1 “Motivation”

Let $G$ be a finite group. A couple weird things happen at the interaction of equivariant and chromatic homotopy theory. One is that $L_{K(n)}S^G \simeq L_{K(n)}S_{hG} - K(n)$-locally, homotopy fixed points and homotopy orbits are the same.

Another goes by the name of the redshift conjecture. This says that algebraic $K$-theory shifts chromatic levels. For example:

**Example 1.**

$$K(H\mathbb{F}_p)^p \simeq H\mathbb{Z}_p \vee \Sigma^{-1}H\mathbb{Z}_p.$$ 

**Example 2.**

$$K(H\mathbb{Z}_p)^p \simeq \Sigma ku^p \times (\text{Im } J \times \mathbb{Z}_p) \times B(\text{Im } J \times \mathbb{Z}_p).$$

**Example 3.** $K(ku)$ is still unknown, but it is known to have a $v_2$-self map.

...and that’s basically all the evidence we have for the following conjecture:

**Conjecture 4** (Redshift conjecture).

$$V(n) \wedge K(L_{K(n)}A) \xrightarrow{\sim} v_{n+1}^{-1}V(n) \wedge K(L_{K(n)}A)$$

where $A$ is a ‘suitably finite’ commutative ring spectrum.

Part of the reason this is such a difficult problem is that algebraic $K$-theory is generally very hard to compute. How do we compute it? Well, there’s a map from algebraic $K$-theory to topological cyclic homology $TC$, and given a map of ring spectra $A \rightarrow B$, these fit into a pullback square

$$
\begin{array}{ccc}
K(A) & \longrightarrow & TC(A) \\
\downarrow & & \downarrow \\
K(B) & \longrightarrow & TC(B).
\end{array}
$$

Topological cyclic homology, in turn, comes from a tower of topological Hochschild homology spectra $THH$. Morally, we say “$TC(A)^p = THH(A)^{S^1}$,” and so we reduce to computing these $S^1$-fixed points on $THH$. In order to do this, you have to fit together data about the $p$-subgroups $C_p^n$ of $S^1$ in a clever way.

Frequently, we get lucky and have

$$THH(A)^{C_p^n} \simeq THH(A)^{tC_p^n},$$

where the $t$ denotes the Tate construction, to be discussed below. There’s a spectral sequence

$$\tilde{H}(C_p^n; THH(A)) \Rightarrow \pi_{\ast} THH(A)^{tC_p^n},$$

where $\tilde{H}$ is Tate cohomology.
As an aside, here’s the definition of Tate cohomology. We recall that for \( M \) a \( G \)-module, \( H_0(G;M) = M_G \) and \( H^0(G;M) = M^G \). There’s a norm map \( N : M_G \to M^G \) given by \( \pi \mapsto \sum_{g \in G} g \pi \), which one observes is invariant and independent of the class of \( x \in M \) mod the action of \( G \). If we take a projective resolution \( P_* \) and an injective resolution \( I_* \) for \( M \) over \( \mathbb{Z}[G] \), and truncate them by replacing \( P_0 \to M \to 0 \) with \( H_0(G;M) \to 0 \) and \( 0 \to M \to I_0 \) with \( 0 \to H^0(G;M) \), then the norm map fits them together into a \( \mathbb{Z} \)-graded complex \( P_* \to I_* \). The homology of this is the Tate cohomology \( \tilde{H}^*(G;M) \). This is a fancy way of saying that

\[
\tilde{H}^*(G;M) = \begin{cases}
H^*(G;M) & * \geq 1 \\
H_{-*+1}(G;M) & * < -1 \\
\ker(N) & *= 0 \\
\coker(N) & *= 1.
\end{cases}
\]

2 The Tate construction

In the topological case, if \( M \) is a \( G \)-spectrum, there are spectral sequences

\[
H^*(G;\pi_*M) \Rightarrow \pi_*M^hG \quad \text{and} \quad H_*(G;\pi_*M) \Rightarrow \pi_*M_hG,
\]

and the Tate construction on \( M \) is a spectrum that fits in between the homotopy orbits and the homotopy fixed points, so as to give you the above spectral sequence.

More precisely, consider the cofiber sequence

\[
EG_+ \to S^0 \to \tilde{E}G.
\]

Nonequivariantly, \( EG_+ \simeq S^0 \), so \( \tilde{E}G \simeq \ast \). On the other hand, \( (EG_+)^G \simeq \ast \), so \( \tilde{E}G^G \simeq S^0 \).

For \( X \) a \( G \)-spectrum, there’s a map \( X \to F(EG_+,X) \), inducing a diagram

\[
EG_+ \wedge X \xrightarrow{\sim} X \xrightarrow{F} \tilde{E}G \wedge X
\]

Taking \( G \)-fixed points gives us a diagram

\[
X_{hG} \xrightarrow{\sim} X^G \xrightarrow{\phi G} \tilde{X}^G
\]

where \( X^{tG} \) is, by definition, the Tate construction on \( X \). Often, the norm map is nullhomotopic, and we get an extension \( X^{hG} \to X^{tG} \to \Sigma X_{hG} \), as we do algebraically with Tate cohomology.

**Proposition 5** (May).

\[ X^{tG} \simeq F(\tilde{E}G, \Sigma EG_+ \wedge X)^G. \]

If \( X \) is finite and has a trivial \( C_p \)-action, then

\[ X^{tC_p} \simeq \text{holim}( (BC_p)^{-n} \wedge \Sigma X). \]

Here \( (BC_p)^{-n} \) is defined via James periodicity, which says that for \( p^{n-k} | r \),

\[ (BC_p)^{2n+r} \simeq \Sigma^r (BC_p)^{2n} \]

where \( X^{m}_n \) is defined for a CW-spectrum \( X \) by crushing out its \((m - 1)\)-skeleton and restricting to the \( n \)-skeleton of the result. Thus, we can crush ‘negative-dimensional skeleta’ of \( BC_p \) by crushing actual skeleta and desuspending. Using this, May’s proposition, and the fact that \( F(X,Y \wedge Z) \simeq F(X,Y) \wedge Z \) for \( Z \) finite, we get the desired statement about \( X^{tC_p} \).
3 \(K(n)^G\) and consequences

**Proposition 6.** If \(G\) is a finite group acting trivially on \(K(n)\), then
\[
K(n)^G \simeq \ast.
\]

**Corollary 7.**
\[
K(n)_*BG \cong K(n)^*BG.
\]

**Proof.** \(K(n)_*BG = \pi_*(K(n) \wedge BG) = \pi_*K(n)_G\), and likewise \(K(n)^*BG = \pi_*K(n)^hG\); since the Tate construction is trivial, these two spectra are equivalent. \(\Box\)

**Lemma 8.** If \(K\) is complex-oriented and \(K_*BG\) is finitely generated over \(K_*\), then
\[
\text{holim}_s K \wedge BG^{(-s\xi)} \simeq \ast,
\]
where \(\xi\) is some complex bundle on \(BG\) and \(BG^{(-s\xi)}\) the Thom construction on \(s\xi\).

**Proof.** \(K_*BG^{(r)} \to K_*BG\) is surjective for sufficiently large \(r\), by the finite generation hypothesis. So there’s a diagram
\[
\begin{array}{ccc}
K_*(BG^{(r)})^{-(s+j)\xi} & \rightarrow & K_*BG^{-s\xi} \\
\downarrow & & \downarrow \\
K_*BG^{-(s+j)\xi} & \rightarrow & K_*BG^{-s\xi}
\end{array}
\]
with the left map surjective. But \((BG^{(r)})^{-(s+j)\xi}\) has a top cell in some finite dimension, and \(BG^{-s\xi}\) has a top cell in some finite dimension. For sufficiently large \(j\), we can thus make the top map zero, so that \(K_*BG^{-(s+j)\xi} \to K_*BG^{-s\xi}\) is zero as well. \(\Box\)

**Lemma 9.** Let \(V\) be a finite dimensional \(G\)-representation and \(K\) a complex-oriented spectrum with \(K_*BH\) finitely generated over \(K\) for all \(H \leq G\). Then \(F(S^\infty V, K \wedge EG_+)\) is equivariantly contractible.

**Proof.**
\[
F(S^\infty V, K \wedge EG_+) \simeq F(\text{holim}_s S^nV, K \wedge EG_+) \simeq \text{holim}_s F(S^nV, K \wedge EG_+) \simeq \text{holim}_s S^{-nV} \wedge K \wedge EG_+.
\]
Now, for any \(H \leq G\), \(V\) is an \(H\)-representation by restriction, and likewise \(EG_+\) is a model for \(EH_+\). Therefore, \((K \wedge EG_+ \wedge S^{-nV})^H \simeq K \wedge BH^{-n\xi_1}\) and for \(n \gg 0\), this is contractible since \(K_*BH\) is finitely generated, using the previous lemma. \(\Box\)

In particular, \(K(n)\) satisfies the above conditions, so \(K(n)^G \simeq \ast\). As a corollary, if \(X\) is type \(n\), then \((L_{K(n)}X)^tG \simeq \ast\), so \((L_{K(n)}X)^{hG} \simeq (L_{K(n)}X)^{hG}\).

**Proposition 10.** If \(K\) is a p-local \(v_n\)-periodic spectrum, with \(v_n\) a unit and \(v_i\) acting nilpotently for all \(0 \leq i \leq n - 1\), then \(K^{tG} \simeq \ast\).

This allows us to access a phenomenon known as **blueshift**, which says that the Tate construction with respect to trivial finite group actions tends to decrease chromatic levels.

**Theorem 11** (Greenlees-Sadofsky). If \(K\) is as above without the nilpotence assumption, then \(t_G K\) is \(v_{n-1}\)-periodic.

**Theorem 12** (Hovey-Sadofsky). If \(X\) is \(E(n)\)-local and \(G\) acts trivially, then \(X^{tG}\) is \(E(n-1)\)-local, and in fact,
\[
\langle (L_nX)^{tG} \rangle = \langle L_{n-1}X \rangle
\]
where angle brackets denote Bousfield classes.
Work of Ando, Morava, and Sadofsky gives us a weak equivalence

$$(E(n)[w]^{\text{C}_p})_{I_{n-1}}^\wedge =: TE \sim \to H\text{W}(\mathbb{F}_p((y))^{\text{sep}}) \otimes_{\mathbb{Z}_p} E_{n-1}.$$ 

Here $E(n)[w]$ is basically $E(n)$ with a $p^{n-1} - 1$th root of $v_{n-1}$, called $w$, added to its homotopy. We then have $\pi_0 TE = W(\mathbb{F}_p((y))^{\text{sep}})[[w_1, \ldots, w_{n-2}]]$, where $w_i$ is the image of $v_i x^{p^i - 1}$.

**Proposition 13.** For $E$ complex-oriented, $\pi_*(E^{\text{C}_p}) = E_*(x)/[p](x)$. 