ana Cannas da Silva, *Lectures on Symplectic Geometry.* Issue 1764 Volume 1764 of Lecture Notes in Mathematics; Springer Science & Business Media, 2001

[4] Pavel Etingof, Edward Frenkel, David Kazhdan, *An analytic version of the Langlands correspondence for complex curves.* arXiv:1908.09677v4 [math.AG] 13 Jul 2021

[5] Pavel Etingof, Edward Frenkel, David Kazhdan, *Analytic Langlands correspondence for* PGL(2) *on* P^1 *with parabolic structures over local fields.* arXiv:2106.05243 [math.AG] 15 May 2022

[6] Pavel Etingof, Edward Frenkel, David Kazhdan, *A general framework for the analytic Langlands correspondence.* arXiv:2311.03743 [math.AG] 13 Feb 2024

[7] Pavel Etingof, Henry Liu, *Hitchin systems and their quantization.* arXiv:2409.09505 [math.AG] 14 Sep 2024

[8] Pavel Etingof, Alexander Varchenko *p-curvature of periodic pencils of flat connections.* arXiv:2401.05652v4 [math.AG] 7 Jan 2025

[9] Nigel Hitchin, *Stable bundles and integrable systems,* Duke Math. J. 54(1): 91–114 (1987). DOI: 10.1215/S0012-7094-87-05408-1

[10] George R. Kempf, *Algebraic Varieties.* Cambridge University Press; 1993.

[11] Molev, A., Ragoucy, E. Higher-order Hamiltonians for the trigonometric Gaudin model. Lett Math Phys 109, 2035–2048 (2019). https://doi.org/10.1007/s11005-019-01170-2

[12] Olsson, M. (2023). Algebraic spaces and stacks (Vol. 62). American Mathematical Society.

[13] Talalaev, D.V. The quantum Gaudin system. Funct Anal Its Appl 40, 73–77 (2006). https://doi.org/10.1007/s10688-006-0012-5

Department of Mathematics, MIT, Cambridge, MA 02139, USA
Email address: hrab@mit.edu, jvulakh@mit.edu