On q-Analogs of the Markov Equation

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July 26th, 2025

Abstract

It is a classical result of Rudakov [1] that exceptional vector bundles on the complex projective plane $\mathbb{P}^2$ correspond to integer solutions of the *Markov equation* $x^2 + y^2 + z^2 = xyz$. Bridgeland [2, 3] shows that exceptional triples on $\mathbb{P}^2$ give *Koszul t-structures* on $\mathcal{D}^b \operatorname{Coh} \mathrm{K}\mathbb{P}^2$, where $\mathrm{K}\mathbb{P}^2$ is the canonical bundle over $\mathbb{P}^2$; these arise by transport of structure along derived equivalences $\mathcal{D}^b \operatorname{Coh} \mathrm{K}\mathbb{P}^2 \simeq \mathcal{D}^b \operatorname{Mod} A$, where A is a Koszul algebra.

We introduce a general framework for vector bundles on $\mathbb{P}^2$ to investigate Koszul t-structures in the Grothendieck group, relying on the introduction of a deformation parameter t. We apply this framework to $\mathrm{K}\mathbb{P}^2$ to reinterpret and refine some results of Bridgeland [2]—including an explicit algorithm that translates between the quivery subcategories of [2] and exceptional triples on $\mathbb{P}^2$—and to $\mathrm{T}^*\mathbb{P}^2$, where we produce an infinite family of candidates for new Koszul t-structures on $\mathcal{D}^b \operatorname{Coh} \mathrm{T}^*\mathbb{P}^2$. These seem to have interesting properties and, in the classical limit, connections to classical objects, not unlike the case of $\mathrm{K}\mathbb{P}^2$ and the Markov equation.

Contents

1. Introduction

The *Markov equation* is the Diophantine equation

$$x^2 + y^2 + z^2 = xyz. \tag{1}$$

A triple of integers (x, y, z) satisfying (1) is called a *Markov triple*, and each of x, y, and z a *Markov number*. The classical method for solving (1) (see [4], for example) is *Vieta jumping*: We view (1) as a quadratic equation in x, allowing us to "jump" from one Markov triple (x, y, z) to another $(yz - x, y, z)$ by replacing x with the other root of the quadratic, namely $yz - x$.

Theorem 1.1 (Markov)**.** *All Markov triples (up to sign) are generated from the solution* $(3, 3, 3)$ *by repeated jumps and rotations* $(x, y, z) \mapsto (z, x, y)$.

Markov numbers have many interesting properties: For example, they are connected to Diophantine approximation through the work of Springborn [5], who uses them to construct *Markov fractions*, which are central to his description of the "least approximable" rational numbers. They also have an algebro-geometric interpretation via vector bundles on the complex projective plane $\mathbb{P}^2$ (see Section 1.1). This paper seeks to reinterpret and generalize this, by the use of an additional "deforming" parameter, which we introduce in Section 1.2. Our results may thus be viewed as q-analogs (i.e., deformations) of the Markov problem.

1.1. A geometric interpretation.

Rudakov [1] showed that the Markov triples are exactly (thrice) the ranks of *exceptional triples* of vector bundles on $\mathbb{P}^2$. Here, *exceptional* refers to a certain semiorthonormality condition. The Euler pairing matrix (see Section 2) of an exceptional triple (E_0, E_1, E_2) of vector bundles takes the following form[1]

$$\mathsf{P}_{\mathbb{P}^2}(E_0, E_1, E_2) := \Big(\chi(E_i, E_j)\Big)_{ij} = \begin{bmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{bmatrix}, \quad x, y, z \in \mathbb{Z}. \tag{2}$$

In particular, the classes of exceptional triples form a basis of the Grothendieck group $\mathrm{K}(\mathcal{D}^b \operatorname{Coh} \mathbb{P}^2) \simeq \mathbb{Z}^3$. Veselov [6] has recently shown that the Markov fractions of

[1] However, the converse is not true: this condition does not imply exceptionality, at least without additional hypotheses.

Springborn are precisely the slopes of these exceptional vector bundles.

Bondal and Polishchuck [7] showed that the Markov equation can be recovered from the unipotence of the Serre duality functor on $\mathcal{D}^b \operatorname{Coh} \mathbb{P}^2$. Linear algebraically, this implies that if A is the Euler pairing matrix of an exceptional triple (as in (2)), then $A^{-1}A^{\intercal}$ is unipotent. One checks by direct computation that the unipotence of $A^{-1}A^{\intercal}$ is equivalent to (x, y, z) being a Markov triple. Moreover, if we write

$$A_3 \coloneqq \langle \tau_1, \tau_2 \mid \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2 \rangle$$

for the Artin braid group on three strands, then:

Theorem 1.2 (Gorodentsev, Rudakov)**.** *There is a* transitive *action of A_3 on the collection of exceptional triples. The generators τ_i induce Vieta jumps on the corresponding Markov triples.*

Remark. De Thanhoffer de Volcsey and Van den Bergh [8] have used the general setup of Bondal and Polishchuck [7] on other spaces to produce a four-variable analog of the Markov equation. This paper attempts to extend the Markov equation in a different direction, that we now describe.

1.2. Koszulity and q-deformations. Bridgeland [9, 2] interprets Markov triples geometrically in a different—but related—way, using the total space $\mathrm{K}\mathbb{P}^2$ of the canonical (i.e., dualizing) line bundle on $\mathbb{P}^2$. It is a noncompact Calabi–Yau threefold. Let $\mathcal{D}_0 \coloneqq \mathcal{D}^b_0 \operatorname{Coh} \mathrm{K}\mathbb{P}^2$ denote the full subcategory of complexes of quasicoherent sheaves on $\mathbb{P}^2$ with coherent cohomology sheaves supported on the zero section $i\colon \mathbb{P}^2 \hookrightarrow \mathrm{K}\mathbb{P}^2$; then $\mathcal{D}_0$ is dual to the full derived category $\mathcal{D} \coloneqq \mathcal{D}^b \operatorname{Coh} \mathrm{K}\mathbb{P}^2$. We are interested in equivalences

$$\mathcal{D}_0 = \mathcal{D}^b_0 \operatorname{Coh} \mathrm{K}\mathbb{P}^2 \simeq \mathcal{D}^b \operatorname{Mod} A$$

where A is a Koszul ring: a (noncommutative) ring whose category of finitely generated modules $\operatorname{Mod} A$ is cohomologically very well-behaved. Any such equivalence is pinned down by identifying the objects in $\mathcal{D}_0$ that correspond to the simple A-modules. Since $\mathrm{K}(\mathcal{D}_0) \simeq \mathrm{K}(\mathcal{D}^b \operatorname{Coh} \mathbb{P}^2) \simeq \mathbb{Z}^3$, there must be exactly three simples in $\operatorname{Mod} A$, which then correspond to a triple (s_0, s_1, s_2) in $\mathcal{D}_0$. Let us call triples in $\mathcal{D}_0$ arising this way *Koszul triples*. Upon passage to $K(\mathcal{D}_0)$ they yield bases, which we will call *Koszul bases.*

Bridgeland [9] shows that if (E_0, E_1, E_2) is an exceptional triple on $\mathbb{P}^2$, then (i_*E_0, i_*E_1, i_*E_2) is a Koszul triple in $\mathcal{D}_0$. Borrowing the terminology of [9], we say a *quivery triple* is one of the form $\Phi(i_*E_0, i_*E_1, i_*E_2)$, where Φ is an automorphism of $\mathcal{D}_0$. So all quivery triples are Koszul. In Theorem 3.2, we will make progress towards a converse result.

Let

$$B_3 \coloneqq \langle r, \tau_0, \tau_1, \tau_2 \mid \tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j,\ r\tau_i r^{-1} = \tau_{i+1},\ r^3 = 1\rangle$$

be the cyclic braid group on 3 strands. The following result produces many quivery triples:

Theorem 1.3 (Bridgeland)**.** *There is a* free *action of B_3 on quivery triples.*

In general, one cannot recover the automorphism Φ from a quivery triple (we do not even know if it is unique). We describe an algorithm to recover *one* Φ for triples in the B_3-orbit of a fixed starting triple in Section 3.3. As a consequence, we are able to recover the Markov numbers and Markov fractions associated to such a triple.

In general, such an action is difficult to compute with—it corresponds to tilting between t-structures for $\mathrm{K}\mathbb{P}^2$. In our case, however, the fact that quivery triples are Koszul allows us to capture the necessary information entirely in the Grothendieck group by the introduction of a deformation parameter t $(= q^{1/3})$. The q-deformed Grothendieck group $\mathrm{K}(\mathcal{D}^b_0 \operatorname{Coh}^t \mathrm{K}\mathbb{P}^2) \simeq \mathbb{Z}[t^{\pm}]^{\oplus 3}$ becomes a module over the ring of Laurent polynomials $\mathbb{Z}[t^{\pm}]$ and comes with a canonical pairing χ with values in $\mathbb{Z}[t^{\pm}]$. The action of B_3 lifts to this deformation, allowing us to compute with it. To recover the non-deformed version, one simply sets $t = 1$.

1.3. Other spaces. Related phenomena occur for many spaces X arising in geometric representation theory [10]. The category $\mathcal{D}^b \operatorname{Coh} X$ often comes equipped with

- an action of a braid group B on $\mathcal{D}^b \operatorname{Coh} X$;
- a family of equivalences $\mathcal{D}^b \operatorname{Coh} X \simeq \mathcal{D}^b \operatorname{Mod} A$, with A a Koszul algebra, permuted by the action of B;
- a lifting of the B-action to a deformation of $\mathcal{D}^b \operatorname{Coh} X$.

This is known for several X, including Springer resolutions (and slices therein), by the work of Bezrukavnikov, Losev, Mirković, Riche, and Rumynin [11, 12, 13]. It has connections to the study of Bridgeland stability conditions, stable envelopes, equivariant quantum cohomology, and homological mirror symmetry.

This paper studies the cotangent bundle $\mathrm{T}^*\mathbb{P}^2$—another example of a space for which similar results are expected to hold. In particular, in Theorems 4.1 and 4.2 we show that there is a large infinite family of "mutations" that produce different *potentially* Koszul (see Section 2.4) bases in $\mathrm{K}(\mathcal{D}^b_0 \operatorname{Coh}^t \mathrm{T}^*\mathbb{P}^2) \simeq \mathbb{Z}[t^{\pm}]^{\oplus 3}$, with explicit formulae (Theorem 4.5) in terms of more familiar classical objects (Chebyshev polynomials), in the classical limit $t = 1$. However, we do not know whether these bases lift to genuine Koszul triples in $\mathcal{D}^b_0 \operatorname{Coh} \mathrm{T}^*\mathbb{P}^2$.

2. Framework

2.1. The Euler pairing. Let X be a complex variety equipped with an action t of $\mathbb{C}^\times$. Then t induces a grading on the Grothendieck group $K := \mathrm{K}(\mathcal{D}^b \operatorname{Coh}^t X)$ (or any full triangulated subcategory thereof), which is thereby endowed with a structure of a $\mathbb{Z}[t^{\pm}]$-module. There is an inner product $\chi\colon K \times K \to \mathbb{Z}[t^{\pm}]$ called the *Euler pairing*, which is defined on the classes of $E, F \in \mathcal{D}^b \operatorname{Coh}^t X$ by the formula

$$\chi([E],[F]) := \sum_{i,w \in \mathbb{Z}} (-1)^i t^w \dim_{\mathbb{C}} \operatorname{Ext}^i(E,F)_w,$$

where the subscript w denotes taking the wth graded piece—the one on which $t \in \mathbb{C}^\times$ acts by t^w. (In particular, the right-hand side depends only on the classes $[E]$ and $[F]$ and not on their representatives E and F.) The Euler pairing is additive in both variables and sesquilinear:

$$\chi(tu, v) = t^{-1}\chi(u,v) \quad \text{and} \quad \chi(u, tv) = t\chi(u,v).$$

If X is Calabi–Yau, then Serre duality gives us another symmetry:

$$\text{if } \chi(u,v) = p(t) \in \mathbb{Z}[t^{\pm}], \text{ then } \chi(v,u) = (-1)^n t^n p(t^{-1}).$$

Remark. We see that setting $t := 1$—i.e., forgetting the $\mathbb{C}^\times$-action—recovers the Euler pairing on the ordinary Grothendieck group $\mathrm{K}(\mathcal{D}^b \operatorname{Coh} X)$.

Notation. All of our varieties are taken to be defined over $\mathbb{C}$. If $\mathcal{D}$ is a triangulated category, we will generally represent objects of $\mathcal{D}$ using upper case letters (e.g. E, S, ...) and their images in the Grothendieck group $\mathrm{K}(\mathcal{D})$, using the corresponding lowercase letters (i.e., $e \coloneqq [E]$, $s \coloneqq [S]$, ...). The indices for tuples, matrices, vectors, and "mutations" (see below) will be taken modulo 3, and labeled 0, 1, 2.

2.2. Exceptional triples on $\mathbb{P}^2$. We begin with a review of Rudakov's work [1] on $\mathbb{P}^2$. A triple (E_0, E_1, E_2) of objects in $\mathcal{D}^b \operatorname{Coh} \mathbb{P}^2$ is called *exceptional* if for any $0 \leq j \leq i \leq 2$,

$$\operatorname{Ext}^*(E_i, E_j) \simeq \begin{cases} \mathbb{C} & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Thus, an exceptional triple gives us a basis (E_0, E_1, E_2) of $\mathrm{K}(\mathcal{D}^b \operatorname{Coh} \mathbb{P}^2) \simeq \mathbb{Z}^3$. In terms of the basis coming from $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$—which is an exceptional triple—the Euler pairing may be expressed as

$$\chi(u, v) = u^{\mathsf{T}} \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} v.$$

In terms of the Euler product, the braid group action referenced in Section 1.1 is explicitly given on the generators $\tau_1, \tau_2 \in A_3$ by

$$\begin{aligned} \tau_1(u_0, u_1, u_2) &= (u_1, -u_0 + \chi(u_0, u_1)u_1, -u_2) \\ \tau_2(u_0, u_1, u_2) &= (-u_0, u_2, -u_1 + \chi(u_1, u_2)u_2), \end{aligned}$$

and on their inverses by:

$$\begin{aligned} \tau_1^{-1}(u_0, u_1, u_2) &= (-u_1 + \chi(u_0, u_1)u_0, u_0, -u_2) \\ \tau_2^{-1}(u_0, u_1, u_2) &= (-u_0, -u_2 + \chi(u_1, u_2)u_1, u_1). \end{aligned}$$

We now make explicit the connection between exceptional triples and solutions to the Markov equation, as in Theorem 1.2. For any $u_0, u_1, u_2 \in \mathrm{K}(\mathcal{D}^b \operatorname{Coh} \mathbb{P}^2)$, let

$$\mathsf{P}_{\mathbb{P}^2}(u_0, u_1, u_2) \coloneqq \Big(\chi(u_i, u_j)\Big)_{ij}$$

be their *pairing matrix*. (We will omit the subscript $\mathbb{P}^2$ when it is clear from context.) So for an exceptional triple (E_0, E_1, E_2), the pairing matrix $\mathsf{P}(e_0, e_1, e_2)$ is always upper triangular with 1s on the diagonal; in that case, multiplying the above-diagonal (i.e. $(0,1)$th, $(0,2)$th, and $(1,2)$th) entries of $\mathsf{P}(e_0, e_1, e_2)$ by 3 gives a solution to the

Markov equation.

By the work of Veselov [6], the Markov fractions of Springborn [5] are just the slopes (i.e., degree/rank) of the exceptional triples (E_0, E_1, E_2). We can extract this data using linear algebra in the Grothendieck group: Let $\mathsf{C}(u_0, u_1, u_2)$ be the matrix whose ith column is $u_i \in \mathrm{K}(\mathcal{D}^b \operatorname{Coh} \mathbb{P}^2)$, expressed in terms of the basis $(e_0, e_1, e_2) := ([\mathcal{O}], [\mathcal{O}(1)], [\mathcal{O}(2)])$. Then it is easy to see that the entries of the column vector

$$\mathsf{C}^{\mathsf{T}}(u_0, u_1, u_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

are exactly the *ranks* of (u_0, u_1, u_2). Similarly, the elements of the column vector

$$\mathsf{C}^{\mathsf{T}}(u_0, u_1, u_2) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

are exactly the *degrees* of (u_0, u_1, u_2). It will be crucial later that the isomorphism class of an *exceptional* vector bundle is uniquely determined by its rank and degree [1]; in particular, the procedure above allows us to recover an exceptional triple from its image in the Grothendieck group.

2.3. Koszul bases on $\mathrm{T}^*\mathbb{P}^2$ and $\mathrm{K}\mathbb{P}^2$. We consider analogous objects on two related spaces, both vector bundles over $\mathbb{P}^2$: the canonical bundle $\mathrm{K}\mathbb{P}^2$ and cotangent bundle $\mathrm{T}^*\mathbb{P}^2$. They are both noncompact Calabi–Yau varieties, equipped with a scaling action of $q \in \mathbb{C}^\times$ on fibers. We normalize the actions by defining $t := q^{1/3}$ and $t := q^{1/2}$, respectively.

If X is either of these spaces, we have the zero section $i \colon \mathbb{P}^2 \hookrightarrow X$ and projection $\pi \colon X \to \mathbb{P}^2$, inducing maps

$$i_* \colon \mathcal{D}^b \operatorname{Coh} \mathbb{P}^2 \to \mathcal{D}_0 \quad \text{and} \quad \pi^* \colon \mathcal{D}^b \operatorname{Coh} \mathbb{P}^2 \to \mathcal{D},$$

where $\mathcal{D} := \mathcal{D}^b \operatorname{Coh}^t X$ denotes the bounded derived category of $\mathbb{C}^\times$-equivariant coherent sheaves on X and $\mathcal{D}_0$ is the full subcategory of complexes whose cohomology sheaves are supported on the zero section. Then $\mathrm{K}(\mathcal{D})$ and $\mathrm{K}(\mathcal{D}_0)$ are both free $\mathbb{Z}[t^{\pm}]$-modules of rank 3.

If (F_0, F_1, F_2) is an exceptional triple on $\mathbb{P}^2$, then applying π^* and i_* gives us bases for $\mathrm{K}(\mathcal{D})$ and $\mathrm{K}(\mathcal{D}_0)$, respectively. Moreover, every exceptional triple has a unique *dual* exceptional triple E_0, E_1, E_2, characterized by

$$\mathrm{Ext}^*(F_i, E_j) \simeq \begin{cases} \mathbb{C} & i = j \\ 0 & i \neq j. \end{cases}$$

A key result is the following; it was proved by Bridgeland [3] for $\mathrm{K}\mathbb{P}^2$, but one checks by computing Ext-groups that it is also true for $\mathrm{T}^*\mathbb{P}^2$.

Theorem 2.1. *Let* $B \coloneqq \mathrm{End}_X(\pi^* F_0 \oplus \pi^* F_1 \oplus \pi^* F_2)$. *Then,*

(a) B *is a Koszul algebra;*

(b) $\mathcal{F} \coloneqq \mathrm{Ext}^*_X(\pi^* F_0 \oplus \pi^* F_1 \oplus \pi^* F_2, -) \colon \mathcal{D} \to \mathcal{D}^b \,\mathrm{Mod}\, B$ *is an equivalence;*

(c) $\mathcal{F}$ *restricts to an equivalence* $\mathcal{F}_0 \colon \mathcal{D}_0 \to \mathcal{D}^b_0 \,\mathrm{Mod}\, B$ *onto the full subcategory of complexes with nilpotent cohomology;*

(d) *the* $S_i \coloneqq i_* E_i \langle i \rangle$ *(here* $\langle i \rangle$ *denotes a twist by the action of* $t \in \mathbb{C}^\times$*) are sent by* $\mathcal{F}_0$ *(and* $\mathcal{F}$*) to the simple objects of* $\mathrm{Mod}\, B$.

Thus, passing to the Grothendieck group, an exceptional collection on $\mathbb{P}^2$ gives us a *Koszul basis* (s_0, s_1, s_2) of $\mathrm{K}(\mathcal{D}_0) \simeq K(\mathcal{D}) \simeq \mathbb{Z}[t^\pm]^{\oplus 3}$, as defined in Section 1.2. We would like to study the collection of Koszul basis; from the discussion so far, we see that the problem of classifying Koszul bases may be viewed as a q-extension of the Markov problem.

2.4. Koszulity in the K-group. We adopt much of the same notation as used in Section 2.2. Fix the basis (e_0, e_1, e_2) of $\mathrm{K}(\mathcal{D}_0)$ coming from the exceptional triple $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$, for which the dual exceptional triple is

$$(\mathcal{O}, \Omega^1(1)[1], \Omega^2(2)[2]) \simeq (\mathcal{O}, \Omega^1(1)[1], \mathcal{O}(-1)[2]).$$

Thus, e_i is the class of $i_* \Omega^i(i)[i]\langle i \rangle$. Now for any basis (s_0, s_1, s_2) of $\mathrm{K}(\mathcal{D}_0)$, we have the *column matrix* $\mathsf{C}(s_0, s_1, s_2)$ and *pairing matrix* $\mathsf{P}(s_0, s_1, s_2)$, with entries in $\mathbb{Z}[t^\pm]$; here C is defined with respect to the basis (e_0, e_1, e_2) fixed above, but P is independent of this choice.

We will occasionally add subscripts (i.e. $\mathsf{P}_{\mathrm{K}\mathbb{P}^2}$), when it is unclear what space we are referring to. Finally, when the column matrix is evaluated at $t = 1$, we denote it by $\mathsf{C}_1(s_0, s_1, s_2)$.

Remark. One might ask why we chose to work with $\mathcal{D}_0$, introducing an extra duality and a support condition, instead of with $\mathcal{D}$? While we could have worked with $\mathcal{D}$ in theory, its pairing matrices have infinitely many terms so computations quickly become unreasonable.

The following (fairly strong) linear algebraic conditions are necessary (but not sufficient) for a basis (s_0, s_1, s_2) to be Koszul (cf. [14]).

- $\mathsf{P}(s_0, s_1, s_2)$ has entries in $\mathbb{Z}[t]$—i.e., no negative powers of t.
- Even powers of t in $\mathsf{P}(s_0, s_1, s_2)$ must have positive coefficients and odd powers of t must have negative coefficients.
- $\mathsf{P}(s_0, s_1, s_2)|_{t=0}$ is the identity matrix.
- The power series expansions (in t) of the entries of $\mathsf{P}^{-1}(s_0, s_1, s_2)$ only have nonnegative coefficients.

We say such a basis is *potentially Koszul*.

3. The canonical bundle

In this section, we specialize to $X \coloneqq \mathrm{K}\mathbb{P}^2$, the total space of the canonical bundle (i.e. $\mathcal{O}(-3)$) on $\mathbb{P}^2$. We prove some results that connect Koszul triples in $\mathcal{D}_0 \coloneqq \mathcal{D}^b_0 \operatorname{Coh} \mathrm{K}\mathbb{P}^2$ with the Markov numbers, reinterpreting parts of Bridgeland's work [3, 2] in this framework. Finally, we also prove some results towards recovering an exceptional triple on $\mathbb{P}^2$ from Koszul triples in the B_3-orbit considered in *loc. cit.*

3.1. Markov bases. With respect to the basis chosen in Section 2.4, the Euler pairing for $\mathrm{K}(\mathcal{D}_0) \simeq \mathbb{Z}[t^{\pm}]^{\oplus 3}$ is given by

$$\chi(u(t), v(t)) = u(t^{-1})^{\intercal} \begin{bmatrix} 1-t^3 & 3t^2 & -3t \\ -3t & 1-t^3 & 3t^2 \\ 3t^2 & -3t & 1-t^3 \end{bmatrix} v(t).$$

Let us denote this matrix—which is just $\mathsf{P}_{\mathrm{K}\mathbb{P}^2}(e_0, e_1, e_2)$—by $A_{\mathrm{K}\mathbb{P}^2}$. Then the image of Bridgeland's braid group action (cf. [2, Section 2.3]) in the equivariant Grothendieck group is given on generators by

$$\begin{aligned}
\tau_1(s_0, s_1, s_2) &= \left(ts_1, t^{-1}(-s_0 + \chi_1(s_1, s_0)s_1), -s_2 + \chi_1(s_1, s_2)s_1\right), \\
r(s_0, s_1, s_2) &= (s_2, s_1, s_0).
\end{aligned}$$

We also have the cyclic variants of τ_1, defined by $r\tau_i r^{-1} = \tau_{i+1}$.

Remark. Observe that the pairing matrix $\mathsf{P}(\tau_i(s_0, s_1, s_2))$ depends only on the pairing matrix $\mathsf{P}(s_0, s_1, s_2)$. So we have a well-defined action of the τ_i on pairing matrices.

Let us say a pairing matrix $\mathsf{P}(s_0, s_1, s_2)$ is *Markov* if it is of the form

$$\begin{bmatrix} 1-t^3 & at^2 & -ct \\ -at & 1-t^3 & bt^2 \\ ct^2 & -bt & 1-t^3 \end{bmatrix},$$

for (a, b, c) a solution to the Markov equation. In that case, we say (s_0, s_1, s_2) is a *Markov* basis. For example, the pairing matrix $A_{\mathrm{K}\mathbb{P}^2}$ is Markov, and so (e_0, e_1, e_2) is a Markov basis.

The following result is shown in [9], but we reprove it in our framework:

Theorem 3.1. *The operations τ_i and r described above indeed produce an action of the (cyclic) braid group*

$$B_3 \coloneqq \langle r, \tau_0, \tau_1, \tau_2 \mid \tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j,\ r\tau_i r^{-1} = \tau_{i+1},\ r^3 = 1\rangle.$$

on the collection of Markov bases and Markov pairing matrices. The B_3-orbit of $A_{\mathrm{K}\mathbb{P}^2} \coloneqq \mathsf{P}(e_0, e_1, e_2)$ is in bijection with solutions to the Markov equation.

Proof. It is clear that r preserves the Markov property. We can check how the τ_i act on Markov pairing matrices:

$$\tau_1\colon \begin{bmatrix} 1-t^3 & at^2 & -ct \\ -at & 1-t^3 & bt^2 \\ ct^2 & -bt & 1-t^3 \end{bmatrix} \mapsto \begin{bmatrix} 1-t^3 & at^2 & -bt \\ -at & 1-t^3 & (ab-c)t^2 \\ bt^2 & -(ab-c)t & 1-t^3 \end{bmatrix}$$

$$\tau_2\colon \begin{bmatrix} 1-t^3 & at^2 & -ct \\ -at & 1-t^3 & bt^2 \\ ct^2 & -bt & 1-t^3 \end{bmatrix} \mapsto \begin{bmatrix} 1-t^3 & ct^2 & -(bc-a)t \\ -ct & 1-t^3 & bt^2 \\ (bc-a)t^2 & -bt & 1-t^3 \end{bmatrix} \tag{3}$$

$$\tau_0\colon \begin{bmatrix} 1-t^3 & at^2 & -ct \\ -at & 1-t^3 & bt^2 \\ ct^2 & -bt & 1-t^3 \end{bmatrix} \mapsto \begin{bmatrix} 1-t^3 & (ac-b)t^2 & ct \\ -(ac-b)t & 1-t^3 & at^2 \\ ct^2 & -at & 1-t^3 \end{bmatrix}$$

Thus, the transformations τ_i are analogous to the traditional Vieta jumps (by considering how the coefficients a, b, c change, up to permutation). The initial matrix $A_{\mathrm{K}\mathbb{P}^2}$ has coefficients $(3, 3, 3)$, which is the "primitive" solution to the Markov equation, so the orbit is indeed in bijection with solutions of the Markov equation.

Let us now show that the braid group relations hold. We have

$$\begin{aligned}
(s_0, s_1, s_2) &\overset{\tau_1}{\longmapsto} \left(ts_1, \frac{1}{t}(-s_0 - ats_1), -s_2\right) \\
&\overset{\tau_0}{\longmapsto} \left(\frac{1}{t}(s_2 - bt^2 s_1), \frac{1}{t}(s_0 + as_1), t^2 s_1\right) \\
&\overset{\tau_1}{\longmapsto} \left(s_0 + ats_1, \frac{1}{t^2}(-s_2 + t^2(b - ac)s_1 - cts_0), -t^2 s_1\right)
\end{aligned}$$

and

$$\begin{aligned}
(s_0, s_1, s_2) &\overset{\tau_0}{\longmapsto} \left(\frac{1}{t}(-s_2 - cs_0), -s_1, ts_0\right) \\
&\overset{\tau_1}{\longmapsto} \left(-ts_1, \frac{1}{t^2}(s_2 + tcs_0 + t(ac - b)), -ts_0\right) \\
&\overset{\tau_0}{\longmapsto} \left(s_0 + ats_1, \frac{1}{t^2}(-s_2 + t^2(b - ac)s_1 - cts_0, -t^2 s_1\right).
\end{aligned}$$

Thus, $\tau_0\tau_1\tau_0 = \tau_1\tau_0\tau_1$ and the other braid relations follow analogously. The relations $r\tau_i r^{-1} = \tau_{i+1}$ are clear. □

Remark. In fact, the above action of the braid group is free [9, Theorem 5.6].

If we assume that τ_i preserves Koszulity, then we can show that all potentially Koszul pairing matrices are Markov, making progress towards classifying Koszul t-structures on $\mathcal{D}_0$.

Theorem 3.2. *Suppose τ_i preserves Koszulity. Then all Koszul bases are Markov.*

Proof. Suppose (s_0, s_1, s_2) is a Koszul basis. For simplicity, we refer to the pairing matrix $\mathsf{P}_{\mathrm{K}\mathbb{P}^2}(s_0, s_1, s_2)$ simply by P and the column matrix $\mathsf{C}_{\mathrm{K}\mathbb{P}^2}(s_0(t), s_1(t), s_2(t))$ simply by $C(t)$.

We will first show that $\det P = (1-t^3)^3$. Note that $P = C(t^{-1})^\intercal A_{\mathrm{K}\mathbb{P}^2} C(t)$, so,

$$\det(P) = \det(C(t^{-1})^\intercal)\det(A_{\mathrm{K}\mathbb{P}^2})\det(C(t)).$$

Since $s_0(t), s_1(t), s_2(t)$ is a basis, $\det(C)$ must be a unit in $\mathbb{Z}[t^\pm]^{\oplus 3}$; that is, $\det(C) = \pm t^n$ for $n \in \mathbb{Z}$. Then, $\det(C(t^{-1})^\intercal) = \pm t^{-n}$. Thus,

$$\det(P) = \det(A_{\mathrm{K}\mathbb{P}^2}) = (1-t^3)^3,$$

as desired.

Now, we will show that s_i must be *spherical*, that is, $\chi(s_i, s_i) = 1 - t^3$ for all i. Note that, since $\chi(s_0, s_0) = s_0(t^{-1})A_{\mathrm{K}\mathbb{P}^2}s_0(t)$, the Laurent polynomial $\chi(s_i, s_i)$ is antipalindromic, and further, $\chi(s_i, s_j) = -t^3\chi(s_j, s_i)(t^{-1})$. The first three conditions for (potential) Koszulity (see Section 2.4) imply that P must take the following form:

$$\begin{bmatrix} 1 - xt + xt^2 - t^3 & at^2 - bt & ft^2 - et \\ bt^2 - at & 1 - yt + yt^2 - t^3 & ct^2 - dt \\ et^2 - ft & dt^2 - ct & 1 - zt + zt^2 - t^3 \end{bmatrix},$$

for $a, b, c, d, e, f, x, y, z \in \mathbb{Z}_{\geq 0}$. The fourth condition for (potential) Koszulity imposes a sign restriction on the coefficients of the power series expansion of P^{-1}. We explicitly compute an element of the matrix P^{-1} using the power series expansion of $\frac{1}{\det(P)} = \frac{1}{(1-t^3)^3}$:

$$(P^{-1})_{2,2} = \left[(1 - xt + xt^2 - x^3)(1 - yt + yt^2 - t^3) - (at^2 - bt)(bt^2 - at)\right](1 + t^3 + t^6 + \cdots)^3.$$

Taking the linear term, we get $-(x + y) \geq 0$. Symmetrically, $-(y + z) \geq 0$ and $-(z + x) \geq 0$. Since x, y, and z are nonnegative, they must all vanish—i.e., the s_i are spherical.

Finally, if τ_1 preserves Koszulity, then the pairing matrix of $(s'_0, s'_1, s'_2) \coloneqq \tau_1(s_0, s_1, s_2)$ must also take on the same form. But, a direct computation reveals that $\chi(s'_0, s'_1) = -b + at^2$, so $b = 0$. Analogously, d and f must also be 0. Finally, we can compute:

$$\det \begin{bmatrix} 1-t^3 & at^2 & -et \\ -at & 1-t^3 & ct^2 \\ et^2 & -ct & 1-t^3 \end{bmatrix} = -t^9+(a^2+c^2+e^2-ace+3)t^6-(a^2+c^2+e^2-ace+3)t^3-1.$$

But we know that the determinant of P is $(1-t^3)^3$, so we must have that a, c, e are solutions to the Markov equation, as desired. □

3.2. Ranks, degrees, and Markov fractions. We have seen that there is an action of B_3 on the collection of Markov triples, and that exceptional triples on $\mathbb{P}^2$ give rise to Markov triples in $\mathcal{D}_0$ via the pushforward i_* (of the dual triple). We will call such triples $\mathcal{D}_0$ *exceptional* as well; not all Markov triples are exceptional, but we might hope that they are "almost" so:

Can we associate an exceptional triple on $\mathbb{P}^2$ to a Markov triple on $\mathrm{K}\mathbb{P}^2$?

More concretely, it is known [1] that exceptional vector bundles on $\mathbb{P}^2$ are determined by their degree and rank, so we would like to recover these invariants from a Markov triple. As a byproduct, their quotient—i.e., the slope—allows us to produce the Markov fractions of Springborn [5] from the geometry of $\mathrm{K}\mathbb{P}^2$.

Recall that by Theorem 1.2, there is an action of A_3 on the collection of exceptional triples on $\mathbb{P}^2$. Thus, we have

$$A_3 \circlearrowright \{\text{exc. trip. } /\mathbb{P}^2\} \xrightarrow{\text{dual}} \{\text{exc. trip. } /\mathbb{P}^2\} \xrightarrow{i_*} \{\text{Markov trip.} \in \mathcal{D}_0\} \circlearrowleft B_3,$$

and a natural question is whether the action of A_3 on the left is the same as its action on the right, where A_3 here is viewed as the subgroup of B_3 generated by τ_1 and τ_2. This is indeed the case:

Theorem 3.3. *The action of A_3 on exceptional triples on $\mathbb{P}^2$ is compatible with the action of B_3 on Markov triples on $\mathrm{K}\mathbb{P}^2$. In particular,* $\mathrm{Orbit}_{A_3}(e_0, e_1, e_2)$ *consists of exceptional triples.*

Proof. Let $M_0 = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right]$, $M_1 = I_3$, and $M_2 = \left[\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$. We will prove the theorem by showing the following:

$$\mathsf{C}_1\left(\alpha_n \cdots \alpha_1(e_0, e_1, e_2)\right) = \mathsf{C}_1^{-1\intercal}\left(i_* \alpha_n^{-1} \cdots \alpha_1^{-1}(e_0, e_1, e_2)\right) M_{r_n}$$

where α_j for $1 \leq j \leq n$ is the mutation $\tau_1^{\pm}$ or $\tau_2^{\pm}$ and $r_n \in \{0,1,2\}$; here the inverses on the right-hand side arise because one takes duals before applying i_*. Let $(u_0, u_1, u_2) \coloneqq \alpha_{n-1}^{-1} \cdots \alpha_1^{-1}(e_0, e_1, e_2)$ and its dual be (s_0, s_1, s_2). Also let $(s_0', s_1', s_2') \coloneqq \alpha_{n-1} \cdots \alpha_1(e_0, e_1, e_2)$.

We proceed by induction on n, where the base case $n = 1$ is clear. Without loss of generality, we may assume $\alpha_n = \tau_1$; the other cases follow analogously. Then,

$$\begin{aligned} \mathsf{C}_1\left(\alpha_n \cdots \alpha_1(e_0, e_1, e_2)\right) &= \mathsf{C}_1(\tau_1(s_0', s_1', s_2')) \\ &= \mathsf{C}_1(i_*s_0, i_*s_1, i_*s_2) M_{r_{n-1}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & \chi_{\mathrm{K}\mathbb{P}^2}(s_1', s_0') & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

where $\mathsf{C}_1(s_0', s_1', s_2') = \mathsf{C}_1(i_*(s_0, s_1, s_2)) M_{r_{n-1}}$ follows from our induction hypothesis. Further,

$$\begin{aligned} \mathsf{C}_1^{-1\intercal}\left(i_*\alpha_n^{-1} \cdots \alpha_1^{-1}(e_0, e_1, e_2)\right) &= \mathsf{C}_1^{-1\intercal}(i_*\tau_1^{-1}(u_0, u_1, u_2)) \\ &= \mathsf{C}_1(i_*s_0, i_*s_1, i_*s_2) \begin{bmatrix} \chi_{\mathbb{P}^2}(u_0, u_1) & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1\intercal} \\ &= \mathsf{C}_1(i_*s_0, i_*s_1, i_*s_2) \begin{bmatrix} 0 & 1 & 0 \\ -1 & \chi_{\mathbb{P}^2}(u_0, u_1) & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

where the second equality comes from the fact that taking the dual and then applying i_* is the same as doing i_* first and then taking the dual.

We now relate the pairing matrices of (s_0, s_1, s_2) and (u_0, u_1, u_2) over $\mathbb{P}^2$. Let $T_{s \to u}$ be the transition matrix from the basis (s_0, s_1, s_2) to (u_0, u_1, u_2). Since (u_0, u_1, u_2) is dual to (s_0, s_1, s_2), we can rewrite our pairing matrix formula in terms of the basis (s_0, s_1, s_2) and get:

$$\begin{aligned} \chi(u_i, s_j) = \delta_{i,j} \ \forall i, j \implies \\ T_{s \to u} \mathsf{P}_{\mathbb{P}^2}(s_0, s_1, s_2) T_{s \to s}^{\intercal} = I_3 \implies \\ T_{s \to u} = \mathsf{P}_{\mathbb{P}^2}^{-1}(s_0, s_1, s_2). \end{aligned}$$

Thus, we have

$$\begin{aligned}\mathsf{P}_{\mathbb{P}^2}(u_0,u_1,u_2) &= T_{s\to u}\mathsf{P}_{\mathbb{P}^2}(s_0,s_1,s_2)T_{s\to u}^{\mathsf{T}} \\ &= \mathsf{P}_{\mathbb{P}^2}^{-1}(s_0,s_1,s_2)P_{\mathbb{P}^2}(s_0,s_1,s_2)\mathsf{P}_{\mathbb{P}^2}^{-1\mathsf{T}}(s_0,s_1,s_2) \\ &= \mathsf{P}_{\mathbb{P}^2}^{-1\mathsf{T}}(s_0,s_1,s_2).\end{aligned}$$

We can also relate the pairing matrices of (i_*s_0, i_*s_1, i_*s_2) and (u_0,u_1,u_2) over their respective spaces using the definition of the Euler product in terms of Ext-groups. We get

$$\begin{aligned}\mathsf{P}_{\mathrm{K}\mathbb{P}^2}(i_*s_0,i_*s_1,i_*s_2) &= \mathsf{P}_{\mathbb{P}^2}(s_0,s_1,s_2) - t^3\mathsf{P}_{\mathbb{P}^2}^{\mathsf{T}}(s_0,s_1,s_2) \\ &= \mathsf{P}_{\mathbb{P}^2}^{-1\mathsf{T}}(u_0,u_1,u_2) - t^3\mathsf{P}_{\mathbb{P}^2}^{-1}(u_0,u_1,u_2).\end{aligned}$$

Since (u_0,u_1,u_2) is an exceptional triple, its pairing matrix is upper triangular. We can then explicitly compute the $(1,0)$th entry of these matrices:

$$\begin{aligned}\chi_{\mathrm{K}\mathbb{P}^2}(s_1',s_0') &= (-1)^{\delta_{a_{n-1},0}}\chi_{\mathrm{K}\mathbb{P}^2}(i_*s_1,i_*s_0) \\ &= (-1)^{\delta_{a_{n-1},0}}\left(-\chi_{\mathbb{P}^2}(u_0,u_1) + \chi_{\mathbb{P}^2}(u_0,u_2)\chi_{\mathbb{P}^2}(u_2,u_1)\right) \\ &= (-1)^{\delta_{a_{n-1},0}+1}\chi_{\mathbb{P}^2}(u_0,u_1).\end{aligned}$$

It is now straightforward to verify the claimed statement. We show the computations explicitly for $r_{n-1}=0$; the other cases are done analogously:

$$\begin{aligned}\mathsf{C}_1\left(\alpha_n\cdots\alpha_1(e_0,e_1,e_2)\right) &= \mathsf{C}_1(i_*s_0,i_*s_1,i_*s_2)M_0\begin{bmatrix}0 & -1 & 0\\ 1 & \chi_{\mathrm{K}\mathbb{P}^2}(s_1',s_0') & 0\\ 0 & 0 & -1\end{bmatrix} \\ &= \mathsf{C}_1(i_*s_0,i_{s_1},i_{s_2})\begin{bmatrix}0 & -1 & 0\\ -1 & -\chi_{\mathbb{P}^2}(u_0,u_1) & 0\\ 0 & 0 & 1\end{bmatrix} \\ &= \mathsf{C}_1(i_*s_0,i_{s_1},i_{s_2})\begin{bmatrix}0 & 1 & 0\\ -1 & \chi_{\mathbb{P}^2}(u_0,u_1) & 0\\ 0 & 0 & -1\end{bmatrix}M_0 \\ &= \mathsf{C}_1^{-1\mathsf{T}}\left(i_*\alpha_n^{-1}\cdots\alpha_1^{-1}(e_0,e_1,e_2)\right)M_1,\end{aligned}$$

as desired. Similarly, if $r_{n-1}=1$, then $r_n=2$, and if $r_{n-1}=2$, then $r_n=1$.

Therefore, we have proven our claim and shown that applying an action in $A_3 \subset B_3$ on $\mathcal{D}^b(\mathrm{Coh}\,\mathrm{K}\mathbb{P}^2)$ corresponds to applying an action in A_3 on exceptional vector bundles over $\mathbb{P}^2$ (through the i_* and duality maps). □

Remark. The reason τ_0 on $\mathrm{K}\mathbb{P}^2$ does not also have a counterpart on $\mathbb{P}^2$ is that the intrinsic ordering of an exceptional triple is not respected by τ_0.

We *could* define an operation "τ_0" on exceptional triples (E_0, E_1, E_2) over $\mathbb{P}^2$ that swaps E_0 and E_2 and then performs Gram–Schmidt to preserve exceptionality. But this can actually be expressed in terms of τ_1 and τ_2: it corresponds to $\tau_2\tau_1\tau_2^{-1}$.

In this sense, the mutation τ_0 that arises on $\mathrm{K}\mathbb{P}^2$ is therefore something new, and comes from the "cyclic" nature of the canonical bundle.

We start by extracting a rank from $\mathrm{Orbit}_{B_3}(e_0, e_1, e_2)$. The following notation will be convenient. For a word $w := a_m a_{m-1} \cdots a_1$ with $a_i \in \{\tau_0^{\pm}, \tau_1^{\pm}, \tau_2^{\pm}\}$ for all i, we let $[a_k]_w$ be the matrix taking the basis $a_{k-1} \cdots a_1(e_0, e_1, e_2)$ to $a_k \cdots a_1(e_0, e_1, e_2)$. We will drop the subscript w if it is clear.

Proposition 3.4. *The ranks of the exceptional triple corresponding to* $(s_0, s_1, s_2) \in \mathrm{Orbit}_{A_3}(e_0, e_1, e_2)$ *are the elements of*

$$\mathsf{C}_1^{-1}(s_0, s_1, s_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Remark. In the sequel, we allow B_3 in place of A_3 and take this as the *definition* of the ranks of (s_0, s_1, s_2). The proof will show that these are still Markov numbers.

Proof. Let $\vec{v}(s_0, s_1, s_2) := \mathsf{C}_1^{-1}(s_0, s_1, s_2) \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]$. We claim that the ith entry $v_i(s_0, s_1, s_2)$ of this vector is $\pm\frac{\chi(s_{i+1}, s_{i-1})}{3}$, where the sign is the same for all i. We proceed by induction—it suffices to show that $(s_0', s_1', s_2') := \tau_0(s_0, s_1, s_2)$ preserves this property; the others follow cyclically. Let (u_0, u_1, u_2) be the basis dual to (s_0, s_1, s_2) with respect to χ. Then, we have:

$$\begin{aligned}\mathsf{C}_1^{-1}(s_0', s_1', s_2') &= [\tau_0]^{-1} C_1^{-1}(s_0, s_1, s_2) \\ &= \begin{bmatrix} \chi(s_0, s_2) & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \mathsf{C}_1^{\intercal}(u_0, u_1, u_2) \\ &= \mathsf{C}_1^{\intercal}(-u_2, -u_1, u_0 + \chi(s_0, s_2)u_2).\end{aligned}$$

By the proof of Theorem 3.1 (specifically, (3)), we know exactly how the pairing matrices of (s_0', s_1', s_2') and (s_0, s_1, s_2) compare. We use this to verify the property claimed for the $v_i(s_0', s_1', s_2')$:

$$\begin{aligned}v_0(s_0', s_1', s_2') &= -u_2 \cdot \left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right] = \frac{\mp\chi(s_0, s_1)}{3} = \frac{\mp\chi(s_1', s_2')}{3} \\ v_1(s_0', s_1', s_2') &= -u_1 \cdot \left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right] = \frac{\mp\chi(s_2, s_0)}{3} = \frac{\mp\chi(s_2', s_0')}{3} \\ v_2(s_0', s_1', s_2') &= (u_0 + \chi_1(s_0, s_2)u_2) \cdot \left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right] \\ &= \frac{\pm\chi(s_1, s_2) \pm \chi(s_0, s_2)\chi(s_0, s_1)}{3} \\ &= \frac{\pm\chi(s_1, s_2) \mp \chi(s_2, s_0)\chi(s_0, s_1)}{3} \\ &= \frac{\mp\chi(s_0', s_1')}{3},\end{aligned}$$

as desired. In particular, this means that the elements of $\vec{v}(s_0, s_1, s_2)$ are exactly Markov numbers (divided by 3), and are therefore equal to the ranks of the corresponding exceptional triple over $\mathbb{P}^2$, since the $\chi(s_{i+1}, s_{i-1})$ and the ranks both transform similarly under the action of B_3. □

We can similarly extract an appropriate degree from $\mathrm{Orbit}_{A_3}(s_0, s_1, s_2)$.

Proposition 3.5. *The degrees of the exceptional triple corresponding to* $(s_0, s_1, s_2) \in \mathrm{Orbit}_{A_3}(e_0, e_1, e_2)$ *are (up to sign) the elements of*

$$\mathsf{C}_1^{-1}(s_0, s_1, s_2) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Proof. The ranks of an exceptional triple (E_0, E_1, E_2) on $\mathbb{P}^2$ can be extracted using

$\mathsf{C}_1^\intercal(e_0, e_1, e_2) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, essentially by definition. From our work in Theorem 3.3, we have

$$\mathsf{C}_1^{-1}(s_0, s_1, s_2) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = M_{r_n} \mathsf{C}_1^\intercal(i_* e_0, i_* e_1, i_* e_2) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Multiplying by M_{r_n} on the left is equivalent to negating some of the rows, so $\mathsf{C}_1^{-1}(s_0, s_1, s_2) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ extracts the degree. □

3.3. Reduction from B_3 to A_3. We now present an algorithm to transform any element of $\mathrm{Orbit}_{B_3}(e_0, e_1, e_2)$ into an element of $\mathrm{Orbit}_{A_3}(e_0, e_1, e_2)$. Doing so allows us to algorithmically describe how to compute the degrees, ranks, and Markov fractions for the entire orbit. We will recursively define a vector $f(s_0, s_1, s_2)$ such that $\mathsf{C}_1^{-1}(s_0, s_1, s_2) f(s_0, s_1, s_2)$ outputs a triple of degrees associated with the triple of ranks we extracted in (the remark after) Proposition 3.4.

Consider some action $\alpha_m \alpha_{m-1} \cdots \alpha_1(e_0, e_1, e_2)$ for $\alpha_i \in \{r, \tau_j, \tau_j^{-1}\}$. Using the relation $r\tau_i r^{-1} = \tau_{i+1}$, we can rewrite this action as $r^n \beta_m \beta_{m-1} \cdots \beta_1$ for $\beta_i \in \{\tau_1^\pm, \tau_2^\pm, (r^{-1}\tau_0)^\pm\}$. Note that doing a rotation at the very end corresponds to permuting the rows of the dual matrix, which does not substantially affect the extraction of ranks and/or degrees. So we may write $f(r^n \beta_m \beta_{m-1} \cdots \beta_1) = f(\beta_m \beta_{m-1} \cdots \beta_1)$, and henceforth only consider $n = 0$.

Let k be the smallest index such that $\beta_k = (r^{-1}\tau_0)^\pm$. Then, set

$$u \coloneqq \beta_m \beta_{m-1} \cdots \beta_1(e_0, e_1, e_2) \quad \text{and} \quad v \coloneqq \beta_m \cdots \beta_{k+1} \tau_1^{-1} \beta_{k-1} \cdots \beta_1(e_0, e_1, e_2).$$

Note that $[\beta_i]_u = [\beta_i]_v$ for $i < k$ because u and v are the same up to the kth operation. Thus, we drop the subscript on our $[\beta_i]$ matrices for $i < k$. We then have the following recursion:

$$f(u) = [\beta_1][\beta_2] \cdots [\beta_{k-1}]([\tau_0]_u [r^{-1}])^\pm [\tau_1^{-1}]_v^{-1} [\beta_{k-1}]^{-1} \cdots [\beta_1]^{-1} f(v),$$

where each matrix is computed at $t = 1$. Note that $[\tau_1^{-1}]_v^{-1}$ is equal to the matrix $[\tau_1]$ that takes $\mathsf{C}_1(\beta_{k-1} \cdots \beta_1)$ to $\mathsf{C}_1(\tau_1 \beta_{k-1} \cdots \beta_1)$, but we write it in this way to clarify exactly what pairing matrix $[\tau_1^{-1}]_v^{-1}$ is defined by.

To prove that this works, we first show that the ranks associated with u and v are the same. It suffices to show that $r^{-1}\tau_0$ and τ_1^{-1} do the same operation to the rank. Indeed, it is easy to verify that $r^{-1}\tau_0$ and τ_1^{-1} correspond to the same Vieta jump on the pairing matrix: $(a, b, c) \mapsto (a, c, ac - b)$. By Proposition 3.4, since the ranks are innately related to the pairing matrix, this implies that they do the same operation to the rank.

So we will associate to u and v the same degrees: $\mathsf{C}_1^{-1}(u)f(u) = \mathsf{C}_1^{-1}(v)f(v)$. We can simplify this further, noting that in fact $[\beta_i]_u = [\beta_i]_v$ for all i because the pairing matrices are the same at every step, so we can drop the subscripts almost everywhere:

$$\begin{aligned} f(u) &= C_1(u)C_1^{-1}(v)f(v) \\ &= ([\beta_1]\cdots[\beta_{k-1}]([\tau_0]_u[r^{-1}])^{\pm}[\beta_{k+1}]\cdots[\beta_m]) \\ &\qquad \cdot ([\beta_m]^{-1}\cdots[\beta_{k+1}]^{-1}[\tau_1^{-1}]_v^{-1}[\beta_{k-1}]^{-1}\cdots[\beta_1^{-1}])f(v) \\ &= [\beta_1]\cdots[\beta_{k-1}]([\tau_0]_u[r^{-1}])^{\pm}[\tau_1^{-1}]_v^{-1}[\beta_{k-1}]^{-1}\cdots[\beta_1]^{-1}f(v), \end{aligned}$$

which is exactly the recurrence we sought.

In particular, using this, we can repeatedly shift the $\tau_0^{\pm}$ term of minimal index to the left until we are left with a word of just $\tau_1^{\pm}$'s and $\tau_2^{\pm}$'s. But, by Proposition 3.5, we know exactly how to extract the degree (and Markov fractions) of these!

Remark. This algorithm explicitly describes an equivalence Φ of derived categories between the quivery subcategory coming from $\mathrm{Orbit}_{B_3}(e_0, e_1, e_2)$ and the exceptional subcategory coming from $\mathrm{Orbit}_{A_3}(e_0, e_1, e_2)$.

Example. We use the recurrence on $u = \tau_0\tau_2$ and $v = \tau_1^{-1}\tau_2$ to provide a concrete example of how the algorithm works on a small case. We have:

$$\begin{aligned} f(\tau_0\tau_2) &= f\left((r^{-1}\tau_0)\tau_2\right) \\ &= [\tau_2][\tau_0]_u[r^{-1}][\tau_1^{-1}]_v^{-1}[\tau_2]^{-1}f(\tau_1^{-1}\tau_2) \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} -6 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}. \end{aligned}$$

Thus, the rank and degree of u are given by:

$$\mathsf{C}_1^{-1}(\tau_0\tau_2)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 3 & -1 \\ -1 & 6 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

and

$$\mathsf{C}_1^{-1}(\tau_0\tau_2)\left[\begin{smallmatrix}3\\1\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}0 & 1 & 0\\ 0 & 3 & -1\\ -1 & 6 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}3\\1\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\1\\3\end{smallmatrix}\right]$$

respectively. We obtain the Markov fractions $\frac{1}{1}$, $\frac{1}{2}$, and $\frac{3}{5}$.

4. The cotangent bundle

We now move on to $X \coloneqq \mathrm{T}^*\mathbb{P}^2$, so we obtain a new Euler product χ on $\mathcal{D}_0 \simeq \mathbb{Z}[t^{\pm}]^{\oplus 3}$, described in the basis fixed in Section 2.4 by:

$$\chi(u(t), v(t)) = u(t^{-1})^{\intercal} \begin{bmatrix} 1+t^2+t^4 & -3t-3t^3 & 3t^2 \\ -3t-3t^3 & 1+10t^2+t^4 & -3t-3t^3 \\ 3t^2 & -3t-3t^3 & 1+t^2+t^4 \end{bmatrix} v(t).$$

As before, we denote this matrix $\mathsf{P}_{\mathrm{T}^*\mathbb{P}^2}(e_0, e_1, e_2)$ by $A_{\mathrm{T}^*\mathbb{P}^2}$. We also introduce the following mutations that take one basis to another:

$$\begin{aligned}
\tau_0(s_0, s_1, s_2) &= (-t^3 s_0, -s_2 + \chi_2(s_0, s_2)s_0, -s_1 + \chi_2(s_0, s_1)s_0) \\
\tau_1(s_0, s_1, s_2) &= (-s_0 + \chi_2(s_1, s_0)s_1, t^2 s_1, -s_2 + \chi_2(s_1, s_2)s_1) \\
\tau_2(s_0, s_1, s_2) &= (-s_1 + \chi_2(s_2, s_1)s_2, -s_0 + \chi_2(s_2, s_0)s_2, -t^3 s_2).
\end{aligned}$$

(We are no longer viewing these operations as part of a braid group.) Note the symmetry between τ_0 and τ_2 and their lack of symmetry with τ_1. As with $\mathrm{K}\mathbb{P}^2$, it will become helpful to view τ_i as the matrix taking $\mathsf{C}(s_0, s_1, s_2)$ to $\mathsf{C}(\tau_i(s_0, s_1, s_2))$.

4.1. A potentially Koszul family. We begin by showing that specific combinations of these mutations on (e_0, e_1, e_2) preserve potential Koszulity.

Theorem 4.1. *Let $B' \coloneqq \{\tau_1^n \tau_{a_m}\tau_{a_{m-1}}\cdots\tau_{a_1} : a_i \in \{0, 2\}\}$. Then, all elements in the B'-orbit of (e_0, e_1, e_2) are potentially Koszul.*

Proof. To show this, we characterize the pairing matrices that we get by applying this action. It is easy to verify that τ_0 and τ_2 do not change the original pairing matrix $A_{\mathrm{T}^*\mathbb{P}^2}$. It suffices then to check how τ_1^n modifies an arbitrary basis (e'_0, e'_1, e'_2) with pairing matrix $A_{\mathrm{T}^*\mathbb{P}^2}$. We claim that

$$\mathsf{P}(\tau_1^n(e'_0, e'_1, e'_2)) = \begin{bmatrix} 1+(x_{n+1}x_n-8)t^2+t^4 & -x_n t - x_{n+1}t^3 & (x_{n+1}x_n-6)t^2 \\ -x_{n+1}t - x_n t^3 & 1+10t^2+t^4 & -x_{n+1}t - x_n t^3 \\ (x_{n+1}x_n-6)t^2 & -x_n t - x_{n+1}t^3 & 1+(x_{n+1}x_n-8)t^2+t^4 \end{bmatrix}$$

where (x_i) is a sequence recursively defined by $x_{i+1} = 10x_i - x_{i-1}$ with $x_0 = x_1 = 3$. We proceed by induction, with the base case $n = 0$ being clear. We show that the first row of $\mathsf{P}(\tau_1^{n+1}(e'_0, e'_1, e'_2))$ satisfies these properties; the other two rows can be checked similarly (and/or using the skew-symmetry of χ). Let $\tau_1^n(e'_0, e'_1, e'_2) =: (s_0, s_1, s_2)$ and $\tau_1^{n+1}(e'_0, e'_1, e'_2) =: (s'_0, s'_1, s'_2)$. We compute

$$\begin{aligned}
\chi(s'_0, s'_0) &= \chi(-s_0 + \chi_2(s_1, s_0)s_1, -s_0 + \chi_2(s_1, s_0)s_1) \\
&= \chi(s_0, s_0) + x_{n+1}t\chi(s_0, s_1) + x_{n+1}t^{-1}\chi(s_1, s_0) + x_{n+1}^2\chi(s_1, s_1) \\
&= 1 + (x_{n+1}x_n - 8)t^2 + t^4 + x_{n+1}t(-x_nt - x_{n+1}t^3) \\
&\quad + x_{n+1}t^{-1}(-x_{n+1}t - x_nt^3) + x_{n+1}^2(1 + 10t^2 + t^4) \\
&= 1 + (x_{n+1}(10x_{n+1} - x_n) - 8)t^2 + t^4 \\
&= 1 + (x_{n+2}x_{n+1} - 8)t^2 + t^4,
\end{aligned}$$

$$\begin{aligned}
\chi(s'_0, s'_1) &= \chi(-s_0 + \chi_2(s_1, s_0)s_1, t^2s_1) \\
&= -t^2\chi(s_0, s_1) - x_{n+1}t\chi(s_1, s_1) \\
&= -t^2(-x_nt - x_{n+1}t^3) - x_{n+1}t(1 + 10t^2 + t^4) \\
&= -x_{n+1}t - (10x_{n+1} - x_n)t^3 \\
&= -x_{n+1}t - x_{n+2}t^3,
\end{aligned}$$

and

$$\begin{aligned}
\chi(s'_0, s'_2) &= \chi(-s_0 + \chi_2(s_1, s_0)s_1, -s_2 + \chi_2(s_1, s_2)s_1) \\
&= \chi(s_0, s_2) + x_{n+1}t\chi(s_0, s_1) + x_{n+1}t^{-1}\chi(s_1, s_2) + x_{n+1}^2\chi(s_1, s_1) \\
&= (x_{n+1}x_n - 6)t^2 + x_{n+1}t(-x_nt - x_{n+1}t^3) \\
&\quad + x_{n+1}t^{-1}(-x_{n+1}t - x_nt^3) + x_{n+1}^2(1 + 10t^2 + t^4) \\
&= (x_{n+1}(10x_{n+1} - x_n) - 6)t^2 \\
&= (x_{n+2}x_{n+1} - 6)t^2,
\end{aligned}$$

which proves our claim.

We now verify the potential Koszulity (see Section 2.4). The hard part is to prove that the power series expansion of the inverse of the pairing matrix has only positive coefficients. By the exact same argument as used in Theorem 3.2,

$$\det(\mathsf{C}(\tau_1^n(e_0', e_1', e_2')) = \det(A_{\mathrm{T}^*\mathbb{P}^2}) = (1-t^2)^4 = (1+t^2+t^4+\cdots)^4.$$

Thus, we can explicitly compute the $(0,0)$th element of $P^{-1}(\tau_1^n(e_0', e_1', e_2'))$ as

$$(1+t^2+\cdots)^4\left((1+10t^2+t^4)(1+(x_{n+1}x_n-8)t^2+t^4)+(x_nt+x_{n+1}t^3)(x_{n+1}t+x_nt^3)\right),$$

which clearly has all nonnegative coefficients. Symmetrically, the $(1,1)$th, $(2,2)$th, $(0,2)$th, and $(2,0)$th elements of $\mathsf{P}^{-1}(\tau_1^n(e_0', e_1', e_2'))$ also satisfy this property. Finally, we explicitly compute the $(0,1)$th element of this inverse pairing matrix:

$$\frac{1}{(1-t^2)^4}\left((x_{n+1}t+x_nt^3)(1+(x_{n+1}x_n-8)t^2+t^4-(x_{n+1}x_n-6)t^2)\right)$$
$$=(1+t^2+t^4+\cdots)^2(x_{n+1}t+x_nt^3),$$

which also clearly has all nonnegative coefficients. Again, symmetrically, the $(1,0)$th, $(1,2)$th, and $(2,1)$th elements of $\mathsf{P}^{-1}(\tau_1^n(e_0', e_1', e_2'))$ also satisfy this property, so we have proven potential Koszulity, as desired. □

Remark. We have verified that the mutations in B' preserve the linear algebra conditions for Koszulity. However, as noted in Section 2.4, these conditions are not sufficient: they do not imply that these triples come from a triple of coherent sheaves in $\mathcal{D}_0^b$ that correspond to the simple modules of some Koszul algebra. In general, proving that a mutation preserves Koszulity is a fairly difficult problem.

Not only do the B'-mutations of (e_0, e_1, e_2) produce potentially Koszul bases, but in fact, each basis produced is distinct:

Theorem 4.2. *Each element of B' produces a different potentially Koszul basis of $\mathrm{K}(\mathcal{D}_0)$ by acting on (e_0, e_1, e_2).*

Proof. Assume otherwise for the sake of contradiction. Then, there exists a nontrivial relation $\tau_1^n\tau_{a_m}\tau_{a_{m-1}}\cdots\tau_{a_1} = \tau_1^{n'}\tau_{b_{m'}}\tau_{b_{m'-1}}\cdots\tau_{b_1}$ for $a_i, b_i \in \{0,2\}$ and (without loss of generality) $m \geq m'$. In the proof of Theorem 4.1, the sequence (x_i) is clearly increasing, so for the pairing matrices $\mathsf{P}(\tau_1^n\tau_{a_m}\tau_{a_{m-1}}\cdots\tau_{a_1}(e_0, e_1, e_2))$ and $\mathsf{P}(\tau_1^{n'}\tau_{b_m'}\tau_{b_{m'-1}}\cdots\tau_{b_1}(e_0, e_1, e_2))$ to be the same, we must have $n = n'$. Thus, we can simplify the relation to $\tau_{a_m}\tau_{a_{m-1}}\cdots\tau_{a_1} = \tau_{b_{m'}}\tau_{b_{m'-1}}\cdots\tau_{b_1}$. Without loss of generality, assume $a_1 = 2$.

Let $d(s)$ and $l(s)$ be the degree and leading coefficient (respectively) of the third Laurent polynomial in the tuple s. We also say that the *height* of a tuple (s_0, s_1, s_2) is the maximal value of $d(s_0), d(s_1)$, and $d(s_2)$. We will prove the following claim: for a basis $(s_0, s_1, s_2) \in \mathrm{Orbit}_{B'}(e_0, e_1, e_2)$ with positive height, the following are all true:

- $l(s_0)$ and $l(s_2)$ have a different sign than $l(s_1)$
- $|l(s_1)| > |l(s_0)| + |l(s_2)|$
- either all of the following hold:
  - $d(s_0) - 1 = d(s_1) = d(s_2) + 1$
  - $|l(s_1)| \geq |l(s_2)| \geq |l(s_0)|$
  - $|3l(s_0)| > |l(s_1)|$

  or all of the following hold:
  - $d(s_2) - 1 = d(s_1) = d(s_0) + 1$,
  - $|l(s_1)| \geq |l(s_0)| \geq |l(s_2)|$
  - $|3l(s_2)| > |l(s_1)|$.

We will show this by induction, proving that τ_0 (and thus, τ_2 by symmetry) preserves these properties. Let

$$\tau_0(s_0, s_1, s_2) = (-t^3 s_0, -s_2 + 3t^2 s_0, -s_1 - 3ts_0) =: (s_0', s_1', s_2').$$

If $d(s_0) - 1 = d(s_1) = d(s_2) + 1$, then clearly $d(s_0') - 1 = d(s_1') = d(s_2') - 1 = d(s_1) + 3$. Further, $l(s_0') = -l(s_0)$, $l(s_1') = 3l(s_0)$, and $l(s_2') = -3l(s_0)$, so the conditions of the leading coefficients are also satisfied.

If $d(s_2) - 1 = d(s_1) = d(s_1) + 1$, we have $l(s_0') = -l(s_0)$, $l(s_1') = -l(s_2) + 3l(s_0)$, $l(s_2') = -l(s_1) - 3l(s_0)$, and $d(s_0') - 1 = d(s_1') = d(s_2') - 1 = d(s_1) + 1$. Importantly, the condition $3|l(s_2)| > |l(s_1)|$ implies that all of these values $l(s_i')$ are nonzero and therefore do actually correspond to the leading coefficient. We can easily verify the sign conditions on $l(s_i')$. Further, using our induction hypothesis, we have

$$
\begin{aligned}
&|l(s_2')| = 3|l(s_0)| - |l(s_1)| > |l(s_0)| + (|l(s_0)| + |l(s_2)| - |l(s_1)|) > |l(s_0)| = |l(s_0')| \\
&|l(s_1')| = 3|l(s_0)| - |l(s_2)| > 3|l(s_0)| - |l(s_1)| > |l(s_2')|, \\
&|l(s_2')| + |l(s_0')| = 3|l(s_0)| - |l(s_1)| + |l(s_0')| > 3|l(s_0)| - |l(s_2)| = |l(s_1')|, \\
&3|l(s_0')| = 3|l(s_0)| > 3|l(s_0)| - |l(s_2)| = |l(s_2')|,
\end{aligned}
$$

so the bounding conditions also hold.

Therefore, we have proven our induction hypothesis. Particularly, we take two things away from this induction: the height of $\tau_0(s_0, s_1, s_2)$ and, symmetrically, $\tau_2(s_0, s_1, s_2)$ is strictly greater than the height of (s_0, s_1, s_2), and applying τ_i forces $d(s_i')$ to be maximal for $i \in \{0, 2\}$.

Let $(r_0, r_1, r_2) \coloneqq \tau_{a_m} \cdots \tau_{a_1}(e_0, e_1, e_2) = \tau_{b_{m'}} \cdots \tau_{b_1}(e_0, e_1, e_2)$. Since $a_1 = 2$, the height of (r_0, r_1, r_2) is positive, and our induction applies. Particularly, $d(r_{a_m})$ and $d(r_{b_{m'}})$ must both be maximal, so $a_m = b_{m'}$. Repeatedly applying this process, we get that $a_{m-i} = b_{m'-i}$ for all $1 \leq i \leq m-1$. Thus, we have the new relation $b_{m'-m} b_{m'-m-1} \cdots b_1 = 1$. But, the degree of $b_{m'-m} b_{m'-m-1} \cdots b_1$ must be at least $m' - m$ (because the height is strictly increasing at each step), so $m' - m = 0$ and we have no nontrivial relation. □

4.2. The classical limit and Chebyshev polynomials. We can actually write a closed form for the (duals of) triples in $\text{Orbit}_{B'}(e_0, e_1, e_2)|_{t=1}$. Before we do so, we describe the sequence (x_i) in terms of Chebyshev polynomials.

Lemma 4.3. *Let $T(r, -5)$ be the rth first-order Chebyshev polynomial computed at -5. Then,*

$$x_r = (-1)^{r+1} \frac{T(r, -5) - T(r-1, -5)}{2}.$$

Proof. We proceed by strong induction, with the base cases $r = 1$ and $r = 2$ being easy to check. Then, since the Chebyshev polynomials follow the recurrence $T(n+1, x) = 2T(n, x) - T(n-1, x)$, we have

$$
\begin{aligned}
x_{r+1} &= 10x_r - 10x_{r-1} \\
&= (-1)^{i+1} \frac{-10T(r, -5) + 10T(r-1, -5) - T(r-1, -5) + T(r, 5)}{2} \\
&= (-1)^{i+1} \frac{T(r+1, -5) - T(r, -5)}{2},
\end{aligned}
$$

as desired. □

Corollary 4.4. *For any r,*

$$\sum_{i=1}^{r}(-1)^i x_i = \sum_{i=1}^{r}\frac{T(i,-5)-T(i-1,-5)}{2} = \frac{T(r,-5)-1}{2}.$$

Observe that, at $t=1$, we have $\tau_0^2=\tau_2^2=1$ (because the pairing matrix is always $A_{\mathrm{T}^*\mathbb{P}^2}$ whenever τ_0 and τ_2 are valid actions), so considering the B'-orbit is equivalent to considering the union of the B''-orbit and B'''-orbit, where

$$B'' := \{\tau_1^n \tau_{a_m}\tau_{a_{m-1}}\cdots\tau_{a_1} \mid a_{2i-1}=0, a_{2i}=2\}$$

and

$$B''' = \{\tau_1^n \tau_{a_m}\tau_{a_{m-1}}\cdots\tau_{a_1} \mid a_{2i-1}=2, a_{2i}=0\}.$$

Theorem 4.5. *For $a_{2i-1}=0$, $a_{2i}=2$, and m even, the dual matrix of*

$$\tau_1^n \tau_{a_m}\tau_{a_{m-1}}\cdots\tau_{a_1}(e_0,e_1,e_2)$$

computed at $t=1$ is

$$\begin{bmatrix} -\binom{m-1}{2} & (m-1)^2-1 & -\binom{m}{2} \\ \frac{T(n,-5)(m^2-m+1)-1}{2} & -T(n,-5)(m^2+2m)+1 & \frac{T(n,-5)(m^2+m+1)-1}{2} \\ -\binom{m+1}{2} & (m+1)^2-1 & -\binom{m+2}{2} \end{bmatrix}.$$

For $a_{2i-1}=0$, $a_{2i}=2$, and m odd, the dual matrix at $t=1$ is

$$\begin{bmatrix} \binom{m+1}{2} & -(m+1)^2+1 & \binom{m+2}{2} \\ \frac{-T(n,-5)(m^2-m+1)+1}{2} & T(n,-5)(m^2+2m)-1 & \frac{-T(n,-5)(m^2+m+1)+1}{2} \\ \binom{m-1}{2} & -(m-1)^2+1 & \binom{m}{2} \end{bmatrix}.$$

For $a_{2i-1}=2$, $a_{2i}=0$, and m even, the dual matrix at $t=1$ is

$$\begin{bmatrix} -\binom{m+2}{2} & (m+1)^2-1 & -\binom{m+1}{2} \\ \frac{T(n,-5)(m^2+m+1)-1}{2} & -T(n,-5)(m^2+2m)+1 & \frac{T(n,-5)(m^2-m+1)-1}{2} \\ -\binom{m}{2} & (m-1)^2-1 & -\binom{m+1}{2} \end{bmatrix}.$$

Finally, for $a_{2i-1} = 2$, $a_{2i} = 0$, and m odd, the dual matrix at $t = 1$ is

$$\begin{bmatrix} \binom{m}{2} & -(m-1)^2+1 & \binom{m-1}{2} \\ \frac{-T(n,-5)(m^2-m+1)+1}{2} & T(n,-5)(m^2+2m)-1 & \frac{-T(n,-5)(m^2+m+1)+1}{2} \\ \binom{m+2}{2} & -(m+1)^2+1 & \binom{m+1}{2} \end{bmatrix}.$$

Proof. We only show the theorem for the first case; the other cases are proved analogously. From the pairing matrices computed in Theorem 4.1, we can express the action $\tau_1^n \tau_{a_m} \tau_{a_{m-1}} \cdots \tau_{a_1}$ on (e_0, e_1, e_2) as matrix multiplication:

$$\begin{aligned} &\mathsf{C}_1 \left(\tau_1^n \tau_{a_m} \tau_{a_{m-1}} \cdots \tau_{a_1}(e_0, e_1, e_2)\right)|_{t=1} \\ &= I_3 \left(\begin{bmatrix} -1 & 3 & -3 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ -3 & 3 & -1 \end{bmatrix} \right)^{\frac{m}{2}} \begin{bmatrix} -1 & 0 & 0 \\ -x_1 & 1 & -x_1 \\ 0 & 0 & -1 \end{bmatrix} \cdots \begin{bmatrix} -1 & 0 & 0 \\ -x_n & 1 & -x_n \\ 0 & 0 & -1 \end{bmatrix}, \end{aligned}$$

where the identity matrix I_3 is just $\mathsf{C}_1(e_0, e_1, e_2)$. Taking the inverse, we get

$$\begin{aligned} &\mathsf{C}_1^{-1} \left(\tau_1^n \tau_{a_m} \tau_{a_{m-1}} \cdots \tau_{a_1}(e_0, e_1, e_2)\right)|_{t=1} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ -x_n & 1 & -x_n \\ 0 & 0 & -1 \end{bmatrix} \cdots \begin{bmatrix} -1 & 0 & 0 \\ -x_1 & 1 & -x_1 \\ 0 & 0 & -1 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ -3 & 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \right)^{\frac{m}{2}}. \end{aligned}$$

We can compute the first part using Lemma 4.3 and corollary 4.4:

$$\begin{aligned} \begin{bmatrix} -1 & 0 & 0 \\ -x_n & 1 & -x_n \\ 0 & 0 & -1 \end{bmatrix} \cdots \begin{bmatrix} -1 & 0 & 0 \\ -x_1 & 1 & -x_1 \\ 0 & 0 & -1 \end{bmatrix} &= \begin{bmatrix} -1 & 0 & 0 \\ \sum_{i=1}^n (-1)^i x_i & 1 & \sum_{i=1}^n (-1)^i x_i \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ T(n,-5) & 1 & T(n,-5) \\ 0 & 0 & -1 \end{bmatrix}. \end{aligned}$$

We now proceed by induction on m (with increments of 2, as m must be even) to finish. The base case $m = 0$ is clear. We have

$$\begin{bmatrix} -1 & 0 & 0 \\ T(n,-5) & 1 & T(n,-5) \\ 0 & 0 & -1 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ -3 & 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \right)^{\frac{m}{2}+1}$$

$$= \begin{bmatrix} -\binom{m-1}{2} & (m-1)^2 - 1 & -\binom{m}{2} \\ \frac{T(n,-5)(m^2-m+1)-1}{2} & -T(n,-5)(m^2+2m)+1 & \frac{T(n,-5)(m^2+m+1)-1}{2} \\ -\binom{m+1}{2} & (m+1)^2 - 1 & -\binom{m+2}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & -3 & 3 \\ 3 & -8 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -\binom{m+1}{2} & (m+1)^2 - 1 & -\binom{m+2}{2} \\ \frac{T(n,-5)(m^2+4m+3)-1}{2} & -T(n,-5)(m^2+6m+8)+1 & \frac{T(n,-5)(m^2+5m+7)-1}{2} \\ -\binom{m+3}{2} & (m+3)^2 - 1 & -\binom{m+4}{2} \end{bmatrix},$$

as desired. This completes our induction and proves the theorem. □

5. Future directions

With an additional assumption, Theorem 3.2 classifies all possible pairing matrices from Koszul bases, but it is possible that multiple bases have the same Markov pairing matrix. Whether all of these Koszul bases are in $\mathrm{Orbit}_B(e_0, e_1, e_2)$ could be a future research direction.

Another area to explore is whether τ_1 (the unusual mutation over $\mathrm{T}^*\mathbb{P}^2$) preserves Koszulity on the level of $\mathcal{D}^b_0 \operatorname{Coh} \mathrm{T}^*\mathbb{P}^2$. The mutation τ_1 on (s_0, s_1, s_2) may be viewed as a flip across s_1, which is complicated when s_1 is neither a spherical nor a $\mathbb{P}$-object (see [15, 16]), which makes it more difficult to understand.

More broadly, the idea of classifying derived equivalences over different spaces is not unique to $\mathrm{K}\mathbb{P}^2$ or $\mathrm{T}^*\mathbb{P}^2$. Understanding such equivalences over (partial) flag varieties is an important step to understanding these spaces in general, which in turn is important for its representation theory implications. Thus, studying Koszul bases over other spaces X is a natural extension.

Acknowledgments

I am very grateful to my mentor Mohit Hulse for introducing me to this field and for his invaluable guidance throughout the entire research process. I would also like to thank Prof. Roman Bezrukavnikov for proposing this project, as well as Dr. Tanya Khovanova and Dr. Jonathan Bloom for their advice on proof writing and explaining math. I am also grateful to my tutor AnaMaria Perez for her advice on presenting and her constant encouragement. I also thank my first week TAs Varun Thvar, Valencia Zhang, and Finan Gammell for teaching me how to use Beamer, and my last week TA Austin Luo for giving me feedback on both my paper and my presentation. I am also grateful to Yale Fan, Merrick Cai, and Iliyas Noman for their very helpful comments on my paper. Finally, I thank the Research Science Institute, the Center of Excellence in Education, and the Massachusetts Institute of Technology for inviting me this summer and Mr. Kalani Oshiro, Mr. Sunny Goyal, Mr. Burak Serdar, and Mr. Ram Rao for sponsoring my time here.

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