

Timothy Chen
Open Cases of the Colmez Conjecture
under the direction of Ayan Nath

Abstract

The Colmez conjecture equates the Faltings height of a complex multiplication (CM) abelian variety with logarithmic derivatives of Artin L -functions. CM abelian varieties are determined by CM fields paired with a choice of CM type; the conjecture is known when the CM field has an abelian Galois group, as well as on average over all CM types of a given signature. Every other result in the literature follows directly from these two, together with the linearity of both sides. In particular, any linear combination of class functions where the Colmez conjecture has been verified is itself verified, so the verified class functions form a subspace. Via induction and inflation, we compute this subspace for all 651 Galois groups up to degree 15 and all 455 of prime degree up to 199. For 592 and 215 of these, respectively, the conjecture can be deduced for all CM types of corresponding CM fields purely from the two known results and linear algebra. For the groups that remain, we characterize the codimension of the verified subspace for both prime degree groups and dihedral groups under their natural action. This result motivates our proof of the Colmez conjecture for two-homogeneous Galois groups, a newly discovered infinite family.

Sylvia Lee

Learning to Grow: Trajectory-Based Discovery of Dynamical Systems using Neural Cellular Automata

under the direction of Ashutosh Tripathi

Abstract

We developed neural cellular automata to simulate growth patterns of biological processes such as morphogenesis and embryogenesis to identify the underlying dynamics. We trained the model to replicate a full growth trajectory from videos of pattern evolutions. We specifically consider a data-limited environment, where experimental data is scarce and the model is given a single video to learn from. We demonstrate that the neural cellular automata is generalizable and robust to noise through exploration of different hyperparameters. We then use sparse symbolic regression methods such as SINDy to interpret the local update function in terms of simpler nonlinear functions.

Katrina Liu
Temporal Preferential Attachment Trees
under the direction of Hang Du

Abstract

Many evolving networks are shaped by both preferential attachment, in which new vertices favor already popular ones, and temporal restriction, in which they can access only recent activity. We study a layered temporal preferential attachment tree where vertices arrive in growing batches and attach to the preceding batch with probability proportional to degree, preserving both effects while avoiding the random normalizing denominator of a sliding-window model. We prove that the proportion of degree- k vertices converges to 2^{-k} and that the maximum degree is asymptotic in probability to $\log_2 N$. Surprisingly, these match the asymptotics of the uniform random recursive tree, suggesting that temporal restriction and preferential attachment balance in the layered model; simulations further illustrate the finite-size behavior of the degree profile and maximum degree.

Lincoln Liu

Heat-like Evolution on Graphs

under the direction of Alexander McWeeney

Abstract

In 2013, Bauer et al. proved a result on graphs analogous to the Li–Yau gradient estimate. In this paper, we analyze similar graphical gradient estimates but extend it to matrices. We define the gradient and Hessian matrix of a function on a graph in accordance to a bilinear gradient form. The maximum principle for the heat equation on graphs is extended to include a gradient term. A matrix form of the maximum principle, analogous to Hamilton’s matrix maximum principle, is proven to hold if and only if a similar gradient term is not too large. Inspired by the discrete Li–Yau inequality, we state a series of discrete statements analogous to Hamilton’s matrix Li–Yau–Harnack inequality on the discrete lattice $\mathbb{Z}^d$, but provide counterexamples for all of these.

Tony Lu

Convergence Behavior of Curve Shortening Flow with Boundary on a Circle

under the direction of Alexander McWeeney

Abstract

The problem of curve shortening flow (CSF), concerning classes of smooth planar curves that satisfy the evolution equation $X_t = -\kappa\mathbf{N} = X_{ss}$, has been studied extensively throughout the literature. The first classical classification theorem for CSF was proven by Gage, Hamilton, and Grayson in the 1980s for convex closed curves. In this paper, we expand upon the work of Langford and Zhu in classifying the end behavior of curves that satisfy CSF with boundary on a circle: in particular, we seek to determine the asymptotic normalized profile of curves that converge to a diameter. Our approach uses a linearization of the CSF equation for graphs to quantify errors and construct a late-time limiting profile that satisfies a heat-type equation with Robin boundary condition.

Michael Luo

Stack-Sorting

under the direction of Ryota Inagaki

Abstract

We prove Zhang's conjecture on West's stack-sorting map, which states that for any positive integer n , the number of 2-stack-sortable permutations of $\{0, \dots, n\}$ ending in 0 is the n th Motzkin number. The proof uses Zhang's stack-sorting tableaux and a bijection to standard Young tableaux with width at most 3. We also generalize West's stack-sorting map to words by defining the maps s_m , which allow m identical elements to sit on top of each other on the stack, but not $m + 1$. We also generalize a result of Chen, Luo, and Zhang by showing for any m and m' , there exist words w for which s_m sorts w much faster than $s_{m'}$.

Grisham Paimagam
Sphere Packing in Grassmannians
under the direction of Siqi Wu

Abstract

This paper analyzes the behavior of packing δ -balls in Grassmannians in the chordal metric. Previous results have achieved a lower bound on the order of $\theta(\delta) \geq 1/2^D$, where $\theta(\delta)$ represents the maximal packing density of δ -balls on a Grassmannian with real dimension D . This study first shows that $\theta(\delta) \geq (1 - o(1)) \frac{D \log D}{2^{D+1}}$ in $\mathbf{Gr}_1(\mathbb{C}^d)$, where δ is constant as $D \rightarrow \infty$. Then, it was shown that $\theta(\delta) \geq (1 - o(1)) \frac{D \log D}{2^{D+1}}$ for $\mathbf{Gr}_k(\mathbb{C}^n)$, where $k = o(n)$ and $\delta^2 \leq \frac{3k}{16 \ln 2}$, where δ is constant as $D \rightarrow \infty$. Finally, it is shown that $\theta(\delta) \geq (1 - \gamma - o(1)) \frac{D \log D}{2^{D+1}}$ in $\mathbf{Gr}_1(\mathbb{C}^d)$ for $1 - 2\delta^2 = \Theta(1/d^\gamma)$, and it becomes apparent that the lack of scale-invariance of Grassmannians impacts the packing bound when δ approaches the maximum radius at a certain asymptotic rate.

Seomgeun Shim

Information Flow in Self Organizing Systems
under the direction of Ashutosh Tripathi

Abstract

This paper explores the use of information theory measurements to characterize information transfer in the Rectangular Ising model, Hexagonal Ising model, and the Block Ising Model. Spin states are simulated through Glauber Dynamics, updating spins based on the local fields. We measure the mutual information of the system over the iterations by introducing a quantity called dynamical mutual information, which is proven to converge to pairwise mutual information when grid is infinitely large. Furthermore, we describe local information in the simulations mathematically, proving the domain walls between (+) and (−) must exist for ferromagnetic and antiferromagnetic Ising models. We analyze the maximal rate of mutual information and find that the peak does not correspond to the critical temperature. Finally, the non-monotonic dynamical mutual information evolution in Block Ising Model is examined.

Sirius Sun
Critical Groups of Signed Subdivided Wheels
under the direction of Ryota Inagaki

Abstract

We find the critical groups of all signed subdivided wheel graphs, which is obtained from an even wheel by adding a vertex on each rim, without connecting to the hub, and the hub taken as the sink. Vertex switching preserves the similarity of the Laplacian matrix, and two subdivided wheels are switching equivalent precisely when their positive edges have the same parity. In the balanced class, the signed graph is switching equivalent to the unsigned subdivided wheel, and Raza's result can be applied unchanged. For the unbalanced class we compute the Smith normal form of the reduced signed Laplacian by eliminating generators one by one; what remains is a two-by-two relation matrix whose entries are determined by a pair of integer sequences satisfying a Pell-type identity. The invariant factors can be written in closed form, yielding a direct sum of two cyclic groups in each case. This extends Raza's computation for unsigned subdivided wheels to the signed setting and fits into the signed chip-firing framework of Cho et al.

Anushka V. Tonapi

On Counting Prime Solutions to Quadratic Forms via
the Hardy-Littlewood Circle Method

under the direction of Siqi Wu

Abstract

We study prime-coordinate solutions of $Q_q(x) = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 - Mq^2 = 0$ as M ranges over residue classes mod 24 and use the Hardy-Littlewood circle method to estimate the major arcs and form conjectures about the minor arcs. We outline the major-arc computation needed to obtain an explicit-constant version of Birch's theorem in this diagonal case, and introduce a conjectural asymptotic for the count of solutions with all coordinates restricted to primes. We use Siegel-Walfisz to observe the distribution of primes in the residue classes, and invoke Hensel's lemma to show that the Gauss sum is analytically equivalent to the singular series in the classical problem. We also provide insight on how our approach differs from the integral case, and we find interesting analogues between the classical approach and ours by looking at the convergence of their respective analytical functions.

Nikola Veselinov

On Supersingular Isogeny Graphs of Drinfeld Modules

under the direction of Ayan Nath

Abstract

We study supersingular isogeny graphs of rank-two Drinfeld modules over $A = \mathbb{F}_q[T]$. For distinct finite primes $\mathfrak{p}$ and $\mathfrak{q}$ of A , we prove that the graph in characteristic $\mathfrak{p}$, with edges given by cyclic $\mathfrak{q}$ -isogenies, is connected. The proof combines Gekeler's ideal-class correspondence with strong approximation to realize the graph as a quotient of the Bruhat–Tits tree of the homothety classes of $A_{\mathfrak{q}}$ -lattices in $(\mathbb{F}_q(T))_{\mathfrak{q}}^2$. We also introduce the completeness number $E(\mathfrak{p})$, the least integer such that the graph is complete for every prime $\mathfrak{q} \neq \mathfrak{p}$ with $\deg \mathfrak{q} \geq E(\mathfrak{p})$, and derive explicit upper bounds using the Ramanujan–Petersson bound for the eigenvalues of operators associated to Brandt matrices over function fields. In particular, if $d = \deg \mathfrak{p}$, then $E(\mathfrak{p}) \leq 2d + 2$, with sharper bounds depending on q and the parity of d . This confirms a conjecture of Micheli and Papikian that the graph becomes complete once $\deg \mathfrak{q}$ is sufficiently large relative to $\deg \mathfrak{p}$. We further obtain parity-dependent lower bounds and prove that $E(\mathfrak{p}) \geq d + \log_q d - K$ for an absolute constant $K > 0$ and sufficiently large odd d , thereby refuting the previously suggested bound $E(\mathfrak{p}) \leq d + 1$. Finally, we prove a criterion yielding an algorithm to compute $E(\mathfrak{p})$.