1 Various forms of nilpotence

The first statement on nilpotence in the homotopy groups of spheres is

**Theorem 1** (Nishida). Every \( \alpha \in \pi_n(S^0) \) for \( n > 0 \) is nilpotent.

**Definition 2.** Let \( E \) be an associative ring spectrum. If \( f : X \to Y \) is such that \( 1_E \land f^\land n : E \land X^\land n \to E \land Y^\land n \) is trivial for some \( n \), then \( f \) is \( E \)-\land nilpotent. If \( f : \Sigma^d X \to X \) is a self-map such that \( 1_E \land f^\land n \) is trivial for some \( n \), then \( f \) is \( E \)-\circ nilpotent. (Here by \( f^\circ n \), we mean the composition \( f \circ \Sigma^d f \circ \cdots \circ \Sigma^{(n-1)d} f \).) If \( R \) is a (not necessarily associative) ring spectrum, \( \alpha \in \pi_* R \), and \( 1_E \land (\mu^n \alpha^n) \) is trivial for any sequence of multiplications \( \mu^n : R^n \to R \), then \( \alpha \) is \( E \)-Hurewicz nilpotent.

If \( \alpha \in \pi_n R \) for \( R \) a ring spectrum, then there’s a self-map \( \overline{\alpha} : \Sigma^n R \to R \land R \to R \), and we define

\[
\overline{\alpha}^{-1} R = \text{hocolim}(R \to \Sigma^{-n} R \to \Sigma^{-2n} R \to \cdots).
\]

(For example, this could be constructed as a mapping telescope.)

**Lemma 3.** \( \overline{\alpha}^{-1} R \land E \simeq * \) iff \( \alpha \) is \( E \)-Hurewicz nilpotent.

2 The main theorem

**Theorem 4** (Nilpotence theorem, Devinatz-Hopkins-Smith).

I. Suppose \( R \) is a connective associative ring spectrum of finite type. Then if \( \alpha \in \pi_n(R) \) is \( MU \)-Hurewicz nilpotent, it is nilpotent in \( \pi_* R \).

II. The same, where \( R \) is any ring spectrum.

III. If \( f : F \to X \) for \( F \) a finite spectrum is \( MU \)-\land nilpotent, then \( f \) is nilpotent.

IV. If we have a sequence

\[
\cdots \to X_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \to \cdots,
\]

and \( f_n \) is \( c_n \)-connected for \( c_n \geq mn + b \) for some \( m, b \), with \( MU(f_n) = 0 \) for all \( n \), then the homotopy colimit of this sequence is contractible.

V. If \( R \) is \( p \)-local, and \( \alpha \in \pi_*(R) \) is \( BP \)-Hurewicz nilpotent, then \( \alpha \) is nilpotent.

VI. If \( R \) is \( p \)-local, and \( \alpha \in \pi_*(R) \) is \( K(n) \)-Hurewicz nilpotent for all \( 0 \leq n \leq \infty \), then \( \alpha \) is nilpotent.

Note that Nishida’s theorem follows as an immediate corollary – since we know that all elements of \( \pi_n S \) for \( n > 0 \) are torsion and that \( \pi_* MU \) is torsion-free, the image of any \( \alpha \in \pi_n S \) in \( \pi_n MU \) must be zero, so that \( \alpha \) is nilpotent by statement III.
Proof of $I \Rightarrow III$. Suppose that $\alpha : F \to X$ is $MU$-nilpotent; replacing $F$ and $X$ with sufficiently high smash powers, we see that $1 \wedge \alpha : MU \wedge F \to MU \wedge X$ is zero. Since $F$ is finite, it has a Spanier-Whitehead dual $DF$, and $\pi : S \to X \wedge DF$ is zero after smashing with $MU$. Since $S$ is a small object, the image of this map is some finite subspectrum of $X \wedge DF$, and any given null homotopy must likewise factor through some finite subspectrum. Let $X'$ be such a subspectrum; after sufficient suspension, we can take $X'$ to be connective. The tensor algebra $T(X') = \bigvee_{n \geq 0} (X')^n$ is then a connective associative ring spectrum of finite type, and if $i : X' \to T(X')$ is the inclusion, then $i_* \pi$ is $MU$-Hurewicz nilpotent in $T(X')$, so by I, $\pi$ is nilpotent, and thus $\alpha$ is nilpotent.

$III \Rightarrow II$. Suppose $\alpha \in \pi_n(R)$ is $MU$-Hurewicz nilpotent, so that $MU \wedge S^n \to MU \wedge MU \wedge R \to MU \wedge R$ is nilpotent. It immediately follows that $\alpha$ is nilpotent.

$III \Rightarrow IV$. See the DHS paper.

$II \Rightarrow V$. This follows from the splitting of $MU_{(p)}$.

$V \Rightarrow VI$. We have $\langle BP \rangle = \langle K(0) \rangle \vee \cdots \vee \langle K(n) \rangle \vee \langle P(n + 1) \rangle$, where angle brackets denote the Bousfield class and $P(n + 1)$ is the cohomology theory with $\langle P(n + 1) \rangle = \mathbb{Z}_{(p)}[v_{n+1}, \ldots]$. Note that $\lim P(n) = H\mathbb{F}_p$. Thus if $f : F \to X$ is $K(n)$-Hurewicz nilpotent for all $n$, then $F \to \lim P(n) \wedge X$ factors through some $P(n) \wedge X$, and it is therefore $BP$-Hurewicz nilpotent, thus nilpotent by $\overline{V}$.

The nilpotence theorem resolves the following conjecture:

**Conjecture 5** (Ravenel’s nilpotence conjecture). If we have $f : \Sigma^d X \to X$ with $MU_*(f) = 0$, and $X$ is finite, then $f$ is composition nilpotent.

Proof. The mapping telescope of $f$ is contractible, so some composite of $f$ is zero.

Outline of proof of the nilpotence theorem. We’ve reduced the theorem to proving I. We have a sequence of maps

$$\ast \simeq \Omega SU(1) \to \Omega SU(2) \to \cdots \to \Omega SU \to BU.$$ 

If $X(n)$ is the Thom spectrum of $\Omega SU(n) \to BU$, then we can write $MU$ as a colimit

$$S^0 = X(0) \to X(1) \to \cdots \to X(\infty) \to MU,$$

where $X(n) \to MU$ is $2n - 1$-connected. Thus if $\alpha \in \pi_d(R)$ has zero $MU$-Hurewicz image, it also has zero $X(n)$-Hurewicz image for sufficiently large $n$.

We thus reduce to showing that if $\alpha$ is $X(n+1)$-Hurewicz nilpotent, then it is $X(n)$-Hurewicz nilpotent. As usual, it suffices to look locally at each $p$; this is sketched out below.

## 3 Homology of various spaces

Fix $L_0 \in \mathbb{C}^n$. We have a map $\Sigma \mathbb{C}^{n-1} \to SU(n)$ given by $(z, L) \mapsto (z^{-1} \pi_{L_0} + \pi_{L_0})(z \pi_L + \pi_L)$. The adjoint maps $\mathbb{C}^{n-1} \to \Omega SU(n)$ are compatible with the inclusions $\mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$, $\Omega SU(n) \hookrightarrow \Omega SU(n+1)$, and thus we get a map $\Sigma \mathbb{C}^n \to \Omega SU \to BU$.

Let $V$ be the virtual bundle $O(1)^{-1} - 1$ over $\mathbb{C}^n$. Then the Thom spectrum $(\mathbb{C}^{n-1})^V$ is $\Sigma^{-2} \mathbb{C}^n$, and we get complex $n$-orientations $\Sigma^{-2} \mathbb{C}^n \to X(n)$ for each $n$ with colimit $\Sigma^{-2} \mathbb{C}^\infty \to MU$. We have $H_*(\mathbb{C}^{n-1}) = \mathbb{Z}[\beta_0, \ldots, \beta_n]$, and such an identification induces $H_*(\Omega SU(n)) = \mathbb{Z}[\beta_0, \ldots, \beta_n]/(\beta_0 - 1)$. The Thom isomorphism gives $H_*(X(n)) = \mathbb{Z}[b_0, \ldots, b_n]/(b_0 - 1)$, where $b_i$ comes from $\Sigma^{-2} \mathbb{C}^{n+1}$.

**Proposition 6.** Suppose $k \leq n$. Then $X(n)_{\ast} \mathbb{C}^k = X(n)_{\ast} \{\beta_1, \ldots, \beta_k\}$, and $X(n)_{\ast} X(k) = X(n)_{\ast} [b_0, \ldots, b_k]/(b_0 - 1)$.

**Theorem 7.** $X(n)_{\ast} X(n)$ is flat over $X(n)$. 
4. Further reductions

There’s a pullback square

\[
\begin{array}{ccc}
F'_{k,n} & \to & \Omega SU(n + 1) \\
\downarrow & & \downarrow \\
J_k S^{2n} & \to & \Omega S^{2n+1}.
\end{array}
\]

Write \( F_{k,n} \) for the Thom spectrum of \( F'_{k,n} \to \Omega SU(n + 1) \to BU \). We have maps

\[ F_{0,n} = X(n) \to F_{1,n} \to \cdots \to X(n + 1), \]

and we define \( G_{k,n} = (F_{p^{k-1},n})_{(p)} \).

**Theorem 8.** If \( X(n + 1) \wedge \alpha^{-1} R \simeq * \) then \( G_{k,n} \wedge \alpha^{-1} R \simeq * \) for large enough \( k \).

**Theorem 9.** \( \langle G_{k,n} \rangle = \langle G_{k+1,n} \rangle \) as Bousfield classes.

Thus, \( G_{0,n} \wedge \alpha^{-1} R \simeq * \), and it follows that \( X(n)_{(p)} \wedge \alpha^{-1} R \simeq * \), as desired.