2017 TALBOT WORKSHOP: OBSTRUCTION THEORY FOR STRUCTURED RING SPECTRA

MENTORED BY MARIA BASTERRA AND SARAH WHITEHOUSE

Prerequisites

We will assume some working knowledge of a “modern” category of spectra, i.e. familiarity with a symmetric monoidal category of spectra with product that descends to the smash product in the stable homotopy category [EKMM97, HSS00]. We will also use freely the language of model category theory [DS95, Hov99, Hir03]. Participants should know about \( k \)-invariants and Postnikov towers [GJ99, Chapter VI]. Familiarity with the rudiments of spectral sequences will also be assumed [Wei94, Chapter 5], [McC01].

Detailed List of Talks

1 Overview talk: [mentors] What are structured ring spectra, why do we care? Historical perspective.

2 Operads: Background [May72], [Val12], Stasheff’s associahedron and \( A_\infty \) operads [Sta63], \( E_n \)-operads: the Boardman-Vogt little \( n \)-cubes operad, \( E_\infty \)-operads: the Barratt-Eccles operad, the linear isometries operad [BV73, May09c]. Simplicial spectra over simplicial operads [GH, section 1.3]

3 Examples of structured ring spectra: Brief reminder of additive infinite loop space theory [May77, Seg74, Tho95]. Multiplicative infinite loop space theory, connective \( E_\infty \)-ring spectra from bipermutative categories, examples arising this way [EM06, May09a]. Thom spectra [May09b, ABG+14, Rud98, CM15].

4 Robinson’s obstruction theory I: The \( A_\infty \) case. Description of strategy. Cooperation algebras and Hochschild cohomology. Obstructions to existence and uniqueness. Applications: Morava \( K \)-theory, completed Johnson-Wilson spectra. [Rob89, Bak91, Rob04, Ang08].

5 Robinson’s obstruction theory II: The \( E_\infty \) case and Gamma cohomology. Spaces of trees and \( \text{Lie}(n) \), as ingredients of an \( E_\infty \)-operad. Notion of \( n \)-stage for an \( E_\infty \)-structure via filtration of this operad. The \( \Xi \) bicomplex and definition of \( \Gamma \)-cohomology. Obstructions to existence of extension of the underlying \( n \)-1-stage of a given \( n \)-stage to an \( n+1 \)-stage. Obstructions to uniqueness. [RW96, RW02, Rob03, Rob04, Ric06].


Date: February 28, 2017.
7 **Gamma cohomology II: more calculations and applications.** Calculations for rings of integer-valued polynomials; applications to uniqueness of the $E_\infty$ structures on $KU$ and $KO$: existence and uniqueness for the Adams summand $E(1)$ of $KU(p)$ and for the $I_n$-adic completions of Johnson-Wilson spectra $E(n)$ \cite{BR05}. A lower bound for coherences on $BP$: at a prime $p$, $BP$ has a $(2p^2 + 2p - 2)$-stage structure; lower bound for coherences on (localized) Johnson-Wilson spectra \cite{Re06}.

8 **Quillen (co)homology:** Homology as total left derived functors of abelianization \cite{Qui67}. The classical theory for associative and commutative rings: André-Quillen (co)homology, the cotangent complex and derivations \cite{Qui70}. The case of simplicial algebras over a simplicial operad \cite{GH04} section 4.

9 **TAQ I: Construction and properties.** Quillen cohomology in the context of spectra. The $E_\infty$ case \cite{Bas99}, \cite{BGR08}. The $A_\infty$ case \cite{Laz04a}. Are there differences between the algebraic and topological versions? Do they agree for Eilenberg-MacLane spectra? Relation to ordinary spectra cohomology. Analogue of Hurewicz theorem. Tools for calculating TAQ \cite{Bas99} section 5. Universal coefficients spectral sequences \cite{BM13}.

10 **TAQ II: Cohomology for operadic algebras.** Derived indecomposables as stabilization \cite{BM05}. Application to suspension spectra of $E_\infty$ ring spaces. The case of $E_n$-$R$-algebras over $H$ for connective commutative $S$-algebras $R$ and $H$ \cite{BM13} section 2. The iterated bar construction computes the derived indecomposables. Related spectral sequences and their multiplicative structure \cite{BM11}.

11 **Obstruction theory for connective spectra:** Constructing Postnikov Towers \cite{Bas99}, \cite{DS06}. Calculations and applications. Lazarev’s work \cite{Laz01}, \cite{Laz04a}. Morava $K$-theory \cite{Ang11}. $E_n$ genera \cite{CM15}.

12 **Application to the Brown-Peterson spectrum** \cite{BM13}. The speaker should coordinate with the person doing talk 10 to discuss the action of Dyer-Lashof operations on the relevant spectral sequence. Discuss existence of $E_4$ structure. In what sense is this unique? What goes wrong for $E_5$? Coordinate with person doing talk 11 on the structure of the Quillen idempotent.

13 **Goerss-Hopkins’ obstruction theory I:** Overview of the strategy and theory; input data as $E_\infty$-coalgebra in $E_\ast E$-comodules; output information on the homotopy type of the moduli space of realizations. Some key ingredients: simplicial spectra over simplicial operads, resolutions, the spiral exact sequence \cite{GH}, \cite{GH04} \cite{DK84}, \cite{DK95}. The speaker should coordinate with the person doing talk 2 for the part of simplicial operads.

14 **Goerss-Hopkins’ obstruction theory II:** Relevant André-Quillen cohomology. The Bousfield-Kan spectral sequence for the homotopy groups of the space of structured maps. Obstructions to realizing maps. Postnikov towers for simplicial algebras, $n$-stages and constructing realizations inductively. Decomposition of moduli spaces. Obstructions to realization and uniqueness \cite{GH}, \cite{GH04}.

15 **Applications:** Lifting the algebraic theory of deformations of height $n$ formal groups to structured ring spectra: Lubin-Tate spectra, the Hopkins-Miller Theorem \cite{Rez98}, Goerss-Hopkins $E_\infty$ version \cite{GH04}. Generalized truncated Brown-Peterson spectra of height 2, $BP(2)$ \cite{EN12}.
16 **Comparison results I:** Gamma homology as stable homotopy of $\Gamma$-modules [PR00]. Gamma homology as TAQ of Eilenberg-MacLane spectra (in the flat case) [BM02]. André-Quillen cohomology for $E_\infty$ differential graded algebras and simplicial $E_\infty$ algebras and relation to TAQ [Man03]. The Goerss-Hopkins obstruction groups are isomorphic to gamma cohomology groups [BR04].

17 **Comparison results II:** Stable homotopy of algebraic theories [Sch01]. For commutative augmented algebras, this is isomorphic to stable homotopy of $\Gamma$-modules. Isomorphism of Atiyah-Hirzebruch spectral sequences [BR04].

18/19 **Related Topics:** We have left two lectures for some related topics, to be chosen by those giving these talks. There are connections to many areas of active research. A few possibilities are listed below. Alternatively, participants could propose their own topic to the mentors.

- **Negative results:** While the question of whether $BP$ has an $E_\infty$ ring spectrum structure is open, there are results which show that there is no such structure with certain good properties one might hope for. In particular, the natural orientation from $MU$ to $BP$ is not an $E_\infty$ map, at least for small primes [JN10]: $BP$ is not an $E_\infty$ core of $MU$ [HKM01].
  - Functor Calculus: TAQ as the derivative of the forgetful functor from a category of commutative algebras to a category of modules [BM02].
  - Also relevant [Ric01, Kuh06, Chi05]
  - Algebraic formulation of $E_n$-homology [Fre10, Fre11]; as functor homology [LR11].
  - Factorization homology [Fra13]
  - Homotopy completion [HH13]; TAQ via circle product of operadic algebras and bimodules; associated filtrations [KP].

20 **Future directions:** [mentors, discussion]

References


[BM02] Maria Basterra and Randy McCarthy, $\Gamma$-homology, topological André-Quillen homology and stabilization, Topology Appl. 121 (2002), no. 3, 551–566.


