CLOSURES OF CONORMAL BUNDLES ON THE FLAG VARIETY

DYNE LEE,
MENTOR: KATIA BOGDANOVA

Abstract. We prove the normality of the closure of the conormal bundle to certain GL_n-orbits on the product of complete flag varieties $\mathcal{B} \times \mathcal{B}$. We relate these orbits to GL_n-orbits on the product of two Grassmannians $\mathrm{Gr}_k \times \mathrm{Gr}_l$ and prove the normality of closures of conormals on this space through vanishing conditions on cohomologies of coherent sheaves on Grassmannians. We also prove normality of certain strata of $\mathcal{B} \times \mathfrak{gl}_n$ defined by A. Knutson and S. Sam in [KS22].

Contents

1. Introduction

Let $G = \mathrm{GL}_n$ be the general linear group over an algebraically closed field of characteristic 0 and $B \subseteq G$ the Borel subgroup of upper triangular matrices. Throughout this paper, we denote by V the vector space k^n. The flag variety $\mathcal{B} = G/B$ is the moduli space of complete flags in V, i.e. chains of subspaces

$$0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = V$$

where V_i is an i-dimensional subspace. Flag varieties are important objects in algebraic geometry and representation theory. G/B carries an action of B, which corresponds to left multiplication by a matrix on every subspace of a flag. The standard flag

$$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \langle e_1, \ldots, e_n \rangle .$$

is fixed under multiplication every element of B.

The B-orbits are parametrized by elements of the Weyl group $W = S_n$ of GL_n. For a permutation $w \in S_n$, a representative of the B-orbit corresponding to w is the flag given by permuting the standard basis elements by σ:

$$0 \subset \left\langle e_{\sigma(1)} \right\rangle \subset \left\langle e_{\sigma(1)}, e_{\sigma(2)} \right\rangle \subset \ldots \subset \left\langle e_{\sigma(1)}, \ldots, e_{\sigma(n)} \right\rangle .$$

These orbits give a decomposition of $\mathcal{B}$ into locally closed strata, called Schubert cells. The closure of a Schubert cell $C(w)$ in $\mathcal{B}$ under the Zariski topology is called a Schubert variety. It is the union of the Schubert cells $C(u)$ where $u \leq w$ under the Bruhat order on S_n.

Similarly, let $\mathcal{B}^2 = (G \times G)/(B \times B)$ be the space of pairs of complete flags. G acts diagonally on $\mathcal{B}^2$. Given a pair of flags (F_1, F_2), we can always find some $g \in G$ such that gF_1 is the standard flag. Since this flag is fixed by B, the G-orbits in this space are again parametrized by elements $w \in W$, with a representative element being (F, wF) where w acts by permuting the standard basis elements. Let $\mathcal{B}^{2,0}_w$ be the orbit corresponding to w and $\mathcal{B}^2_w$ be its closure in $\mathcal{B}$.

Consider the conormal bundle $T^*_{\mathcal{B}^2_w}\mathcal{B}^2$ to the closure of a G-orbit in $\mathcal{B}^2$. This variety is smooth, but its closure in $T^*(\mathcal{B}^2)$ can become singular. We are interested in the normality of the closure for different elements $w \in W$. Geometrically, normality provides a measure of how singular a variety is, and the normality of this closure has consequences in representation theory.

The Grassmannian $\mathrm{Gr}(d,s)$ is the moduli space of d-dimensional subspaces in k^s. It is a smooth variety of dimension $d(s-d)$. When $s = n$, we write Gr_d for $\mathrm{Gr}(d,n)$. It is isomorphic to the homogeneous space G/P_d, where P_d is the parabolic subgroup of block upper triangular matrices with two blocks of size d and $n-d$. Since the Weyl group W_{P_d} of P_d is $S_d \times S_{n-d}$, B-orbits in G/P_d correspond to elements of $S_n/(S_d \times S_{n-d})$ (See [Bri04]).

Suppose $k+l \le n$, and let $\mathrm{pr} : \mathcal{B}^2 \to \mathrm{Gr}_k \times \mathrm{Gr}_l$ be the map forgetting everything except the k-dimensional subspace of the first flag and the l-dimensional subspace of the second flag. Considering the action of G on this space, as above, we can always standardize the first subspace to be $\langle e_1, \dots, e_k\rangle$. Thus G-orbits in $\mathrm{Gr}_k \times \mathrm{Gr}_l$ are parametrized by $W_{P_k}\backslash W/W_{P_l}$. These double cosets are of the form X_i for $0 \le i \le \min(k,l)$, where X_i contains the permutations where i of the first l elements are sent to the first k positions. This corresponds to the G-orbit of pairs of subspaces with exactly i-dimensional intersection.

Let D_m be the orbit of pairs of subspaces with m-dimensional intersection. The preimage of $\overline{D_m}$ along pr is the union of some G-orbits in $\mathcal{B}^2$, one of which is open and dense in $\mathrm{pr}^{-1}(\overline{D_m})$. This open orbit corresponds to the element of the Weyl group which is last in the Bruhat order among these orbits. We denote this element by v_m. Then since the closure of $\mathcal{B}^{2,0}_{v_m}$ is $\mathrm{pr}^{-1}(D_m)$, the closures of the conormals to $\mathcal{B}^{2,0}_{v_m}$ and $\mathrm{pr}^{-1}(D_m)$ are isomorphic. Thus it suffices to show that $\overline{T^*_{\mathrm{pr}^{-1}(D_m)}\mathcal{B}^2}$ is normal, which occurs when

$$\overline{T^*_{D_m}\mathrm{Gr}_k \times \mathrm{Gr}_l}$$

is normal.

An explicit description of v_m is as follows: Let s_i be the simple reflection transposing i and $i+1$ and w_0 the longest element of the Weyl group (w_0 is the permutation which reverses the order of the elements $1,\dots,n$). We denote $\sigma_j = s_{k-j+1}s_{k-j+2}\dots s_{n-l+m-j}$. Then

$$v_m = \sigma_1\sigma_2\dots\sigma_m w_0.$$

When $k+l=n$, we have $v_1 = s_k w_0$. $\mathcal{B}^2_{v_1}$ is the preimage of $\overline{D_1}$ along pr. The computation in this case was done by Roman Bezrukavnikov.

In Section 2, we provide an alternative argument and generalize to the case where $m=2$. It has been conjectured that the closure of the conormal to the orbit corresponding to $w \in S_n$ is normal if $w_0 w$ avoids the permutations 3412 and 4231. The $m=2$ case provides a counterexample to the converse of this conjecture, since it yields w such that $w_0 w$ does not avoid 3412. Given our reduction to the general case to a combinatorial question, we conjecture that the closure of the conormal on $\mathrm{Gr}_k \times \mathrm{Gr}_l$ is in fact normal for all values of m.

In Section 3, we prove the normality of a related variety, given by relaxing one of the conditions in the description of the closure of a conormal. Knutson and Sam describe a stratification of $G/B \times \mathrm{Mat}_n$ in [KS22], where one of the strata is $T^*(G/B)$. The geometric properties of this stratification are useful in studying this cotangent bundle. Knutson and Sam use Frobenius splittings to prove properties of the strata, including normality, and we give a more elementary proof of normality in a specific case.

Acknowledgements

This research was conducted through MIT SPUR 2025. I would like to thank my mentor Katia Bogdanova for her continuous support and guidance, as well as Roman Bezrukavnikov for suggesting this problem. We also thank Jonathan Bloom and Roman Bezrukavnikov for organizing SPUR and providing helpful insights throughout the program.

2. Conormal bundles on the Product of Grassmannians

For $k+l \leq n$, the G-orbits in $\mathrm{Gr}_k \times \mathrm{Gr}_l$ are the sets D_m consisting of the pairs of subspaces whose intersection is m-dimensional, for $0 \leq m \leq \min(k,l)$. When $m=0$, the closure of D_0 is the entire space, so the normality of the closure of the conormal follows immediately. We proceed for $m>0$.

Lemma 2.1. *(1) The cotangent space $T^*(\mathrm{Gr}_k \times \mathrm{Gr}_l)$ at the point (V_k, V_l) is identified with*

$$\mathrm{Hom}(V/V_k, V_k) \oplus \mathrm{Hom}(V/V_l, V_l).$$

(2) The fiber of the conormal sheaf $T^*_{D_m}(\mathrm{Gr}_k \times \mathrm{Gr}_l)$ at the point $(V_k, V_l) \in D_m$ is identified with

$$\mathrm{Hom}(V/(V_k+V_l), V_k \cap V_l).$$

Proof. (1) The tangent space to Gr_k at V_k is isomorphic to $\mathrm{Hom}(V_k, V/V_k)$ (See [GW20, Chapter 8.9]). Hence the cotangent space to Gr_k at this point is $\mathrm{Hom}(V_k, V/V_k)$, and the statement follows.

(2) For all $x \in D_m$, there is a short exact sequence

$$0 \to T(D_m)_x \to T(\mathrm{Gr}_k \times \mathrm{Gr}_l)_x \to T_{D_m}(\mathrm{Gr}_k \times \mathrm{Gr}_l)_x \to 0.$$

The middle term is the dual of the space in (1), where $x=(V_k,V_l)$. We compute $T(D_m)_x$. Consider the tangent space as a set of $k[\epsilon]$-valued points of D_m, where $\epsilon^2=0$. A point in this tangent space is identified with a pair of projective submodules $\mathcal{V}_k, \mathcal{V}_l \subset V \otimes k[\epsilon]$ of dimensions k and l with $\mathcal{V}_k \cap \mathcal{V}_l$ projective of dimension m, such that $V \otimes k[\epsilon]/\mathcal{V}_k$, $V \otimes k[\epsilon]/\mathcal{V}_l$, and $V \otimes k[\epsilon]/(\mathcal{V}_k \cap \mathcal{V}_l)$ are projective as well. If V_k and V_l are the subspaces of V associated to the point $x \in D_m$, $\mathcal{V}_k/\epsilon$ must be isomorphic to V_k, and similarly for V_l. Furthermore, the map

$$\mathcal{V}_k \cap \mathcal{V}_l \to \mathcal{V}_k/\epsilon \cap \mathcal{V}_l/\epsilon$$

must be surjective.

We note that $\mathcal{V}_k$ is the image of a $k[\epsilon]$-linear map $V_k \otimes k[\epsilon] \to V \otimes k[\epsilon] = V \oplus \epsilon V$, and likewise for $\mathcal{V}_l$. Given a map $w : V_k \to V$, there is a unique map

$$\tilde{w} : V_k \otimes k[\epsilon] \to V \oplus \epsilon V$$

such that $\tilde{w}|_{V_k} = \iota \oplus \epsilon w$, where $\iota : V_k \to V$ is the inclusion map. Then $\mathrm{Im}(\tilde{w})$ is some projective module $\mathcal{V}_k$. The same statement holds for V_l. Thus we have a map

$$\begin{aligned} \mathrm{Hom}(V_k, V) \oplus \mathrm{Hom}(V_l, V) &\to T(\mathrm{Gr}_k \times \mathrm{Gr}_l)_x, \\ (w_k, w_l) &\mapsto (\mathrm{Im}(\tilde{w}_k), \mathrm{Im}(\tilde{w}_l)). \end{aligned}$$

Its kernel is the pairs of maps (w_k, w_l) such that $w_k(V_k) \subset V_k$ and $w_l(V_l) \subset V_l$. Imposing the surjectivity condition on $\mathcal{V}_k \cap \mathcal{V}_l$, the tangent space $T(D_m)_x$ is identified with pairs $(f, f') \in \mathrm{Hom}(V_k, V/V_k) \oplus \mathrm{Hom}(V_l, V/V_l)$ which agree on $V_k \cap V_l$. The image of $V_k \cap V_l$ under f and f' must lie in

$$V/V_k \cap V/V_l = V/(V_k+V_l).$$

Now the quotient of $T(\mathrm{Gr}_k \times \mathrm{Gr}_l)_x$ by $T(D_m)_x$ can be identified with

$$\mathrm{Hom}(V_k \cap V_l, V/(V_k+V_l)).$$

The dual of this is the fiber of the conormal bundle at x. □

Let $O_m \subset \mathfrak{g}$ be the set of nilpotent matrices of rank m. Its closure in Mat_n is the set of nilpotent matrices of rank at most m.

Lemma 2.2. *Let $Y = T^*_{D_m}(\mathrm{Gr}_k \times \mathrm{Gr}_l)$. The closure $\overline{Y} \subset T^*(\mathrm{Gr}_k \times \mathrm{Gr}_l)$ is identified with the variety*

$$Z = \{(V_k, V_l, x) \in \mathrm{Gr}_k \times \mathrm{Gr}_l \times \overline{O} : \mathrm{Im}(x) \subset V_k, V_l \subset \mathrm{Ker}(x), \dim(V_k \cap V_l) \geq m\}.$$

Remark 2.3. k and l are always greater than or equal to m, and we may observe that they are also at most $n - m$. Thus the above conditions on V_k and V_l are valid.

Proof. Viewing the cotangent space as in Lemma 2.1, we may embed Y into $T^*(\mathrm{Gr}_k \times \mathrm{Gr}_l)$ so that (V_k, V_l, x) corresponds to the point (x, x) in the cotangent space at (V_k, V_l), where we can view x as a map $V/V_k \to V_k$ since $\mathrm{Im}(x) \subset V_k \subset \mathrm{Ker}(x)$. Now the conormal to D_m consists of the triples (V_k, V_l, x) where $\mathrm{Im}(x) \subset V_k, V_l \subset \mathrm{Ker}(x)$, $V_k \cap V_l$ has dimension m, and x is nilpotent of rank at most m by Lemma 2.1 (2).

We show that the closure of the above space is Z. Consider the maps from Y and Z to $\overline{O}$, each forgetting everything but x. For a given $x \in \overline{O}$ of rank m, the fiber over x in Y is the set of pairs of subspaces (W_{k-m}, W_{l-m}) in $\mathrm{Ker}(x)/\mathrm{Im}(x)$ of rank $k - m$ and $l - m$ respectively such that $W_{k-m} \cap W_{l-m} = 0$. The fiber over x in Z is the same set of pairs without the condition that the subspaces do not intersect. It suffices to prove that the first fiber is dense in the second.

It remains to show that the set X of pairs of subspaces (V_i, V_j) such that $V_i \cap V_j = 0$ is dense in $\mathrm{Gr}_i \times \mathrm{Gr}_j$ for $i + j \leq n$. Suppose that U is an open subset of $\mathrm{Gr}_i \times \mathrm{Gr}_j$ which does not intersect X. Let $(W_i, W_j) \in U$. $W_i \cap W_j$ is spanned by some basis $v_1, \ldots, v_m$. There is some $u \in V_n$ such that u is not in W_i or W_j. Extending $v_1, \ldots, v_m$ to a basis for W_i, it is possible to replace each v_k with $v_k + c_k x$ for some constant c_k such that the new subspace has trivial intersection with W_j and is contained in U. □

Lemma 2.4. *Let $f : X \to Y$ be a morphism of ringed spaces, and suppose that we have a complex of $\mathcal{O}_X$-modules*

$$0 \to M^{-n} \to \ldots \to M^{-1} \to M^0 \to 0.$$

If we have

$$R^i f_*(M^{-i}) = 0$$

for all $0 < i < n$, then

$$H^0(f_* M^\bullet) \to H^0(Rf_* M^\bullet)$$

is surjective.

Proof. There is a spectral sequence E such that

$$E_1^{p,q} = R^q f_* M^p$$

and the sequence converges to $H^{p+q}(Rf_* M^\bullet)$. On the E_1 page, there are maps

$$E_1^{-1,0} \to E_1^{0,0} \to E_1^{1,0},$$

which are the maps

$$f_* M^{-1} \to f_* M^0 \to 0.$$

Thus $E_2^{0,0} = H^0(f_* M^\bullet)$. For $E_2^{0,0} \to H^0(Rf_* M^\bullet)$ to be surjective, it suffices for $E_1^{-i,i} = R^i \pi_* M^{-i}$ to vanish for all $i > 0$. □

Lemma 2.5. *Let $f : X \to Y$ be a morphism of ringed spaces, and suppose that we have a complex of $\mathcal{O}_X$-modules*

$$0 \to M_{-n} \to \ldots \to M_{-1} \to M_0 \to 0.$$

If we have

$$R^{j+1} f_*(H^{-j}(M^\bullet)) = 0$$

for all $j > 0$, then

$$H^0(Rf_* M^\bullet) \to f_*(H^0(M^\bullet))$$

is surjective.

Proof. There is a spectral sequence converging to $H^{p+q}(Rf_*M^\bullet)$ with

$$E_2^{p,q} = R^p f_* H^q(M^\bullet).$$

Thus we have $E_2^{0,0} = f_*(H^0(M^\bullet))$, and there are maps

$$E_2^{-2,1} \to E_2^{0,0} \to E_2^{2,-1}.$$

The leftmost term is always 0, and the condition that $R^2 f_*(H^{-1}(M^\bullet)) = 0$ implies that the rightmost term is 0. Repeating the above on every page, the given condition ensures that there are maps

$$0 \to E_r^{0,0} \to 0.$$

Since $E_r^{0,0}$ converges to $H^0(Rf_*M^\bullet)$, the desired statement follows. □

Theorem 2.6. *The variety Z is normal for $m = 1$ and $m = 2$.*

Proof. We construct the following variety $\tilde{Z}$:

$$\tilde{Z} = \{(L, V_k, V_l, H, x) \in \mathrm{Gr}_m \times \mathrm{Gr}_k \times \mathrm{Gr}_l \times \mathrm{Gr}_{n-m} \times \overline{O}_m : \mathrm{Im}(x) \subset L \subset V_k, V_l \subset H \subset \mathrm{Ker}(x)\}.$$

Let

$$X = \{(L, V_k, V_l, H) \in \mathrm{Gr}_m \times \mathrm{Gr}_k \times \mathrm{Gr}_l \times \mathrm{Gr}_{n-m} : L \subset V_k, V_l \subset H\}$$

with the projection map from $\tilde{Z}$ forgetting x. Then $\tilde{Z}$ is the total space of the vector bundle $\mathcal{H}om(L, V/H)$ on X with fibers isomorphic to $\mathrm{Hom}(L, V/H)$. Thus $\tilde{Z}$ is smooth. We have a map $p : \tilde{Z} \to Z$ sending (L, V_k, V_l, H, x) to (V_k, V_l, x). This map is proper.

Thus using the Stein factorization of p through the normalization of Z in $\tilde{Z}$, to show that Z is normal, it suffices to show that $\mathcal{O}_Z \simeq p_*\mathcal{O}_{\tilde{Z}}$. The map $\tilde{O}_Z \to p_*\mathcal{O}_{\tilde{Z}}$ is injective. We now show that it is also surjective.

We define two more spaces

$$\begin{aligned} \tilde{Q} &= \{(L, H, x) \in \mathrm{Gr}_m \times \mathrm{Gr}_{n-m} \times \overline{O}_m : \mathrm{Im}(x) \subset L \subset H \subset \mathrm{Ker}(x)\}, \\ Q &= \{(L, H) \in \mathrm{Gr}_m \times \mathrm{Gr}_{n-m} : L \subset H\}. \end{aligned}$$

We now have a commutative diagram

$$\begin{array}{ccccc} X & \xhookrightarrow{s} & Q \times \mathrm{Gr}_k \times \mathrm{Gr}_l & & \\ \uparrow & & \uparrow{\scriptstyle\phi} & \searrow & \\ \tilde{Z} & \xhookrightarrow{j} & \tilde{Q} \times \mathrm{Gr}_k \times \mathrm{Gr}_l & & \mathrm{Gr}_k \times \mathrm{Gr}_l \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle\varpi} & \nearrow & \\ Z & \xhookrightarrow{g} & \overline{O}_m \times \mathrm{Gr}_k \times \mathrm{Gr}_l & & \end{array} \tag{2.7}$$

where the upper square is Cartesian. Since we wish to show $\tilde{O}_Z \to p_*\mathcal{O}_{\tilde{Z}}$ is surjective, and g is a closed embedding, it suffices to show that $g_*\tilde{O}_Z \to g_*p_*\mathcal{O}_{\tilde{Z}}$ is surjective. Noting that $\overline{O}_m \times \mathrm{Gr}_k \times \mathrm{Gr}_l$ is normal (see [KP79] for the normality of $\overline{O}_m$) and that ϖ is a proper morphism with connected nonempty fibers, the pushforward of the structure sheaf along ϖ is the structure sheaf on $\overline{O}_m \times \mathrm{Gr}_k \times \mathrm{Gr}_l$ by Stein factorization. Using the lower commutative square in the diagram, it suffices to show that

$$\varpi_*\mathcal{O}_{\tilde{Q} \times \mathrm{Gr}_k \times \mathrm{Gr}_l} \to \varpi_* j_* \mathcal{O}_{\tilde{Z}}$$

is surjective. We proceed by constructing a complex of sheaves on $\tilde{Q} \times \mathrm{Gr}_k \times \mathrm{Gr}_l$ ending with the map

$$\mathcal{O}_{\tilde{Q} \times \mathrm{Gr}_k \times \mathrm{Gr}_l} \to j_*\mathcal{O}_{\tilde{Z}}$$

and considering the pushforward of this complex along ϖ.

First consider the variety $X_U \subset Q \times \mathrm{Gr}_k$ of triples (L, U, H) such that $L \subset U \subset H$. There is a projection $X_U \to \mathrm{Gr}_m \times \mathrm{Gr}_k$, whose image S is the set of (L, U) with $L \subset U$. There is a section of the vector bundle $E_{U,1} = \mathcal{H}om(L, V/U)$ on $\mathrm{Gr}_m \times \mathrm{Gr}_k$ whose zero locus is S. The rank of this bundle is $r_1 = m(n-k)$, which is equal to the codimension of S in $\mathrm{Gr}_m \times \mathrm{Gr}_k$. Thus we have a Koszul resolution

$$0 \to \wedge^{r_1} E_{U,1}^\vee \to \ldots \to E_{U,1}^\vee \to \mathcal{O}_{\mathrm{Gr}_m \times \mathrm{Gr}_k} \to 0,$$

where the rightmost cohomology H^0 is $\mathcal{O}_S$ and the complex is exact everywhere else.

Similarly, we obtain a Koszul resolution for the set of (U, H) with $U \subset H$ in $\mathrm{Gr}_k \times \mathrm{Gr}_{n-m}$ using the vector bundle $E_{U,2} = \mathcal{H}om(U, V/H)$. Tensoring the two resolutions, we obtain a complex $M_U^\bullet$ with $H^0(M_U^\bullet) = \mathcal{O}_{X_U}$. However, since this is a Koszul complex of a vector bundle $E_{U,1} \oplus E_{U,2}$ of rank nm and the codimension of X_U in $Q \times \mathrm{Gr}_k$ is $m(n-m)$, the complex also has cohomology in degrees up to $-m^2$.

We repeat the above replacing Gr_k with Gr_l and tensor the resulting complexes to obtain a complex

$$M^\bullet = 0 \to \wedge^r E^\vee \to \ldots \to E^\vee \to \mathcal{O}_{Q \times \mathrm{Gr}_k \times \mathrm{Gr}_l} \to 0$$

with $H^0(M) = s_*\mathcal{O}(X)$. Pulling back this complex along ϕ results in a complex of sheaves on $\tilde{Q} \times \mathrm{Gr}_k \times \mathrm{Gr}_l$ with the cohomology at degree 0 being $j_*\mathcal{O}_{\tilde{Z}}$. It now suffices to show that further taking the pushforward of this complex along ϖ preserves H^0.

Suppose that the conditions of Lemma 2.4 and Lemma 2.5 hold for $\phi^* M^\bullet$ and ϖ. Then we have that

$$H^0(\varpi_* \phi^* M^\bullet) \to \varpi_*(H^0(\phi^* M^\bullet))$$

is surjective. Since the pushforward of the structure sheaf along ϖ is the structure sheaf on $\overline{O}_m \times \mathrm{Gr}_k \times \mathrm{Gr}_l$, this yields our desired surjectivity. Thus Propositions 2.8 and 2.9 suffice to prove the claim. □

Proposition 2.8. *Let $M^\bullet$ and ϖ be defined as above. For all m, the following higher pushforwards vanish:*

$$R^i \varpi_*(\phi^* M^{-i}) = 0$$

Proof. Recall the maps defined as in 2.7. We first consider

$$R^i \varpi_*(\phi^* M^{-i}).$$

The complex $M^\bullet$ is obtained by taking the tensor product of four Koszul complexes. M^{-i} is the direct sum of terms of the form

$$\wedge^{i_1}(\mathcal{L}_m \boxtimes Q_k^\vee) \otimes \wedge^{i_2}(\mathcal{L}_k \boxtimes Q_{n-m}^\vee) \otimes \wedge^{i_3}(\mathcal{L}_m \boxtimes Q_l^\vee) \otimes \wedge^{i_4}(\mathcal{L}_l \boxtimes Q_{n-m}^\vee),$$

where $\mathcal{L}_j$ and Q_j denote the tautological bundle and the tautological quotient bundle on Gr_j and $i_1 + i_2 + i_3 + i_4 = i$. Furthermore, since ϖ acts as the identity on Gr_k and Gr_l, the bundles defined on these Grassmannians do not affect the higher pushforwards, and we may replace $(\mathcal{L}_m \boxtimes Q_k^\vee)$ by $({\mathcal{L}_m}^{\oplus \operatorname{rank} Q_k^\vee})$. We also rewrite the sheaf in terms of Schur functors, using the fact ([Kap85, Lemma 0.5]) that

$$\wedge^i(A \otimes B) = \bigoplus_{|\lambda| = i} \Sigma^\lambda A \otimes \Sigma^{\lambda^*} B.$$

Thus it suffices to show that

$$R^i \varpi_*(\Sigma^{\lambda_1} \mathcal{L}_m \otimes \Sigma^{\lambda_3} \mathcal{L}_m \otimes \Sigma^{\lambda_2} Q_{n-m}^\vee \otimes \Sigma^{\lambda_4} Q_{n-m}^\vee) = 0$$

for all $|\lambda_j| = i_j$. Using the Littlewood-Richardson rule for the decomposition of products of Schur functors (described in [FH91, Appendix A.1]), we can further decompose the above term into a direct sum of products

$$\Sigma^\lambda \mathcal{L}_m \otimes \Sigma^\mu Q_{n-m}^\vee,$$

where $|\lambda| + |\mu| = i + 1$. Since $\overline{O}_m$ is affine, we can further pushforward along the map sending $\overline{O}_m$ to a point, and it suffices to show that

$$H^i(\tilde{Q}, \Sigma^\lambda \mathcal{L}_m \otimes \Sigma^\mu Q_{n-m}^\vee) = 0.$$

From the rightmost component of 2.7 and the projection formula applied to ϕ, the above is equal to

$$H^i(Q, \Sigma^\lambda \mathcal{L}_m \otimes \Sigma^\mu Q_{n-m}^\vee \otimes f_* \mathcal{O}_{\tilde{Q}})$$

where $f : \tilde{Q} \to Q$. Since $\tilde{Q}$ is the total space of the vector bundle $\mathcal{L}_m \otimes Q_{n-m}^\vee$ on Q,

$$f_* \mathcal{O}_{\tilde{Q}} = \mathrm{Sym}^*(\mathcal{L}_m^\vee \otimes Q_{n-m}).$$

We now observe that it is sufficient to prove that the cohomology of

$$\Sigma^\lambda \mathcal{L}_m \otimes \Sigma^\mu Q_{n-m}^\vee$$

on Q lies in degree strictly less than $|\lambda| + |\mu|$. This is because tensoring this bundle with $\mathrm{Sym}^j(\mathcal{L}_m^\vee \otimes Q_{n-m})$ is equivalent to decreasing $|\lambda|$ and $|\mu|$ by j.

To compute the above cohomology on Q, we write $\Sigma^\lambda \mathcal{L}_m \otimes \Sigma^\mu Q_{n-m}^\vee$ as a pushforward of a line bundle on the flag variety G/B. The following is adapted from Kapranov's argument in [Kap85] for analogous bundles on one Grassmannian. Let F_1 be the space of flags of dimensions $(1, 2, \ldots, m)$ inside V^n. This is the total flag space of $\mathcal{L}_m$. Similarly, let F_2 be the total flag space of Q_{n-m} and F_3 the space of flags covering the remaining dimensions $(m+1, \ldots, n-m-1)$. G/B is the product $F_1 \times F_2 \times F_3$. For nonincreasing integer sequences $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_m)$, we have line bundles $\mathcal{O}(-\lambda_m, \ldots, -\lambda_1)$ and $\mathcal{O}(\mu_1, \ldots, \mu_m)$ on F_1 and F_2. Taking the tensor product of these bundles with the trivial bundle on F_3, we obtain the line bundle

$$\mathcal{O}(-\lambda_m, \ldots, -\lambda_1, 0, \ldots, 0, \mu_1, \ldots, \mu_m)$$

on G/B whose pushforward to Q is $\Sigma^\lambda \mathcal{L}_m \otimes \Sigma^\mu Q_{n-m}^\vee$. Thus it suffices to compute the cohomology of the above line bundle on G/B, for which we use the Borel-Weil-Bott Theorem. The theorem states that the line bundle $\mathcal{O}(a_1, \ldots, a_n)$ has cohomology only in degree equal to the length of the permutation σ which puts $(a_1, \ldots, a_n) + \rho$ in nonincreasing order, where $\rho = (n, n-1, \ldots, 1)$. Furthermore, if $(a_1, \ldots, a_n) + \rho$ has repeated terms, the bundle has no cohomology.

Let

$$\chi = (-\lambda_m, \ldots, -\lambda_1, 0, \ldots, 0, \mu_1, \ldots, \mu_m).$$

We proceed by induction on $|\lambda| + |\mu|$. When this is equal to 1, so that either $\lambda_1 = 1$ or $\mu_1 = 1$ and all other terms are 0, there is at most one term in $\chi + \rho$ that is not in decreasing order with the other terms. When either $|\lambda|$ or $|\mu|$ is increased by 1, the length of σ increases by at most 1. Thus $\mathcal{O}(\chi)$ has cohomology in degree at most $|\lambda| + |\mu| - 1$, which is the desired condition.

□

Proposition 2.9. *With $M^\bullet$ and ϖ defined as above for $m \leq 2$, we have*

$$R^{i+1}\varpi_*(H^{-i}(\phi^* M^\bullet)) = 0.$$

Proof. We first compute $H^{-i}(\phi^* M_U^\bullet)$. Recalling that $M_U^\bullet$ is the tensor product of the Koszul complexes using the vector bundles $E_{U,1}$ and $E_{U,2}$, $M_U^\bullet$ is the Koszul complex of a bundle obtained by restricting $E_{U,2}$ to the subset Y of $Q \times \mathrm{Gr}_k$ where $L \subset U$. Let this be $E_U|_Y$, where $E_U = \mathcal{L}_k^\vee \otimes Q_{n-m}$. Then the complex is

$$\ldots \to \wedge^2 E_U|_Y^\vee \to E_U|_Y^\vee \to \mathcal{O}_Y \to 0.$$

We define another bundle

$$E'_U = (\mathcal{L}_k/\mathcal{L}_m)^\vee \otimes Q_{n-m},$$

where the first component has fibers isomorphic to U/L. We then have a short exact sequence

$$0 \to E'_U \to E_U|_Y \to \mathcal{M} \to 0,$$

where $\mathcal{M} = \mathcal{L}_m^\vee \otimes Q_{n-m}$. It follows that

$$H^{-i}(M_U^\bullet) = \wedge^i(\mathcal{M}|_{X_U}),$$

where $X_U \subset Q \times \mathrm{Gr}_k$ is the set of (L, U, H) with $L \subset U \subset H$. Since the same statement is true for Gr_l, we have

$$H^{-i}(\phi^* M^\bullet) = \bigoplus \wedge^j(\mathcal{M}|_X) \otimes \wedge^{i-j}(\mathcal{M}|_X).$$

Now to prove the required vanishing, we fix a point $(U, W) \in \mathrm{Gr}_k \times \mathrm{Gr}_l$. Now the condition $L \subset U, W \subset H$ on Q is equivalent to picking L, H so that $L \subset U \cap W$ and $U + W \subset H$. Thus we can define the product of two Grassmannians

$$P = \mathrm{Gr}(L, U \cap W) \times \mathrm{Gr}(H/(U + W), V/(U + W)).$$

and consider a commutative diagram

$$\begin{array}{ccc}
P & \hookrightarrow & X \\
\downarrow & & \downarrow \\
Q \times \{U\} \times \{W\} & \hookrightarrow & Q \times \mathrm{Gr}_k \times \mathrm{Gr}_l \\
\downarrow & & \downarrow \\
\{U\} \times \{W\} & \hookrightarrow & \mathrm{Gr}_k \times \mathrm{Gr}_l
\end{array}$$

Let $\mathcal{L}_1$ be the tautological bundle on $\mathrm{Gr}(L, U \cap W)$ and Q_2 the quotient bundle on $\mathrm{Gr}(H/(U + W), V/(U + W))$. It suffices to show that

$$H^{|\alpha|+1}(P, \Sigma^\alpha \mathcal{L}_1 \boxtimes \Sigma^\beta Q_2^\vee) = 0$$

for all α, β obtained from the decomposition of

$$\wedge^j(\mathcal{L}_1 \boxtimes Q_2^\vee) \otimes \wedge^{i-j}(\mathcal{L}_1 \boxtimes Q_2^\vee).$$

Concretely, α results from combining two partitions $\lambda \vdash j$ and $\mu \vdash (i - j)$ following the Littlewood-Richardson rule, while β comes from combining λ^* and μ^*.

In particular, $|\alpha| = |\beta|$, and they must be Young diagrams inside a $m \times 2m$ rectangle since they come from tensor products of partitions fitting inside an $m \times m$ rectangle–otherwise the corresponding Schur functors of $\mathcal{L}_1$ or Q_2 are zero. Then rewriting this as products of cohomologies of $\Sigma^\alpha \mathcal{L}_1$ and $\Sigma^\beta Q_2$ on their respective Grassmannians using the Kunneth formula, for $m = 1$ and $m = 2$, we can verify by checking all possible cases that the desired vanishing holds:

The $m = 1$ case is straightforward since there are only two possibilities for α and β. For $m = 2$, we consider the possibilities for a partition α in a 2×4 rectangle: If $\alpha = (a, 0)$, we first consider the cohomology of $\Sigma^\alpha \mathcal{L}_1$. Let $r = \dim U \cap W$. Then $(-\alpha, 0, \dots, 0) + \rho = (r, r - 1 - a, r - 2, \dots, 2, 1)$. Now for there to be no repetitions in this sequence, we must have $r \leq a + 1$. Furthermore, β has at least $\frac{a}{2}$ rows in order to come from the transposed partitions that were combined to make α. $a = 1$ is straightforward. When $\alpha = 2$, we must have $r \leq 3$, so the length of the permutation putting $\alpha + \rho$ in order is at most 1, and our choices for β are $(2, 0)$ and $(1, 1)$, both of which do not cause problems. Similarly, we check the possibilities for $a = 3$ and $a = 4$.

Repeating this process for $\alpha = (a, 1), (a, 2), (a, 3)$, or $(4, 4)$, we see that the sum of the lengths of the permutations putting $(-\alpha, 0, \dots, 0) + \rho$ and $(0, \dots, 0, \beta)$ in decreasing order is never equal to $|\alpha| + 1$, so the desired cohomology vanishing holds. □

Remark 2.10. The cohomology vanishing condition in the previous proposition can be straightforwardly checked for other special cases where $m = \min(k, l)$ or $m = \min(k, l) - 1$, so our variety is also normal in these cases.

A slightly different approach to the second vanishing condition above is as follows: It suffices to check that

$$H^{i+1}(P, \wedge^i(\mathcal{L}_1 \boxtimes Q_2^\vee \oplus \mathcal{L}_1 \boxtimes Q_2^\vee)) = 0$$

for $i > 0$. We have

$$\mathcal{L}_1 \boxtimes Q_2^\vee \oplus \mathcal{L}_1 \boxtimes Q_2^\vee = (\mathcal{L}_1 \boxtimes Q_2^\vee) \otimes \underline{k}^{\oplus 2},$$

where $\underline{k}^{\oplus 2}$ is the trivial vector bundle of rank 2. Using the Schur functor decomposition of the exterior power, we have

$$\wedge^i(\mathcal{L}_1 \boxtimes Q_2^\vee \oplus \mathcal{L}_1 \boxtimes Q_2^\vee) = \bigoplus \Sigma^\lambda(\mathcal{L}_1 \boxtimes Q_2^\vee) \otimes \Sigma^{\lambda^*}(\underline{k}^{\oplus 2}),$$

where $|\lambda| = i$ and λ has rows of length at most 2 (otherwise $\Sigma^{\lambda^*}\underline{k}^{\oplus 2} = 0$). We can decompose Schur functors of tensor products by the following:

$$\Sigma^\lambda(V \otimes W) = \bigoplus (\Sigma^{\mu_1} V \otimes \Sigma^{\mu_2} W)^{\oplus c_{\mu_1,\mu_2}},$$

where c_{μ_1,μ_2} is nonzero if and only if V_λ has nonzero multiplicity in $V_{\mu_1} \otimes V_{\mu_2}$, where V_λ denotes the irreducible representation of S_i corresponding to $|\lambda|$ ([FH91, Exercise 6.11]). Thus our question for general m reduces to the following statement:

Conjecture 2.11. *Let μ_1, μ_2 be Young diagrams with $|\mu_1| = |\mu_2| = i$ and $V_\lambda \hookrightarrow V_{\mu_1} \otimes V_{\mu_2}$. If σ_1 and σ_2 are the permutations putting $(-\mu_1, 0, \ldots, 0) + \rho$ and $(-\mu_2, 0, \ldots, 0) + \rho$ into decreasing order and $l(\sigma_1) + l(\sigma_2) = i + 1$, then $\lambda_1 > 2$.*

The above can be verified for $m \leq 2$, again by checking all possible cases for μ_1 and μ_2.

3. Normality of Knutson-Sam varieties

Recall from the previous section that the closure of the conormal bundle to an orbit D_m can be described as

$$Z = \{(V_k, V_l, x) \in \mathrm{Gr}_k \times \mathrm{Gr}_l \times \overline{O} : \mathrm{Im}(x) \subset V_k, V_l \subset \mathrm{Ker}(x), \dim(V_k \cap V_l) \geq m\}.$$

We define a new variety by removing the condition on $\mathrm{Ker}(x)$:

$$Y = \{(V_k, V_l, x) \in \mathrm{Gr}_k \times \mathrm{Gr}_l \times \overline{O} : \mathrm{Im}(x) \subset V_k, V_l, \dim(V_k \cap V_l) \geq m\}.$$

The normality of Y is a consequence of the normality of the strata on $\mathcal{B} \times \mathrm{Mat}_n$ described by Knutson and Sam in [KS22]. The cotangent space to the flag variety $\mathcal{B}$ at a flag $\{F_i\}$ is given by the set of matrices N such that $NF_i \subset F_{i-1}$ for all i. Thus the cotangent bundle $T^*\mathcal{B}$ is identified with a stratum of $\mathcal{B} \times \mathrm{Mat}_n$, and studying this stratification yields results about $T^*\mathcal{B}$.

Y is the closure of the stratum of $\mathcal{B} \times \mathrm{Mat}_n$ given by the following conditions as in [KS22, Proposition 1.1], with $V_k = E_i$ and $V_l = F^j$:

$$\mathrm{rk}(E^n = \mathbb{A}^n \to \mathbb{A}^n \to \mathbb{A}^n/E_i) = 0$$
$$\mathrm{rk}(F^n = \mathbb{A}^n \to \mathbb{A}^n \to \mathbb{A}^n/F^j) = 0$$
$$\mathrm{rk}(E^n = \mathbb{A}^n \to \mathbb{A}^n \to \mathbb{A}^n/E_0 = \mathbb{A}^n) = m$$
$$\mathrm{rk}(\mathbb{A}^n \to \mathbb{A}^n \to \mathbb{A}^n/(E_i + F^j)) - \dim(E_i \cap F^j) = -m.$$

We give an alternate approach for proving the normality of Y through a similar method to the previous section, using cohomologies of sheaves appearing in a Koszul complex.

Lemma 3.1. *Let $f : X \to Y$ be a morphism of ringed spaces, and suppose that we have an exact sequence of $\mathcal{O}_X$-modules*

$$0 \to M_0 \to \ldots \to M_{n-1} \to M_n \to 0.$$

If we have

$$R^i f_* M_{n-i-1} = 0$$

for all $i > 0$, the map

$$f_* M_{n-1} \to f_* M_n$$

is surjective.

Proof. There is a spectral sequence E such that

$$E_1^{p,q} = R^q f_* M^p$$

and the sequence converges to $H^{p+q}(Rf_* M^\bullet)$. On the E_1 page, there are maps

$$E_1^{-1,0} \to E_1^{0,0} \to E_1^{1,0},$$

which are the maps

$$f_* M^{-1} \to f_* M^0 \to 0.$$

Thus $E_2^{0,0} = H^0(f_* M^\bullet)$. On the E_2 page, there are maps

$$E_2^{-2,1} \to E_2^{0,0} \to E_2^{2,-1}.$$

The last term above is always 0. Furthermore, the condition that $Rf_* M^{-2} = 0$ implies that $E_1^{-2,1} = 0$ and therefore that $E_2^{-2,1} = 0$. Repeating the above for higher pages, we have

$$E_r^{-r,r-1} \to E_r^{0,0} \to E_r^{r,-r+1}.$$

The rightmost term is always 0, and the leftmost term is 0 if $R^{r-1} f_* M^{-r} = 0$. Now observing that $E_r^{0,0}$ converges to $H^0(Rf_* M^\bullet)$, $E_2^{0,0}$ must also be $H^0(Rf_* M^\bullet)$. $\square$

Proposition 3.2. *The variety Y is normal for all $m \geq 1$.*

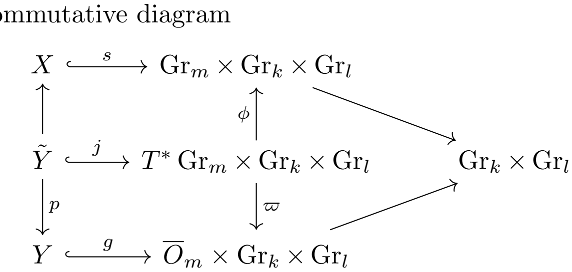
Proof. We construct a resolution of singularities $\tilde{Y} \to Y$ as follows: Let $\tilde{Y} \subset \mathrm{Gr}_k \times \mathrm{Gr}_l \times \mathrm{Gr}_m \times \mathfrak{g}$ be the set of points (V_k, V_l, L, x) such that $L \subset V_k \cap V_l$ and $\mathrm{Im}(x) \subset L \subset \mathrm{Ker}(x)$. $\tilde{Y} \simeq T^* \mathrm{Gr}_m \times_{\mathrm{Gr}_m} X$ for $X \subset \mathrm{Gr}_k \times \mathrm{Gr}_l \times \mathrm{Gr}_m$ being the set of (V_k, V_l, L) such that $L \subset V_k \cap V_l$. Thus $\tilde{Y}$ is smooth, and the map $p : \tilde{Y} \to Y$ forgetting L is proper and birational.

We now consider the Stein factorization of p through the normalization Y' of Y in $\tilde{Y}$. Since $\tilde{Y}$ is smooth, Y' is normal. Thus it suffices to show that $\mathcal{O}_Y \simeq p_* \mathcal{O}_{\tilde{Y}}$, since then $Y' = \underline{\mathrm{Spec}}_Y(\mathcal{O}_Y) = Y$. Since p is surjective, the map $\mathcal{O}_Y \to p_* \mathcal{O}_{\tilde{Y}}$ is injective. We show that it is also surjective.

Let $g : Y \to \overline{O} \times \mathrm{Gr}_k \times \mathrm{Gr}_l$ be the (closed) embedding. It suffices to show that

$$g_* \mathcal{O}_Y \to g_* p_* \mathcal{O}_{\tilde{Y}}$$

is surjective. We have a commutative diagram



where ϖ acts as the moment map $T^* \operatorname{Gr}_m \to \overline{O}_m$ and the identity on the Grassmannians. Using the commutative square at the bottom of the diagram, since ϖ also has connected fibers, if

$$\varpi_* \mathcal{O}_{T^* \operatorname{Gr}_m \times \operatorname{Gr}_k \times \operatorname{Gr}_l} \to \varpi_* j_* \mathcal{O}_{\tilde{Y}} \tag{3.3}$$

is surjective, the desired map is surjective.

Let $\mathcal{L}$ be the tautological bundle on Gr_m. Consider the vector bundle

$$E = \mathcal{L}^\vee \boxtimes Q_k \boxtimes \mathcal{O} \oplus \mathcal{L}^\vee \boxtimes \mathcal{O} \boxtimes Q_l$$

on $\operatorname{Gr}_m \times \operatorname{Gr}_k \times \operatorname{Gr}_l$, where Q_k and Q_l are the tautological quotient bundles on the respective Grassmannians.

First considering the bundle $\mathcal{L}^\vee \boxtimes Q_k$ on $\operatorname{Gr}_m \times \operatorname{Gr}_k$, we have maps

$$\mathcal{L} \boxtimes \mathcal{O} \to V^n \otimes \mathcal{O} \boxtimes \mathcal{O} \to Q_k \boxtimes \mathcal{O},$$

where V^n is the trivial bundle. Since sections of $\mathcal{L}$ are points $v \in V_m$ for m-dimensional subspaces V_m, the zeros of the composed map $\mathcal{L} \boxtimes \mathcal{O} \to Q_k \boxtimes \mathcal{O}$ are the points where $V_m \subset V_k$. Then we have a map $\mathcal{O} \boxtimes \mathcal{O} \to \mathcal{L}^\vee \boxtimes Q_k$ with the same zero locus. Repeating the above for the second summand of E, we obtain a section of E whose zero locus is X.

The dimension of $\operatorname{Gr}_m \times \operatorname{Gr}_k \times \operatorname{Gr}_l$ is $m(n-m) + k(n-k) + l(n-l)$, and the dimension of X is $m(n-m) + (k-m)(n-k) + (l-m)(n-l)$. Thus $\operatorname{codim}_{\operatorname{Gr}_m \times \operatorname{Gr}_k \times \operatorname{Gr}_l} X = m(2n-k-l)$, which is also the rank of E. Hence there exists a Koszul resolution

$$0 \to \wedge^r E^* \to \ldots \to E^* \to \mathcal{O}_{\operatorname{Gr}_m \times \operatorname{Gr}_k \times \operatorname{Gr}_l} \to s_* \mathcal{O}_X \to 0.$$

We pull back this resolution along ϕ to obtain a resolution for $j_* \mathcal{O}_{\tilde{Y}}$, with terms

$$M_i = \wedge^{r-i} \phi^*(E^*)$$

for $0 \le i \le r$ and $M_{r+1} = j_* \mathcal{O}_{\tilde{Y}}$. Gr_m and $T^* \operatorname{Gr}_m$ are smooth, and the map between them has fibers of constant dimension. Thus ϕ is flat, and the pullback along ϕ is exact. We wish to show that the last map in the resolution is still surjective under a pushforward along ϖ. We can further pushforward along the map projecting $\overline{O}_m \times \operatorname{Gr}_k \times \operatorname{Gr}_l \to \operatorname{Gr}_k \times \operatorname{Gr}_l$, and we note that this is the same as pushing along ϕ and the projection from $\operatorname{Gr}_m \times \operatorname{Gr}_k \times \operatorname{Gr}_l \to \operatorname{Gr}_k \times \operatorname{Gr}_l$. By Lemma 3.1, it suffices to show that

$$R^i \phi_*(\wedge^i E^*) = 0$$

for all $i > 0$. The pushforward of the structure sheaf of $T^* \operatorname{Gr}_m$ to Gr_m is $\operatorname{Sym}^*(T \operatorname{Gr})$. Hence by the projection formula, the question now reduces to showing that the following higher pushforwards vanish for all $i > 0$:

$$R^i \phi_*(\wedge^i E^* \otimes \operatorname{Sym}^*(T \operatorname{Gr})).$$

Now further pushing $\operatorname{Gr}_m \to \mathrm{pt}$, it suffices to show that

$$H^i(\wedge^i E^* \otimes \operatorname{Sym}^*(T \operatorname{Gr})) = 0.$$

Writing $E^* = \mathcal{L} \otimes \mathcal{F}$ and $T \operatorname{Gr} = \mathcal{L}^\vee \otimes Q$, the above tensor product can be expressed using Schur functors:

$$\wedge^i E^* \otimes \operatorname{Sym}^j(T \operatorname{Gr}) = \bigoplus_{|\lambda|=i, |\mu|=j} \Sigma^\lambda \mathcal{L} \otimes \Sigma^{\lambda^*} \mathcal{F} \otimes \Sigma^\mu \mathcal{L}^\vee \otimes \Sigma^\mu Q.$$

Since $\mathcal{F}$ is preserved under all pushforwards and all higher pushforwards of $\mathcal{F}$ vanish, it suffices to have the vanishing for $\Sigma^\lambda \mathcal{L} \otimes \Sigma^\mu \mathcal{L}^\vee \otimes \Sigma^\mu Q$. The following lemma finishes the proof. □

Lemma 3.4. *Let $\mathcal{L}$ be the tautological bundle on Gr_m and Q the quotient bundle. For nonincreasing nonnegative integer sequences λ, μ with $|\lambda| = i$,*

$$H^i(\operatorname{Gr}_m, \Sigma^\lambda \mathcal{L} \otimes \Sigma^\mu \mathcal{L}^\vee \otimes \Sigma^\mu Q) = 0$$

for all $i > 0$.

Proof. Kapranov showed in [Kap85] that the cohomology of $\Sigma^\lambda \mathcal{L}^\vee \otimes \Sigma^\mu Q^\vee$ on Gr_m is equal to the cohomology of the line bundle $\mathcal{O}(\lambda, \mu)$. By the Borel-Weil-Bott Theorem, this line bundle has cohomology in degree only equal to the length of σ, where σ is a permutation that puts

$$\chi + \rho = (\lambda, \mu) + (n, n-1, \ldots, 1)$$

into decreasing order. If there are repetitions in $\chi + \rho$, the bundle has no cohomology. This condition only works in our favor, so we will proceed while allowing repetitions.

First consider the case where $\mu = 0$. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ (λ can have at most m positive rows since $\mathcal{L}$ has rank m). Then we have

$$\Sigma^\lambda \mathcal{L} = \Sigma^{-\lambda} \mathcal{L}^\vee,$$

where $-\lambda$ denotes the sequence given by multiplying all terms of λ by -1 and reversing the order. Thus we obtain the sequence

$$\chi + \rho = (n - \lambda_m, \ldots, n - m + 1 - \lambda_1, n - m, \ldots, 1).$$

Suppose that the t-th term causes i_t inversions. If $i_t > 0$,

$$(\chi + \rho)_t = n - m + t - \lambda_t \le n - m - i_t.$$

Then $\lambda_t \ge t + i_t$. Summing this over all t where $i_t > 0$, we have

$$|\lambda| \ge \sum t + \sum i_t.$$

$\sum i_t$ is the total number of inversions, and $\sum t > 0$ (otherwise, there are no inversions and the bundle has cohomology only at degree 0). Thus we have $|\lambda|$ strictly greater than the number of inversions, so the bundle has no cohomology at degree $|\lambda|$.

Now consider the bundle $\Sigma^{-\lambda} \mathcal{L}^\vee \otimes \Sigma^{-\mu} Q^\vee$. Adding the second factor changes only the last m terms of $\chi + \rho$, and it decreases these terms. Thus the number of inversions can only decrease, so this bundle also has no cohomology at degree $|\lambda|$.

Finally, consider $\Sigma^{-\lambda} \mathcal{L}^\vee \otimes \Sigma^{-\mu} Q^\vee \otimes \Sigma^\mu \mathcal{L}^\vee$. We decompose $\Sigma^\mu \mathcal{L}^\vee \otimes \Sigma^{-\lambda} \mathcal{L}^\vee$ in the following manner: For any integer M, we have the identity

$$\Sigma^{-\lambda} \mathcal{L}^\vee = (\det \mathcal{L}^\vee)^{\otimes -M} \otimes \Sigma^{M - \lambda} \mathcal{L}^\vee. \tag{3.5}$$

given in [GN25]. Thus we can take a sufficiently large M so that $M - \lambda$ contains only nonnegative terms ($M = \lambda_1$ suffices). Now using the Littlewood-Richardson rule, we decompose $\Sigma^{M-\lambda} \mathcal{L}^\vee \otimes \Sigma^\mu \mathcal{L}^\vee$ into a direct sum of $\Sigma^\gamma \mathcal{L}^\vee$, where $\gamma = (\mu_j + c_j)$, under the following conditions:

$$\sum_{j=1}^m c_j = |M - \lambda| \mu_j + c_j \ge (M - \lambda)_j$$
$$= \lambda_1 - \lambda_{m-j+1}.$$

The second condition comes from the rule that in the Young diagram resulting from adding $|M - \lambda|$ to μ, no column can contain two boxes from the same row of $M - \lambda$. Reapplying 3.5, now subtracting M from $M - \lambda$, the terms in the decomposition of $\Sigma^\mu \mathcal{L}^\vee \otimes \Sigma^{-\lambda} \mathcal{L}^\vee$ are $\Sigma^{\gamma - M} \mathcal{L}^\vee$. Tensoring this with $\Sigma^{-\mu} Q^\vee$ yields the sequence

$$\chi + \rho = (n + \mu_1 + c_1 - \lambda_1, \ldots, n - m + 1 + \mu_m + c_m - \lambda_1, n - m, \ldots, m + 1, m - \mu_m, \ldots, 1 - \mu_1).$$

Because we have $\mu_j + c_j \ge \lambda_1 - \lambda_{m-j+1}$, we see that the $m - t + 1$-th term from the left in $\chi + \rho$ is greater than or equal to the corresponding term in the sequence we obtained in the first case, $n - t + 1 - \lambda_{m-t+1}$. Thus this case also only decreases the number of inversions, so the desired bundle has no cohomology at degree $|\lambda|$. □

Remark 3.6. There is an analogous variety to Y defined by removing the condition on $\mathrm{Im}(x)$ rather than $\mathrm{Ker}(x)$:

$$\{(V_k, V_l, x) : V_k, V_l \subset \mathrm{Ker}(x), \dim(V_k \cap V_l) \ge m\}.$$

A similar argument to Proposition 3.2 shows the normality of this variety.

References

[Bri04] M. Brion. *Lectures on the geometry of flag varieties*. 2004. arXiv: math/0410240 [math.AG].

[FH91] W. Fulton and J. Harris. *Representation Theory. A First Course*. Graduate Texts in Mathematics. Springer New York, NY, 1991.

[GN25] L. Guseva and A. Novikov. *Full exceptional collections on the isotropic Grassmannians*. 2025. arXiv: 2505.13005 [math.AG].

[GW20] U. Görtz and T. Wedhorn. *Algebraic Geometry I: Schemes*. Springer Studium Mathematik - Master. Springer Spektrum, Wiesbaden, 2020.

[Kap85] M. M. Kapranov. *On the derived category of coherent sheaves on Grassmann manifolds. Mathematics of the USSR-Izvestiya*, 24: 183–192, 1985.

[KP79] H. Kraft and C. Procesi. *Closures of Conjugacy Classes of Matrices are Normal. Inventiones mathematicae*, 53: 227–247, 1979.

[KS22] A. Knutson and S. V. Sam. *A Bruhat Atlas for the Mehta–van der Kallen Stratification of* T^*GL_n/B. *Transformation Groups*, 30: 283–298, 2022.