

# Toric Actions on Lens Spaces and Applications to Ricci Solitons

Papon Lapate and Michael B. Law

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## 1 Introduction

In this paper, we will classify all metrics on 3-dimensional Lens spaces with isometry group of high dimension. This topic arises in an approach to study the Ricci flow, which we describe below.

A *Ricci flow* on a smooth manifold  $M$  is a collection of Riemannian metric  $g_t$  on  $M$  for time  $t$  in an open interval such that  $\frac{\partial g_t}{\partial t} = -2\text{Ric}_{g_t}$ . The study of Ricci flow is an important topic in geometry, and one way to create a Ricci flow is through a particular kind of complete Riemannian manifolds called the *Ricci solitons*.

Ricci solitons are important because, under a mild condition, they induce a self-similar Ricci flow; that is, the Ricci flow for which (assuming  $g_0$  is defined) each  $g_t$  is a pullback of  $g_0$  scaled by a factor that depends on  $t$ . In particular, if that scaling factor is linear in  $t$ , then we obtain an equation of the form  $\text{Ric}_g + \frac{1}{2}\mathcal{L}_V g = \lambda g$ . A complete Riemannian manifold  $(M, g)$  that satisfies this equation for some  $\lambda \in \mathbb{R}$  and some vector field  $V$  is then called a Ricci soliton. In the case where the Ricci soliton is steady (that is, if  $\lambda = 0$ ) and  $V$  is complete, the flow moves by isometries defined by  $V$ .

If we impose the condition that  $V = \text{grad } f$  for some smooth function  $f$  so that  $\text{Ric}_g + \text{Hess}_f = \lambda g$ , then we say that the triple  $(M, g, f)$  is a *Gradient Ricci Soliton* (GRS). GRSs are the focus of many research as they are easier to work with than Ricci solitons in general, and they have many significant properties. For example, Zhang proved in [17] that for any GRS  $(M, g, f)$ ,  $\text{grad } f$  is a complete vector field, and so all GRSs define a solution to the Ricci flow with initial state  $(M, g)$ . In particular, if the GRS is steady or shrinking (that is, if  $\lambda \geq 0$ ), then the Ricci flow is defined for all negative time and thus results from the blow-up limit at singularity of a finite-time Ricci flow [7]. Moreover, Perelman proved in Section 3 of [16] that every shrinking Ricci soliton is a GRS.

In [10], Law studied a particular type of GRSs called the asymptotically  $\mathcal{S}$ -cylindrical steady GRSs (where  $\mathcal{S} = \mathbb{S}^{n-1}/\Gamma$  is a spherical space of dimension 1 less than the GRSs). Intuitively, they are GRSs  $(M, g, f)$  that look like  $\mathcal{S} \times \mathbb{R}$  as  $f \rightarrow \infty$ . More formally, we have the following definition:

**Definition 1.** For a spherical space  $\mathcal{S}$ , an asymptotically  $\mathcal{S}$ -cylindrical steady GRS is a GRS  $(M, g, f)$  such that for any sequence of points  $p_n \in M$  such that  $\lim_{n \rightarrow \infty} f(p_n) = \infty$ , sequence  $r_n \in \mathbb{R}^+$  such that  $r_n R(p_n) = \frac{n-1}{2} + o(1)$  (where  $R$  is the scalar curvature), and  $t \in (0, 1)$ , the sequence  $(M, r_n^{-1} \Phi_{r_n t}^* g, p_n)$  of pointed Riemannian manifolds (where  $\Phi$  denotes the flow given by  $\text{grad } f$ ) converges to

$$(\mathcal{S} \times \mathbb{R}, (n-2)(2-2t)\bar{g}_{\mathcal{S}} + dz^2, *),$$

where  $\bar{g}_{\mathcal{S}}$  is a metric on  $\mathcal{S}$  with constant sectional curvature 1.

Conditions similar to this have been studied in other works, such as in Proposition 2.5 of [5], and there are important nontrivial examples of asymptotically  $\mathcal{S}$ -cylindrical steady GRS, such as the Bryant soliton (for  $\mathcal{S} = \mathbb{S}^{n-1}$ ) [6] and the Appleton solitons (for some Lens spaces of odd dimension  $\mathcal{S}$ ) [2].

In [10], Law classified all asymptotically  $\mathcal{S}$ -cylindrical steady GRSs  $(M, g, f)$  for some spherical space forms  $\mathcal{S}$ . For example, when  $\mathcal{S} = \mathbb{S}^{n-1}$ , Law proved that the only such GRS is the Bryant soliton. Another important case that Law considered is when  $\mathcal{S} = L(p, 1)$ . (Here, for any coprime positive integers  $p$  and  $q$ ,  $L(p, q)$  is the Lens space  $\mathbb{S}^3/\mathbb{Z}_p$ , where  $m \in \mathbb{Z}_p$  acts on  $\mathbb{S}^3 \subset \mathbb{C}^2$  by sending  $(z, w)$  to  $(e^{2\pi i/p} z, e^{2q\pi i/p} w)$ .) In this case, any asymptotically  $\mathcal{S}$ -cylindrical steady GRS must be either a quotient of the Bryant soliton or an Appleton soliton.

To prove the main result of the paper, Law showed that each sufficiently far level set of  $f$  (which is diffeomorphic to  $\mathcal{S}$ ), its isometry group has dimension at least  $\dim \mathcal{I}(\mathcal{S}, \bar{g}_{\mathcal{S}})$  (where  $\mathcal{I}$  denotes the isometry group). Moreover, the identity component of the isometry group of that level set can be extended to the whole manifold. Therefore, this approach encourages a complete classification of Riemannian metrics on  $\mathcal{S}$  with a high dimension of isometry as it would allow one to write an explicit formula for  $g$  outside of a compact subspace of  $M$ . (In fact, Ricci solitons in general are expected to be highly symmetric because the shape of a Riemannian manifold is expected to improve during Ricci flow during negative time.)

By itself, the topic of Riemannian manifolds with a high dimension of isometry is quite well studied. The most highly symmetric Riemannian manifolds are the *symmetric spaces* – those for which, at each point, there exists an isometry acting as  $-\text{Id}$  on the tangent space at that point – and the *homogeneous spaces* – those for which the isometry group is transitive. Going down one level, we have Riemannian manifolds of cohomogeneity 1, on which there is a Lie group action with principal orbit of codimension one. Some results which are important for our paper are the fact that the only 3-dimensional Lens spaces that are homogeneous are those of the form  $L(p, 1)$  (which follows immediately from Proposition 3.4 of [8] and the fact that  $\pi_1(L(p, 1)) \approx \mathbb{Z}_p \approx \pi_1(L(p, q))$ ) and properties of cohomogeneity 1 actions [13].

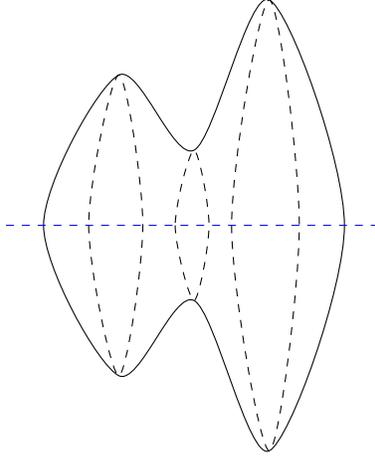


Figure 1: An example of a cohomogeneity 1 manifold. The isometry group is given by rotations around the horizontal axis.

Our main goal is to study Riemannian metrics on Lens space of the form  $L(p, q)$  for  $q \not\equiv \pm 1 \pmod{p}$  with the isometry group of high dimension. To that end, we obtain the following parametrization for such metrics:

**Theorem 2.** *Suppose that  $\mathcal{I}(L(p, q), g)$  has dimension two, then, up to  $\mathbb{T}^2$ -equivariant diffeomorphism,  $g$  is of the form  $dt^2 + a(t)dv^2 + b(t)dvdw + c(t)dw^2$  on the principal orbit, where  $t$  is the first coordinate in  $([0, 1] \times \mathbb{T}^2 / \sim) \approx L(p, q)$ ,  $v$  and  $w$  are coordinates for the two  $\mathbb{S}^1$  components in  $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$ , and  $a, b, c : (0, 1) \rightarrow \mathbb{R}$  are smooth functions satisfying the following conditions:*

- $b^2(t) < 4a(t)c(t)$  for all  $t \in (0, 1)$ .
- $a(t)$ ,  $b(t)$ , and  $c(t)$  can be extended to even smooth functions on  $(-1, 1)$  such that  $\lim_{t \rightarrow 0} \frac{a(t)}{t^2} = 1$ ,  $b(0) = 0$ , and  $c(0) > 0$ .
- Define

$$a_1(t) := a(t)p^2 + b(t)pq + c(t)q^2,$$

then  $a_1(1-t)$ ,  $b(1-t)$ , and  $c(1-t)$  can be extended to even smooth functions on  $(-1, 1)$  such that  $\lim_{t \rightarrow 0} \frac{a_1(1-t)}{t^2} = 1$ ,  $b(1) = \frac{-2q}{p}c(1)$ , and  $c(1) > 0$ .

We also aim to apply this result to the study of asymptotically  $L(p, q)$ -cylindrical steady GRSs in the same way as [10]. We are able to reduce the problem to the case where  $M$  is simply connected (by considering the universal cover of  $M$ ). However, since the isometric action on  $L(p, q)$  does not extend to an action on  $M$  of cohomogeneity 1, we only obtain a weaker description for the possible underlying space  $M$  and the quotient of  $M$  by the toric action (assuming that  $M$  is simply connected); namely, that  $M/\mathbb{T}^2$  is a topological

2-manifold with boundary and  $M$  is a quotient of the product  $(M/\mathbb{T}^2) \times \mathbb{T}^2$  with some identifications along  $\partial(M/\mathbb{T}^2) \times \mathbb{T}^2$ .

In Section 2, we will discuss results regarding isometric action of high dimension on Lens space, ending with the proof of Theorem 1 above. The bulk of this discussion will be about toric actions on Lens spaces. As mentioned above, this will extend to toric actions on asymptotically  $L(p, q)$ -cylindrical steady GRSs. We will state some of the results regarding such actions in Section 3. Finally, in Section 4, we will discuss some possible approaches for classifying asymptotically  $L(p, q)$ -cylindrical steady GRSs.

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## 2 Highly Symmetric Metrics on Lens Space

Intuitively, to get a large symmetry group on a Lens space, we should consider the quotient of the round metric on  $\mathbb{S}^3$ . In which case, [12] has the following result:

**Theorem 3.** *Let  $g_{p,q}$  denotes the metric on  $L(p, q)$  which is the quotient of the round metric on  $\mathbb{S}^3$ .*

- *If  $p > 2$  and  $q \equiv \pm 1 \pmod{p}$ , then the isometry group of  $(L(p, q), g_{p,q})$  has dimension 4.*
- *Otherwise, the isometry group of  $(L(p, q), g_{p,q})$  has dimension 2.*

The case  $q \equiv \pm 1$  has already been studied by Law in [10]. For the other cases, isometry of dimension 2 is, in fact, the most we can hope for:

**Theorem 4.** *Suppose that  $p > 2$  and  $q \not\equiv \pm 1 \pmod{p}$ . For any metric  $g$  on  $L(p, q) = \mathbb{S}^3/\mathbb{Z}_p$ , the isometry group of  $(L(p, q), g)$  has dimension at most 2.*

*Proof.* Assume the contrary that  $\dim \mathcal{I}(L(p, q), g) \geq 3$

As discussed in the introduction,  $(L(p, q), g)$  cannot be homogeneous according to Proposition 3.4 of [8]. Let  $\tilde{g}$  be the lifting of the metric  $g$  to  $\mathbb{S}^3$ . Since  $\mathbb{S}^3$  is simply connected, we see that the normalizer  $\tilde{N}$  of  $\mathbb{Z}_p$  in  $\mathcal{I}(\mathbb{S}^3, \tilde{g})$  acts non-homogeneously on  $\mathbb{S}^3$  and satisfies  $\tilde{N}/\mathbb{Z}_p = \mathcal{I}(L(p, q), g)$ . In particular, the identity component  $N$  of  $\tilde{N}$  is a connected Lie group of dimension at least 3 acting non-homogeneously on  $\mathbb{S}^3$ .

Suppose that the action of  $N$  has a 1-dimensional orbit, then consider any point  $x$  in one such orbit and the 2-dimensional slice  $S_x$  at  $x$ . Since  $\dim Nx = 1$ , the subgroup  $N'$  of  $N$  that fixes  $x$  has dimension at least 2. Since  $N'$  acts isometrically on  $\mathbb{S}^3$ , it acts on  $T_x(Nx)$  in at most 2 ways. Moreover, its action

on  $S_x$  is a subgroup of  $O(2)$ -action. Since  $\dim O(2) = 1 < \dim N'$ , we see that there is a nontrivial element in  $N'$  (and thus in  $N$ ) that fixes  $x$  and  $T_x \mathbb{S}^3$ . In particular, this means that  $N$  doesn't act effectively on  $\mathbb{S}^3$ , which is a contradiction. Thus, all orbits of the  $N$ -action on  $\mathbb{S}^3$  are 0-dimensional or 2-dimensional. In particular, since the action of  $N$  is effective, the action is of cohomogeneity 1.

Let  $X$  be the principal orbit of the  $N$ -action on  $\mathbb{S}^3$  (so  $X$  is a quotient of  $N$ ), then by Theorem 4 of [13],  $\mathbb{S}^3/N$  is either  $\mathbb{S}^1$  or an interval. Since  $\mathbb{S}^3$  is simply connected and compact, the quotient must be a compact interval. Then Theorem 4 of [13] implies that  $\mathbb{S}^3 \approx ([0, 1] \times X)/\sim$ , where  $\sim$  is the reduction of  $\{0\} \times X$  and  $\{1\} \times X$  into singular orbits of the form  $X/\mathbb{S}^m$  for some  $M \in \mathbb{Z}_{\geq 0}$ . Since all orbits have dimension 0 or 2, we must have  $m \in \{0, 2\}$  for both singular orbits.

Write  $X$  as the union of  $M_0 := ([0, 2/3] \times X)/\sim$  and  $M_1 := ((1/3, 1] \times X)/\sim$ . By the Mayer-Vietoris sequence, we know that

$$H_1(M_0 \cap M_1) \rightarrow H_1(M_0) \oplus H_1(M_1) \rightarrow H_1(M_0 \cup M_1)$$

is exact. Moreover,  $M_0 \cup M_1 = \mathbb{S}^3$  is simply connected, and thus the first map must be surjective. Therefore,  $H_1(M_0 \cap M_1) \rightarrow H_1(M_i)$  is also surjective for all  $i \in \{0, 1\}$ .

If  $(\{0\} \times X)/\sim \approx X/\mathbb{S}^0$ , then  $M_0$  is a  $\mathbb{R}^1$ -bundle over  $X/\mathbb{S}^0$  and thus  $H_1(M_0) \approx H_1(X/\mathbb{S}^0)$ . However,  $M_0 \cap M_1 = (1/3, 2/3) \times X$  and so  $H_1(M_0 \cap M_1) \approx H_1(X)$ . Thus, the first map is the canonical map  $H_1(X) \rightarrow H_1(X/\mathbb{S}^0)$ , which gives us a contradiction as this map is not surjective. Hence,  $(\{0\} \times X)/\sim$  is  $X/\mathbb{S}^2$ . Since this has dimension 0 and  $(\{0\} \times X)/\sim$  is connected, it must be a single point  $p_0$ , and so  $X \approx \mathbb{S}^2$ . Similarly,  $(\{1\} \times X)/\sim$  is also a single point  $p_1$ .

Finally, since  $N$  is in the normalizer of  $\mathbb{Z}_p$  in  $\mathcal{I}(\mathbb{S}^3, \tilde{g})$  and  $N$  fixes  $p_0$ , every action in  $N$  sends  $m \cdot p_0$  to a point in  $\mathbb{Z}_p \cdot p_0$  for any  $m \in \mathbb{Z}_p$ . However, this is only possible if  $m \cdot p_0 \in \{p_0, p_1\}$  for all  $m \in \mathbb{Z}_p$ , which is not possible since  $p > 2$  and  $m \cdot p_0$  is different for each  $m$ . Thus, we get a contradiction and we are done.  $\square$

Due to this theorem, we will focus our attention on the metric  $g$  on  $L(p, q)$  with isometry of dimension exactly 2. In this case, the identity component of  $\mathcal{I}(L(p, q), g)$  is a compact, connected Lie group of dimension 2 and thus must be the torus  $\mathbb{T}^2$ . Therefore, we need to classify all effective and smooth actions of  $\mathbb{T}^2$  on the Lens space. Fortunately, there is only one such action:

**Theorem 5.** *All effective and smooth actions of  $\mathbb{T}^2$  on  $L(p, q) = \mathbb{S}^3/\mathbb{Z}_p$  are  $\mathbb{T}^2$ -equivariantly diffeomorphic. In particular, they are homeomorphic to the action of  $\mathbb{T}^2/\mathbb{Z}_p$  on  $\mathbb{S}^3/\mathbb{Z}_p$  given by  $(a, b)(z, w) = (az, bw)$ .*

*Proof.* First, note that by the Principal Orbit Theorem and the fact that  $\mathbb{T}^2$  is abelian, there exists a dense and open subset of  $L(p, q)$  fixed by the same nontrivial subgroup of  $\mathbb{T}^2$ , which is the isotropy group of every point in each

principal orbit. By continuity, this subgroup fixes the entire space, and thus must be trivial by the effectivity of the  $\mathbb{T}^2$ -action. Hence,  $\mathbb{T}^2$  also acts effectively on each principal orbit. In particular, there is an orbit of dimension 2.

Since there is an orbit of dimension one less than that of  $L(p, q)$ , the action is of cohomogeneity 1. By Theorem 4 of [13],  $L(p, q)/\mathbb{T}^2$  is either  $\mathbb{S}^1$  or an interval. Since  $L(p, q)$  has finite fundamental group and is compact, the quotient must be a compact interval. Then Theorem 4 of [13] implies that  $L(p, q) \approx ([0, 1] \times \mathbb{T}^2) / \sim$ , where the relation  $\sim$  reduces  $\{0\} \times \mathbb{T}^2$  and  $\{1\} \times \mathbb{T}^2$  further into  $\{0\} \times \mathbb{T}^2 / Y_0$  and  $\{1\} \times \mathbb{T}^2 / Y_1$  for some spheres  $Y_0$  and  $Y_1$  (of not necessarily the same dimension).

Consider  $L(p, q)$  as the union of open subsets  $M_0 := ([0, 2/3] \times \mathbb{T}^2) / \sim$  and  $M_1 := ((1/3, 1] \times \mathbb{T}^2) / \sim$ . If  $Y_0 \approx \mathbb{S}^0$ , then  $M_0$  is (homeomorphic to) a  $\mathbb{R}^1$ -bundle over  $\mathbb{T}^2/\mathbb{S}^0 \approx \mathbb{T}^2$ , and so  $\pi_1(M_0) \approx \mathbb{Z}^2$ . On the other hand, if  $Y_0 \approx \mathbb{S}^1$ , then  $M_0$  is (homeomorphic to) a  $\mathbb{R}^2$ -bundle over  $\mathbb{T}^2/\mathbb{S}^1 \approx \mathbb{S}^1$ , and so  $\pi_1(M_0) \approx \mathbb{Z}^1$ . Similarly, if  $Y_1 \approx \mathbb{S}^{i_1}$ , then  $\pi_1(M_1) \approx \mathbb{Z}^{2-i_1}$ .

Now, by the Mayer-Vietoris sequence, we know that the sequence

$$H_1(M_0 \cap M_1) \rightarrow H_1(M_0) \oplus H_1(M_1) \rightarrow H_1(M_0 \cup M_1)$$

is exact. Each term in this sequence has the following value:

- $H_1(M_0 \cap M_1) = H_1((1/3, 2/3) \times \mathbb{T}^2) = \mathbb{Z}^2$ .
- $H_1(M_0) \oplus H_1(M_1) = \mathbb{Z}^{4-i_0-i_1}$ , where  $i_0$  and  $i_1$  are the dimensions of  $Y_0$  and  $Y_1$ .
- $H_1(M_0 \cup M_1) = H_1(L(p, q)) = \mathbb{Z}_p$ .

The sequence cannot be exact if  $i_0 + i_1 \leq 1$ . Therefore,  $Y_0 \approx Y_1 \approx \mathbb{S}^1$ . This means that  $M_0$  and  $M_1$  are solid tori.

Write  $\mathbb{T}^2$  as  $\mathbb{S}^1 \times \mathbb{S}^1$  in such a way that  $Y_0$  is the first  $\mathbb{S}^1$  component. Then the map  $H_1(M_0 \cap M_1) \rightarrow H_1(M_0) \oplus H_1(M_1)$  sends  $(1, 0), (0, 1) \in \mathbb{Z}^2 = H_1(M_0 \cap M_1)$  to  $(0, a_1), (1, a_2) \in \mathbb{Z}^2 = H_1(M_0) \oplus H_1(M_1)$ . Since the map has cokernel  $\mathbb{Z}_p$ , we get that the map sends  $(1, 0)$  and  $(0, 1)$  to  $(0, p)$  and  $(1, q')$  for some  $q'$ , which we can assume to be in the range  $[0, p)$  by an appropriate change of basis.

This means that the meridian  $\delta$  and (preferred) longitude  $\gamma$  from the first solid torus is identified with  $p\gamma + b_1\delta$  and  $q'\gamma + b_2\delta$  from the second solid torus. This implies that  $b_1 \equiv q'^{-1} \pmod{p}$  and so the union of the two solid tori is  $L(p, q'^{-1})$ . Therefore,  $q' \equiv \pm q^{\pm 1} \pmod{p}$ .

Finally, by reversing the orientation of  $\mathbb{T}^2$  appropriately and/or switching the two solid tori, we may assume that  $b_1 \equiv q'^{-1} \equiv q$ . Hence, we are done.  $\square$

Note that the homeomorphism can actually be promoted to a diffeomorphism via the slice theorem; see Proposition 6.33 of [1]. Moreover, we can explicitly write down the metrics on  $L(p, q)$  with high dimension of isometry group:

**Theorem 6.** *Suppose that  $\mathcal{I}(L(p, q), g)$  has dimension two, then, up to diffeomorphism,  $g$  is of the form  $dt^2 + a(t)dv^2 + b(t)dvdw + c(t)dw^2$  on the principal*

orbit, where  $t$  is the first coordinate in  $([0, 1] \times \mathbb{T}^2 / \sim) \approx L(p, q)$ ,  $v$  and  $w$  are coordinates for the two  $\mathbb{S}^1$  components in  $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$ , and  $a, b, c : (0, 1) \rightarrow \mathbb{R}$  are smooth functions.

*Proof.* By Theorem 5 and the discussion before, there is an effective and smooth  $\mathbb{T}^2$ -action on  $L(p, q)$ , and we can write  $L(p, q)$  in the form  $([0, 1] \times \mathbb{T}^2 / \sim)$  (with identification  $\sim$  as detailed in the proof of Theorem 5) so that  $\mathbb{T}^2$  acts in the obvious way on the second component.

Since  $\partial_v$  and  $\partial_w$  are invariant under the  $\mathbb{T}^2$ -action, any  $\mathbb{T}^2$ -invariant metric on  $\mathbb{T}^2 \approx \{t\} \times \mathbb{T}^2$  (for  $t \in (0, 1)$ ) must be of the form  $adv^2 + bvdvdw + cdw^2$  for some constant  $a, b, c$ . Finally, by equation (6.27) of [1], the metric on  $L(p, q)$  itself must be of the form given in the theorem statement.  $\square$

In order to extend this metric to singular orbits, we need appropriate boundary conditions for  $a, b, c$ . Assume that  $v$  corresponds to the  $\mathbb{S}^1$  component that fixes every point on the singular orbit at 0. We can identify a neighborhood of a point in that singular orbit with  $\mathbb{D}^2 \times (-1, 1) \subset \mathbb{R}^3$ , where the point  $(r, \theta, z)$  (in the cylindrical coordinate) belongs to the orbit  $\{r\} \times \mathbb{T}^2$ . With this identification,  $\partial_t$  is  $\cos \theta \partial_x + \sin \theta \partial_y$ ,  $\partial_v$  is  $-r \sin \theta \partial_x + r \cos \theta \partial_y$ , and  $\partial_w$  is  $\partial_z$ . We see that the metric at  $(r, \theta, z)$  satisfies

- $1 = \langle \partial_t, \partial_t \rangle = \cos^2 \theta \langle \partial_x, \partial_x \rangle + 2 \cos \theta \sin \theta \langle \partial_x, \partial_y \rangle + \sin^2 \theta \langle \partial_y, \partial_y \rangle$
- $0 = \langle \partial_t, \partial_v \rangle = -r \cos \theta \sin \theta \langle \partial_x, \partial_x \rangle + r (\cos^2 \theta - \sin^2 \theta) \langle \partial_x, \partial_y \rangle + r \cos \theta \sin \theta \langle \partial_y, \partial_y \rangle$
- $0 = \langle \partial_t, \partial_w \rangle = \cos \theta \langle \partial_x, \partial_z \rangle + \sin \theta \langle \partial_y, \partial_z \rangle$
- $a(r) = \langle \partial_v, \partial_v \rangle = r^2 \sin^2 \theta \langle \partial_x, \partial_x \rangle - 2r^2 \cos \theta \sin \theta \langle \partial_x, \partial_y \rangle + r^2 \cos^2 \theta \langle \partial_y, \partial_y \rangle$
- $b(r) = \langle \partial_v, \partial_w \rangle = -r \sin \theta \langle \partial_x, \partial_z \rangle + r \cos \theta \langle \partial_y, \partial_z \rangle$
- $c(r) = \langle \partial_w, \partial_w \rangle = \langle \partial_z, \partial_z \rangle$

Thus, the metric around  $\{0\} \times (-1, 1)$  is given by

$$dx^2 + dy^2 + \frac{1}{r^2} \left( \frac{a(r)}{r^2} - 1 \right) (y^2 dx^2 - xy dx dy + x^2 dy^2) - \frac{b(r)}{r^2} (y dx dz - x dy dz) + c(r) dz^2$$

Let  $A(x, y)$  be the coefficient of  $dx^2$  in the metric above, then  $A(x, y) = \frac{y^2}{r^2} \left( \frac{a(r)}{r^2} - 1 \right)$  at all points except the origin. It is clear that  $A(x, y) = A(-x, -y)$ . Thus,  $\frac{a(r)}{r^2}$  can be extended to the even smooth function  $A(0, r) + 1$  on  $\mathbb{R}$ . Moreover, because  $\frac{y^2}{r^2}$  range from 0 to 1 as  $\theta$  changes, we need  $\lim_{r \rightarrow \infty} \frac{a(r)}{r^2} = 1$  for  $A(x, y)$  to be smooth at the origin. On the other hand, if  $\lim_{r \rightarrow \infty} \frac{a(r)}{r^2} = 1$  and  $\frac{a(r)}{r^2}$  can be extended to an even smooth function over  $\mathbb{R}$ , then one can check that

$\frac{1}{r^2} \left( \frac{a(r)}{r^2} - 1 \right)$  is also a smooth function, which suffices to make the coefficients of  $dx^2$ ,  $dx dy$ , and  $dy^2$  in the metric above smooth.

Similarly, one can check that the coefficients of  $dx dz$  and  $dy dz$  in the metric above is smooth iff  $b(r)$  can be extended to a smooth even function on  $\mathbb{R}$  that takes the value 0 at  $r = 0$ . Finally, we also need  $c(r)$  to be even (or, more precisely, extendable to an even smooth function) and  $c(0)$  to be positive so that the metric is positive-definite.

For the boundary conditions on the other side, let  $G(x, y)$  denotes the  $\mathbb{S}^1$  subgroup of  $\mathbb{T}^2$  that corresponds to  $(x, y) \in \mathbb{Z}^2 = H_1(\mathbb{T}^2)$  (so that  $G(1, 0)$  is the isotropy group of the singular orbit at  $t = 0$  and  $G(p, q)$  is the isotropy group of the singular orbit at  $t = 1$ ). Choose any positive  $n, m$  such that  $pm - qn = 1$  and let  $v'$  and  $w'$  be the coordinates corresponding to  $G(p, q)$  and  $G(m, n)$ . Then  $g$  (as given in Theorem 5) can also be written as  $dt^2 + a_1(t)dv'^2 + b_1(t)dv'dw' + c_1(t)dw'^2$  where

$$a_1(t) := a(t)p^2 + b(t)pq + c(t)q^2 \quad (1)$$

$$b_1(t) := 2a(t)pm + b(t)(pn + qm) + 2c(t)qn \quad (2)$$

$$c_1(t) := a(t)m^2 + b(t)mn + c(t)n^2 \quad (3)$$

Note that  $a_1(1-t)$ ,  $b_1(1-t)$ , and  $c_1(1-t)$  must satisfy the same boundary conditions as  $a(t)$ ,  $b(t)$ , and  $c(t)$ ; namely, that they can be extended to an even smooth function such that  $\lim_{t \rightarrow 0} \frac{a_1(1-t)}{t^2} = 1$ ,  $b_1(1) = 0$ , and  $c_1(1) > 0$ . Moreover, one can see that these three conditions together do not depend on the choice of  $m$  and  $n$ ; since  $b_1(t) = \left( b(t) + \frac{2c(t)q}{p} \right) + \frac{2m}{p} a_1(t)$  and  $c_1(t) = \frac{c(t)}{p^2} + \frac{m}{p} b_1(t) - \frac{m^2}{p^2} a_1(t)$ , we see that the boundary condition we want is true iff it is true for  $m = 0$  and  $n = \frac{1}{p}$  as well. Putting everything together, plus the fact that  $g$  still needs to be positive-definite on the principal orbits, we get our main result that we stated in the introduction:

**Theorem 7.** *Suppose that  $\mathcal{I}(L(p, q), g)$  has dimension two, then, up to  $\mathbb{T}^2$ -equivariant diffeomorphism,  $g$  is of the form  $dt^2 + a(t)dv^2 + b(t)dv dw + c(t)dw^2$  on the principal orbit, where  $t$  is the first coordinate in  $([0, 1] \times \mathbb{T}^2 / \sim) \approx L(p, q)$ ,  $v$  and  $w$  are coordinates for the two  $\mathbb{S}^1$  components in  $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$ , and  $a, b, c : (0, 1) \rightarrow \mathbb{R}$  are smooth functions satisfying the following conditions:*

- $b^2(t) < 4a(t)c(t)$  for all  $t \in (0, 1)$ .
- $a(t)$ ,  $b(t)$ , and  $c(t)$  can be extended to even smooth functions on  $(-1, 1)$  such that  $\lim_{t \rightarrow 0} \frac{a(t)}{t^2} = 1$ ,  $b(0) = 0$ , and  $c(0) > 0$ .
- Define

$$a_1(t) := a(t)p^2 + b(t)pq + c(t)q^2,$$

then  $a_1(1-t)$ ,  $b(1-t)$ , and  $c(1-t)$  can be extended to even smooth functions on  $(-1, 1)$  such that  $\lim_{t \rightarrow 0} \frac{a_1(1-t)}{t^2} = 1$ ,  $b(1) = \frac{-2q}{p}c(1)$ , and  $c(1) > 0$ .

### 3 Toric Action on Ricci Soliton

Now, we come back to the problem of identifying asymptotically  $\mathcal{S}$ -cylindrical GRSs when  $\mathcal{S}$  is a Lens space  $L(p, q)$  for some coprime positive integers  $p > 2$  and  $q \not\equiv \pm 1 \pmod{p}$ . Note that there is no known example of such space. The intent of this section is to narrow down the possibility of such space.

First, we state some known properties of the asymptotically  $\mathcal{S}$ -cylindrical GRSs  $(M, g, f)$  (for general spherical form  $\mathcal{S}$ ):

- $f$  is roughly linear far away from a fixed point. (Proven in Lemma 3.14 of [4].)
- All sufficiently far level sets of  $f$  (that is, the set  $\{f = x\}$  for large  $x$ ) are diffeomorphic to  $\mathcal{S}$ . (Proven in Corollary 2.5 of [10].)
- On sufficiently far level sets of  $f$ ,  $g|_{\{f=x\}}$  is roughly proportional to  $x\bar{g}_{\mathcal{S}}$ . Therefore,  $M$  looks like paraboloid for large  $f$ . (Proven in Section 2.2 of [18].)

Now, we note that we can reduce this problem to the case where  $M$  is simply connected by the following theorem:

**Theorem 8.** *Let  $(\tilde{M}, \tilde{g}, \tilde{f})$  be the universal cover of  $(M, g, f)$ . Then  $(\tilde{M}, \tilde{g}, \tilde{f})$  is an asymptotically  $L(p', q)$ -cylindrical steady GRS for some  $p'$  dividing  $p$ .*

*Proof.* The level sets of  $\tilde{f}$  have almost constant sectional curvature (because the level sets of  $f$  do), and  $(M, g)$  has one end by [14]. Hence, the usual blow-down sequence centered at points going to infinity will converge in the Cheeger–Gromov sense to a family of shrinking quotient-cylinders  $\mathcal{S} \times \mathbb{R}$ , where  $\mathcal{S}$  is a (connected) spherical space form. Therefore,  $(\tilde{M}, \tilde{g}, \tilde{f})$  is an asymptotically  $\mathcal{S}$ -cylindrical steady GRS, so its tangent flow at infinity is  $\mathcal{S} \times \mathbb{R}$ . By Theorem 1.7 in Bamler [3],  $\mathcal{S}$  must be a covering space for  $L(p, q)$ . Hence  $\mathcal{S} = L(p', q)$  for some  $p'$  dividing  $p$ , as claimed.  $\square$

Now, suppose  $(M, g, f)$  is simply connected and let  $\mathcal{S}_x$  denotes the level set  $\{f = x\}$ . Since Theorem 4 implies that  $\dim \mathcal{I}(\mathcal{S}_x, g|_{\mathcal{S}_x}) \leq \dim \mathcal{I}(\mathcal{S}, g_{\mathcal{S}})$ , Section 4 of [10] implies that, for all sufficiently large  $x$ ,  $\dim \mathcal{I}(\mathcal{S}_x, g|_{\mathcal{S}_x}) = 2$  and all isometries in the identity component of  $\mathcal{I}(\mathcal{S}_x, g|_{\mathcal{S}_x})$  can be extended to isometries on  $M$ . In particular, this means that  $g|_{\mathcal{S}_x}$  is of the form given in Theorem 7 and there is a toric action on  $M$  whose principal orbits have trivial isotropy group.

Construct a manifold  $M'$  as the union of two copies of  $M$  that intersect on  $M_{>x} := M|_{\{f>x\}}$ , then  $M'$  can also be viewed as the gluing of two copies of the compact manifold with boundary  $M \setminus M_{<x}$  along the boundary. Thus,  $M'$  is a compact manifold with a toric action coming from the toric action on  $M$ .

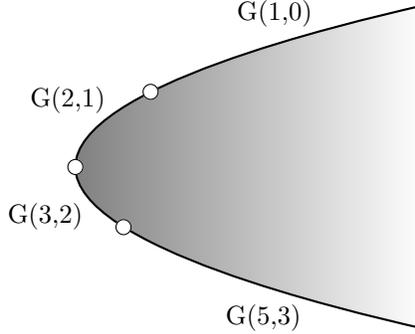


Figure 2: A simplified diagram showing an example of  $M/\mathbb{T}^2$ .  $M$  can be viewed as this figure with each point replaced by a torus, but with identifications as given by the labels on the boundary (see Theorem 9).

Moreover, the sequence

$$H_1(M)^{\oplus 2} \rightarrow H_1(M') \rightarrow H_0(M_{>x}) \rightarrow H_0(M)^{\oplus 2}$$

is exact. However, the first term is zero since  $M$  is simply connected, and the last map is injective since  $M_{>x} \approx \mathbb{R} \times L(p, q)$  is connected. Thus,  $M'$  is also simply connected. Thus, by Lemma 5.2 and Theorem 1.10 of [15],  $M' \rightarrow M'/\mathbb{T}^2$  has a cross-section which is a topological manifold with boundary. (Note that Lemma 5.2 is needed to ensure that there is no orbit with finite nontrivial isotropy group.) This cross-section can be restricted to  $M$  and so we have the following theorem:

**Theorem 9.**  $M/\mathbb{T}^2$  is a topological 2-manifold with boundary. Moreover,  $M$  is of the form  $((M/\mathbb{T}^2) \times \mathbb{T}^2)/\sim$ , where the identification  $\sim$  on  $\partial(M/\mathbb{T}^2) \times \mathbb{T}^2$  is given as follow: write  $\partial(M/\mathbb{T}^2) \approx \mathbb{R}$ , then for some real numbers  $t_1, \dots, t_n \in \mathbb{R}$  and pairs of coprime integers  $(a_0, b_0), \dots, (a_n, b_n) \in \mathbb{Z}^2$  such that  $(a_0, b_0) = (1, 0)$ ,  $(a_n, b_n) = (p, q)$ , and  $\begin{vmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{vmatrix} = \pm 1$ , the identification  $\sim$  reduces  $\{t_i\} \times \mathbb{T}^2$  to a point and reduces  $(t_i, t_{i+1}) \times \mathbb{T}^2$  to  $(t_i, t_{i+1}) \times (\mathbb{T}^2/G(a_i, b_i))$  (where  $t_0 = -\infty$  and  $t_{n+1} = \infty$ ).

We give an intuitive explanation for why this theorem is true. For sufficiently far level sets  $\mathcal{S}_x$  (with large  $x$ ), the action of  $\mathbb{T}^2$  on them has two principal orbits  $\mathbb{T}^2/G(1, 0)$  and  $\mathbb{T}^2/G(p, q)$ , which are represented by  $(\{t\} \times \mathbb{T}^2)/\sim$  for large (positive and negative)  $t \in \mathbb{R}$ . Since the boundary of  $M/\mathbb{T}^2$  (and thus the union of all singular orbits) is connected, we need a way to transition from  $\mathbb{T}^2/G(1, 0)$  to  $\mathbb{T}^2/G(p, q)$  along the singular orbits. To do this, we need to change the isotropy group as we travel along  $\partial(M/\mathbb{T}^2) \approx \mathbb{R}$ , which is why we need the pairs  $(a_i, b_i)$  for  $0 < i < n$ , and the determinant condition is needed so that  $M$  is Euclidean around the fixed points (where transitions between singular orbits with different isotropy groups happen).

## 4 Possible Future Approach

Thus far, for a simply connected asymptotically  $\mathcal{S}$ -cylindrical GRS  $(M, g, f)$ , we obtain a general description of what  $M$  looks like as a manifold. However, the description we get here is not as restrictive as the one in [10]; in particular, we don't get a clear picture of what the level set for a small value of  $f$  looks like, especially near a fixed point of the toric action. As such, to gain further information, it may be wise to deal with the differential equation for Ricci solitons more directly.

One possible way to simplify the Ricci soliton equation is by reducing the calculation to the cross-section of the toric action (which has a metric induced by  $g$ ). Such approach has been done on some Ricci solitons with a toric action, such as on *Einstein manifolds* (Ricci solitons for which  $\text{Ric}_g = \lambda g$ ) [11] and on *gravitational instantons* (complete Riemannian 4-manifolds satisfying the Einstein equation; they are Ricci solitons since they have  $\text{Ric} = 0$ ) [9].

In [11], Liu derived a differential equation regarding the orbit volume function  $\phi$  on a cross-section  $\Sigma$  of the toric action. In particular, for the case of steady Einstein manifolds, Liu proved that  $\Delta_\Sigma \phi = 0$  (special case  $\Lambda = 0$  of Proposition 2); in other words, the orbit volume is constant across all orbits. In our case, however, it turns out that  $\Delta_\Sigma \phi$  depends on  $\text{grad } f$  and thus the situation is more complicated.

In [9], Kunduri and Lucietti showed that asymptotically flat toric gravitational instantons correspond one-on-one to rod structures – the data including the shape of the orbit space of the toric action, the isotropy groups data on the boundary of that orbit space, and the distance between fixed points ( $t_{i+1} - t_i$  in the notation of our Theorem 9). We may hope to get similar result in our case, although it is improbable that our existence result will be the same; otherwise, we would be able to get different metrics on diffeomorphic GRSs, but there is no known example of this happening.

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