Symmetric Matrix

\[ A = \begin{bmatrix} a_{ij} \end{bmatrix}_{1 \leq i, j < \infty}, a_{ij} = a_{ji} \in \mathbb{R}. \]

\[ A_n = \begin{bmatrix} a_{ij} \end{bmatrix}_{1 \leq i, j < n} \]
If $\lambda_{n,1} \geq \lambda_{n,2} \geq \ldots \geq \lambda_{n,n}$ is the spectrum of $A_n$ counting multiplicities then we have the interlacing property

$$\lambda_{n+1,1} \geq \lambda_{n,1} \geq \lambda_{n+1,2} \geq \lambda_{n,2} \geq \lambda_{n+1,3} \geq \ldots$$

Orthogonal Polynomials

Let $-\infty \leq a < b \leq \infty$ and let $\mu$ be a positive Borel measure on $(a, b)$ such that

$$\int_a^b x^{2k} d\mu(x) < \infty$$

for all $k = 1, 2, \ldots$

$\{f_j\}_{j \geq 0}$ the Gram-Schmidt orthonormalization in $L^2((a, b), \mu)$ of $1, x, x^2, \ldots$. Or any family of nonzero multiples of such.

The zeros of $f_n$ are all real and distinct and those of $f_n$ and $f_{n+1}$ strictly interlace.
\( a = -\infty, b = \infty \) and \( d\mu = e^{-x^2}dx \). Hermite Polynomials \( H_n, n \geq 1 \):

\[
x, x^2 - \frac{1}{2}, x^3 - \frac{3}{2}x, x^4 - 3x^2 + \frac{3}{4}, \ldots
\]

\[
A = \begin{bmatrix}
0 & \sqrt{\frac{1}{2}} & 0 & 0 & 0 & \cdots \\
\sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{2}{2}} & 0 & 0 & \cdots \\
0 & \sqrt{\frac{2}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 & \cdots \\
0 & 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{4}{2}} & \cdots \\
0 & 0 & 0 & \sqrt{\frac{4}{2}} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\]

\[H_n(x) = \det[x - A_n]\]

**Theorem 1** Suppose that we have \( n \geq 1, \lambda_{n,1} > \lambda_{n,2} > \ldots > \lambda_{n,n} \) strictly interlacing. Then there exists a at most one (up to conjugation by a diagonal matrix with entries \( \pm 1 \)) tridiagonal \( \infty \times \infty \) real, symmetric matrix, \( A \), such
that the eigenvalues of $A_n$ are $\{\lambda_{n,i} | i = 1, ..., n\}$. Furthermore, if $A$ is a tridiagonal $\infty \times \infty$ real, symmetric matrix whose truncated spectra have the interlacing property then the functions $1, \det(xI - A_1), \det(xI - A_2), ..., \det(xI - A_k), ...$ form a family of orthogonal polynomials for an appropriate measure.

**Theorem 2** Let for each $1 \leq n \leq m$, $\lambda_{n,1} > \lambda_{n,2} > ... > \lambda_{n,n}$ be given with

$$
\lambda_{n+1,1} > \lambda_{n,1} > \lambda_{n+1,2} > \lambda_{n,2} > ... \lambda_{n,n} > \lambda_{n+1,n+1}
$$

for $n + 1 \leq m$. Then there exist exactly $2^{m-2}$ real symmetric $m \times m$ matrices whose truncations have this given spectrum and at most one (up to conjugation by a diagonal $m \times m$ matrix with entries $\pm 1$) is tridiagonal.

**Theorem 3** Let for each $1 \leq n \leq m$, $\{\lambda_{n,1}, ..., \lambda_{n,n}\} = \Lambda_n$ be distinct complex numbers and assume that $\Lambda_{n+1} \cap \Lambda_n = \emptyset$ then the set of all $X \in M_m(\mathbb{C})$ such that the spectrum of $X_n$ is $\Lambda_n$ is an subvariety isomorphic with $(\mathbb{C}^\times)^{m-2}$. 
\( \mathfrak{g} \) a Lie algebra over \( \mathbb{C} \) with a nondegenerate, symmetric, bilinear, \( \mathfrak{g} \)-invariant form.

Identify \( \mathfrak{g} \) with \( \mathfrak{g}^* \) and thereby \( S(\mathfrak{g}) \) with \( S(\mathfrak{g}^*) = P(\mathfrak{g}) \). \( U(\mathfrak{g}) \) filtered by the standard filtration \( U^j \subset U^{j+1} \) one has

\[
Gr U(\mathfrak{g}) \cong S(\mathfrak{g})
\]
as \( P(\mathfrak{g}) \). If \( f, g \in P(\mathfrak{g}) \) are homogeneous of degree \( j, k \) respectively then there are \( u \in U^j, v \in U^k \) such that

\[
u + U^{j-1} \to f, v + U^{k-1} \to g
\]

Then \( [u, v] \in U^{j+k-1} \) and modulo \( U^{j+k-2} \) we get an element of \( P^{j+k-1} \). Independent of all of the choices, \( \{f, g\} \). The Poisson-Bracket.

\[
\{f, gh\} = \{f, g\}h + g\{f, h\}
\]

This implies that there is a vector field, \( X_f \). on \( \mathfrak{g} \) with polynomial coefficients such that \( X_f g = \{f, g\} \).
\( g_n = M_n(\mathbb{C}) \) imbedded in \( g_{n+1} \) as the upper left corner.

Take \( f_{j,k,n}(X) = \text{tr} X_k^j, j = 1, 2, \ldots, k, k = 1, \ldots, n \). These polynomials Poisson commute and are functionally independent.

\[ GZ_n \] is the algebra generated by \( f_{j,k,n}(X) = \text{tr} X_k^j, j = 1, 2, \ldots, k, k = 1, \ldots, n - 1 \).

**Theorem 4** If \( f \in GZ_m \) then \( X_f \) generates a global one parameter group of holomorphic transformations. Thus, in particular, product of the groups generated by the \( X_{f_{j,k,n}} \) acts on \( M_n(\mathbb{C}) \) holomorphically yielding holomorphic action of the additive group of \( \mathbb{C}^{\binom{n}{2}} \) on \( M_n(\mathbb{C}) \).

**Theorem 5** If \( X \in \Omega_m \) then \( AX \subset \Omega_m \) and the orbit, \( AX \) is the set of all \( Y \in M_m(\mathbb{C}) \) such that the spectra of the truncations of \( Y \) are the same as those of \( X \). Furthermore, the restriction of the action of \( A \) to \( \Omega_n \) yields an algebraic action of \( (\mathbb{C}^\times)^{\binom{m}{2}} \).
The last part of this theorem involves work form Mark Colarusso’s thesis. He and Sam Evens have a series of papers that give a detailed study of the properties of these flows with applications.

The analogous flow on Hermitian matrices was studied by Guilleman and Sternberg.


g simple Lie algebra over \( \mathbb{C} \) of rank \( l \). \( \mathfrak{b} \) a Borel subalgebra.

\( \wedge^l \mathfrak{g} \) as a \( \mathfrak{g} \)-module in the usual way.

\( \mathcal{C} \) the Casimir operator corresponding to the Killing form of \( \mathfrak{g} \).

The \( \mathcal{C} \) eigenvalues on \( \wedge^l \mathfrak{g} \) are all less than or equal to \( l \).
Theorem 6  The largest eigenvalue of $\mathcal{C}$ on $\wedge^l \mathfrak{g}$ is $l$. The highest weight spaces of the $l$-eigenspace, $V_l$, in $\wedge^l \mathfrak{g}$ are the lines $\wedge^l \mathfrak{a}$ with $\mathfrak{a}$ an abelian ideal in $\mathfrak{b}$ of dimension $l$. Let $V_l$ be this eigenspace.
Ranee Brylinski:

\( G_l(\mathfrak{g}) \) the Grassmannian of \( l \) dimensional subspaces of \( \mathfrak{g} \).

\( \mathfrak{h} \) a Cartan subalgebra.

\( G \) the adjoint group of \( \mathfrak{g} \).

The closure of \( G\mathfrak{h} \) contains all abelian ideals in \( \mathfrak{b} \) of dimension \( l \).

We proved.

\( e \) a regular nilpotent of \( \mathfrak{g} \).

**Theorem 7** If \( x \) is a regular element of \( \mathfrak{g} \) contained in \( \mathfrak{b} \) then \( \mathfrak{g}^e \) is contained in the closure of \( B\mathfrak{g}^x \).

**Theorem 8** The closure of \( B\mathfrak{g}^e \) contains all abelian ideals of \( \mathfrak{b} \) of dimension \( l \).
Considering the action of $G$ on $G_l(g)$ then each orbit $GV$ contains a closed orbit $GW$ in its closure. The stabilizer of $W$ is a parabolic subgroup of $G$. Hence it contains a Borel subgroup. One checks that if $\dim W = l$ and $BW = W$ then $W \subset [\mathfrak{b}, \mathfrak{b}]$. Thus it is an ideal in $\mathfrak{b}$.

**Theorem 9** $l$–dimensional ideals in $\mathfrak{b}$ are abelian.

Critical to the proofs are Kostant’s cyclic elements that appear in two of the three papers listed. Let $f_1, \ldots, f_l$ be a minimum set of homogeneous generators for $P(g)^G$ ordered by degree. An element, $x \in g$ is called cyclic if

$$f_i(x) = 0, i < l, f_l(x) \neq 0.$$ 

Note that if $c$ is a Coxeter element of $W(h)$ then the order of $c$ is the Coxeter number

$$h = d_l = \deg f_l > d_j = \deg f_j, j < l.$$ 

Let $\zeta$ be a primitive $h$–th root of 1. Then if $z \in \mathfrak{h}, z \neq 0$ is such that $cz = \zeta z$ then

$$f_i(z) = f_i(cz) = \zeta^{d_i} f_i(z).$$

Thus $z$ is a cyclic element.
The singular set $X$ the set of $x \in \mathfrak{g}$, $\dim \mathfrak{g}^x > l$.

The zero set of the $l \times l$ minors of $\text{Jac}(f_1, \ldots, f_l)$.

$$\mathcal{I}(X) = \{ f \in P(\mathfrak{g}) | f(X) = 0 \}.$$  

Do the $l \times l$ minors of $\text{Jac}(f_1, \ldots, f_l)$ generate $\mathcal{I}(X)$?

(..., ...) the Killing form.

If $f \in P(\mathfrak{g})$ then define $\nabla f(x)$ for $x \in \mathfrak{g}$ by

$$(\nabla f(x), y) = df_x(y), y \in \mathfrak{g}.$$  

$$\{ \nabla f_1(x), \ldots, \nabla_l f(x) \}$$
is a basis of $g^x$ if $x$ is regular. Noting that if $f \in P(g)^G$ then

$$g \nabla f(x) = \nabla f(gx).$$

If $x$ is regular semisimple then

$$C(\nabla f_1(x) \wedge \cdots \wedge \nabla_l f(x)) = l \nabla f_1(x) \wedge \cdots \wedge \nabla_l f(x).$$

Thus the result is true for all $x$. If $M$ is the span of the $l \times l$ minors of $\text{Jac}(f_1, \ldots, f_l)$ then $Cf = lf$ for $f \in M$.

**Theorem 10** $M$ is equivalent with $V_l$ as a $G$–module.

If $\mathfrak{h} \subset g$ is a Cartan subalgebra then let

$$\pi_{\mathfrak{h}} = \prod_{\alpha > 0} \alpha$$

for some choice of positive roots relative to $\mathfrak{h}$.

$$\mathcal{I} = \{f \in P(V) | \pi_{\mathfrak{h}}|f|_{\mathfrak{h}}\}.$$ 

Then $M \subset \mathcal{I}$. 
\[ Ric - \frac{1}{2} Rg + \Lambda g = \frac{8\pi G}{c^4} T \]

\( \Lambda \) the cosmological constant.

\( g \) a Lorentzian (- - - +) metric on a 4 manifold \( M \).

Associated to \( g \) is the Hodge \(*\)-operator mapping \( \wedge^k T(M)^* \) \( \rightarrow \) \( \wedge^{4-k} T(M)^* \).
Maxwell's equations $\omega \in \wedge^2 T(M)^*$:

$$d\omega = d \ast \omega = 0.$$ 

These equations are conformally invariant.

Examples:

$M = \mathbb{R}^4, g = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$.

$H = 2 \times 2$ Hermitian matrices

$$X = \begin{bmatrix} x_4 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_4 - x_3 \end{bmatrix}$$

with the metric $\langle X, X \rangle = \det(X)$.

$M = U(2) \cong S^1 \times S^3$. $T_I(M) = iH$ and the left invariant metric $\langle X, X \rangle = -\det(X)$.

$$F : H \to M,$$
\[ F(X) = \frac{I + iX}{I - iX}, \]
\[ (F^* \langle \ldots, \ldots \rangle)_X = 4(1 + 2 \sum_{i=1}^{4} x_i^2 + (X, X)^2)(\ldots, \ldots)_X. \]

Two incarnations of \( U(2, 2) \):

\[
J_1 = \begin{bmatrix} 0 & iI \\ -iI & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.
\]

\[
G_i = \{ g \in GL(4, \mathbb{C}) | gJ_ig^* = J_i \}.
\]

\[
L = \frac{1}{\sqrt{2}} \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix}
\]

\[
\sigma(g) = LgL^* = LgL^{-1}, \quad \sigma : G_1 \rightarrow G_2.
\]

If

\[
\sigma(g) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]
\[ gZ = (AZ + B)(CZ + D)^{-1} \]

for \( ZZ^* < I \).

This realizes \( U(2) \) as the Shilov boundary of \( ZZ^* < I \) and \( U(2, 2) \) as a group of conformal transformations of \( M \).

\[
Maxw \subset \Omega^2(M)_\mathbb{C}
\]

the space of all smooth solutions to Maxwell’s equations with the \( C^\infty \) topology. Then this representation of \( U(2, 2) \) is an admissible, Smooth, Fréchet module of moderate growth. It breaks up into four irreducible submodules. \( Maxw_0^\pm, Maxw_2^\pm \).

Note

\[
H^2(U(2), \mathbb{C}) = 0, \\
H^3(U(2), \mathbb{C}) = \mathbb{C}, H_3(U(2), \mathbb{Z}) = \mathbb{Z}SU(2).
\]
This implies that if $\eta \in \Omega^3(\mathbb{C})$ and $d\eta = 0$ then of $g \in U(2, 2)$ then

$$\int_{SU(2)} \eta = \int_{gSU(2)} \eta = \int_{SU(2)} g^* \eta.$$ 

If $\alpha, \beta \in Maxw$ then $\alpha = d\mu$. Set

$$\langle \alpha, \beta \rangle = \int_{SU(2)} \mu \wedge \overline{\beta}.$$ 

The formula is independent of the choices and

$$\langle Maxw^\varepsilon_{p,q}, Maxw^\nu_{r,s} \rangle = 0$$

unless $\varepsilon = \nu$ and $(p, q) = (r, s)$. Furthermore,

$$\varepsilon \langle \ldots, \ldots \rangle_{Maxw^\varepsilon_{p,q}} > 0.$$ 

If $V$ is the underlying $(g, K)$–module of one of the components of $Maxw$ then

$$H^2(g, K, V) = \mathbb{C}.$$ 

It also has a realization in the continuation of the holomorphic discrete series or the antiholomorphic discrete series.
$SU(2, 2)$ is also the quaternionic real form of $SL(4, \mathbb{C})$ and so these representations can be realized as first cohomology of holomorphic or antiholomorphic line bundles on the open orbit in $\mathbb{P}^3$ that has no non-constant functions. This corresponds to the twister transform.