Quantization, the orbit method, and unitary representations

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Representation Theory, Geometry, and Quantization:
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Outline

Quantum mechanics

Physical system $\longleftrightarrow$ complex Hilbert space $\mathcal{H}$

States $\longleftrightarrow$ lines in $\mathcal{H}$

Observables $\longleftrightarrow$ linear operators $\{A_j\}$ on $\mathcal{H}$

Expected value of obs $A \longleftrightarrow \langle Av, v\rangle$

Energy $\longleftrightarrow$ special skew-adjoint operator A_0

Time evolution $\longleftrightarrow$ unitary group $t \mapsto \exp(tA_0)$

Observable A conserved $\longleftrightarrow [A_0, A] = 0$

Moral of the story: quantum mechanics is about Hilbert spaces and Lie algebras.

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Unitary representations of a Lie group G

Unitary repn is Hilbert space $\mathcal{H}_\pi$ with action

$$G \times \mathcal{H}_\pi \to \mathcal{H}_\pi, \qquad (g, v) \mapsto \pi(g)v$$

respecting inner product: $\langle v, w\rangle = \langle \pi(g)v, \pi(g)w\rangle$.

π is irreducible if has exactly two invt subspaces.

Unitary dual problem: find $\widehat{G}_u$ = unitary irreps of G.

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One of Kostant's greatest contributions was understanding the power of the analogy

unitary repns	$\leftrightsquigarrow$	quantum mech systems
Hilb space, Lie alg of ops		Hilb space, Lie alg of ops

Unitary repns are hard, but quantum mech is hard too. How does an analogy help?

Physicists have a cheat sheet!

There is an easier version of quantum mechanics called classical mechanics. Theories related by

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A little bit of background

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$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

and a nondegeneracy condition.

Any smooth function f on M defines

Hamiltonian vector field $\xi_f = \{f, \cdot\}$.

Example: M = cotangent bundle.

Example: M = Kahler manifold.

Example: M = conjugacy class of $n \times n$ matrices.

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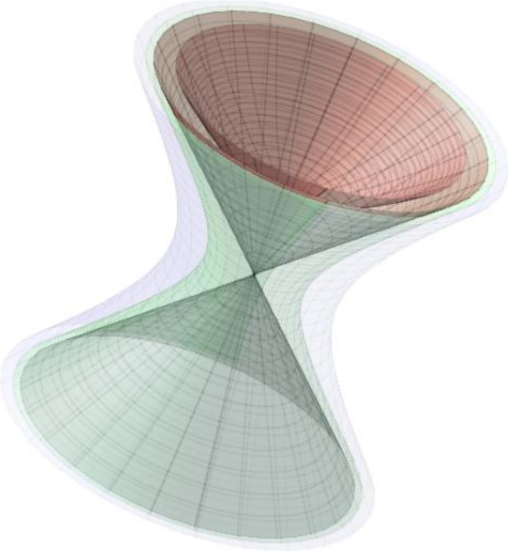
Some conjugacy classes of 2×2 real matrices



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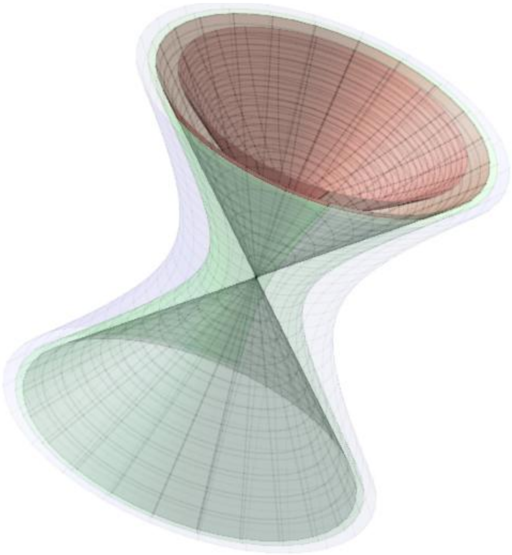

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Moral of the story: classical mechanics is about symplectic manifolds and Poisson Lie algebras.

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Should make irreducible unitary correspond to homogeneous Hamiltonian.

Must make sense of ↑↓. Physics ↑↓ not our problem.

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What's a Hamiltonian G-space?

M manifold with Poisson bracket $\{,\}$ on smooth functions

$\{f, *\} \rightsquigarrow \xi_f \in \mathrm{Vect}(M)$ Hamiltonian vector field

G action on $X \rightsquigarrow$ Lie alg hom $\mathfrak{g} \to \mathrm{Vect}(M)$, $Y \mapsto \xi_Y$.

M is a Hamiltonian G-space if this Lie algebra map lifts

$$\begin{array}{ccc} & & C^\infty(M) \\ & \nearrow & \downarrow \\ \mathfrak{g} & \to & \mathrm{Vect}(M) \end{array} \qquad \begin{array}{ccc} & & f_Y \\ & \nearrow & \downarrow \\ Y & \to & \xi_Y \end{array}$$

Map $\mathfrak{g} \to C^\infty(M)$ same as moment map $\mu\colon M \to \mathfrak{g}^*$.

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Method of coadjoint orbits

Recall: Hamiltonian G-space X comes with (G-equivariant) moment map $\mu\colon X \to \mathfrak{g}^*$.

Kostant's theorem: homogeneous Hamiltonian G-space $=$ covering of G-orbit on $\mathfrak{g}^*$.

Kostant's rep theory $\leftrightsquigarrow$ physics analogy now leads to

Kirillov-Kostant philosophy of coadjt orbits:

$$\{\text{irr unitary reps of } G\} =_{\text{def}} \widehat{G} \overset{?}{\leftrightsquigarrow} \mathfrak{g}^*/G. \qquad (\star)$$

MORE PRECISELY... restrict right side to "admissible" orbits (integrality cond). Expect to find "almost all" of $\widehat{G}$: enough for interesting harmonic analysis.

Method of coadjoint orbits

Recall: Hamiltonian G-space X comes with (G-equivariant) moment map $\mu \colon X \to \mathfrak{g}^*$.

Kostant's theorem: homogeneous Hamiltonian G-space $=$ covering of G-orbit on $\mathfrak{g}^*$.

Kostant's rep theory $\leftrightsquigarrow$ physics analogy now leads to Kirillov-Kostant philosophy of coadjt orbits:

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Evidence for orbit method

With the caveat about restricting to admissible orbits. . .

$$\widehat{G} \quad \overset{?}{\leftrightsquigarrow} \quad \mathfrak{g}^*/G. \qquad (\star)$$

$(\star)$ true for G simply connected nilpotent (Kirillov)

General idea $(\star)$, without physics motivation, due to Kirillov.

$(\star)$ true for G type I solvable (Auslander-Kostant).

$(\star)$ for algebraic G reduces to reductive G (Duflo).

Case of reductive G is still open.

Actually $(\star)$ is false for connected nonabelian reductive G.

But there are still theorems close to $(\star)$.

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So concentrate on reductive groups. . .

Two ways to study representations for reductive G:

1. start with coadjt orbit, seek representation. Hard.
2. start with representation, seek coadjt orbit. Easy.

Really need to do both things at once. Having started to do mathematics in the Ford administration, I find this challenging. (Gave up chewing gum at that time.)

Reductive Lie group G = closed subgp of $GL(n, \mathbb{R})$ which is closed under transpose, and $\#G/G_0 < \infty$.

From now on G is reductive.

$\text{Lie}(G) = \mathfrak{g} \subset n \times n$ matrices. Bilinear form

$$T(X, Y) = \text{tr}(XY) \Rightarrow \mathfrak{g} \overset{G\text{-eqvt}}{\simeq} \mathfrak{g}^*$$

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First example: hyperbolic orbits

$G = GL(n,\mathbb{R})$, $n = p + q$, $x > y$ real numbers

$$\mathcal{O}_{p,q}(x,y) =_{\text{def}} \text{diagonalizable matrices with eigvalues } x \text{ (mult } p\text{) and } y \text{ (mult } q\text{)}.$$

Define $\text{Gr}(p,n)$ = Grassmann variety of p-dimensional subspaces of $\mathbb{R}^n$.

$\mathcal{O}_{p,q}$ is Hamiltonian G-space of dimension $2pq$.

$$\mathcal{O}_{p,q}(x,y) \to \text{Gr}(p,n), \qquad \lambda \mapsto x \text{ eigenspace}$$

exhibits $\mathcal{O}_{p,q}(x,y)$ as affine bundle over $\text{Gr}(p,n)$

General reductive G: $\mathcal{O} \subset \mathfrak{g}^*$ hyperbolic if elements are diagonalizable with real eigenvalues.

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We pause for a word from our sponsor. . .

Classical physics example:

configuration space X = manifold of positions.

State space $T^*(X) =$ symplectic manifold of positions and momenta.

Quantization

$\mathcal{H} = L^2(X)$

= square-integrable half-densities on X

= wave functions for quantum system.

Size of wave function $\leftrightsquigarrow$ probability of configuration.
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Kostant-Kirillov idea:

Hamiltonian G-space $M \approx T^*(X) \implies$

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Hyperbolic representations

Two $GL(n,\mathbb{R})$-equivariant real line bdles on $\mathrm{Gr}(p,n)$:

1. $\mathcal{L}_1$: fiber at p-diml $S \subset \mathbb{R}^n$ is $\bigwedge^p S$;
2. $\mathcal{L}_2$: fiber at S is $\bigwedge^{n-p}(\mathbb{R}^n/S)$.

Real numbers x and $y \rightsquigarrow$ Hermitian line bundle

$$\mathcal{L}(x,y) = \mathcal{L}_1^{ix} \otimes \mathcal{L}_2^{iy}.$$

Unitary representations of $GL(n,\mathbb{R})$ associated to coadjoint orbits $O_{p,q}(x,y)$ are

$$\pi_{p,q}(x,y) = L^2(\mathrm{Gr}(p,n), \mathcal{L}(x,y)).$$

Same techniques (still for reductive G) deal with all hyperbolic coadjoint orbits.

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Real numbers x and y $\rightsquigarrow$ Hermitian line bundle

$$\mathcal{L}(x, y) = \mathcal{L}_1^{ix} \otimes \mathcal{L}_2^{iy}.$$

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Same techniques (still for reductive G) deal with all hyperbolic coadjoint orbits.

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Second example: elliptic orbits

$G = GL(2n, \mathbb{R})$, $x > 0$ real number

$$\mathcal{O}_e(x) =_{\text{def}} \text{real matrices } \lambda \text{ with } \lambda^2 = -x^2 I$$
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$\mathcal{O}_e(x)$ is Hamiltonian G-space of dimension $2n^2$.

Define a complex manifold

$$X = \text{complex structures on } \mathbb{R}^{2n}$$
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Last condition is open, so X open in $\text{Gr}_{\mathbb{C}}(n, 2n)$.

$$\mathcal{O}_e(x) \to X, \qquad \lambda \mapsto ix \text{ eigenspace}$$

is isomorphism $\mathcal{O}_e(x) \simeq X$

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First, complex manifold X: real space $T_x X$ has complex structure j_x: real linear aut, $j_x^2 = -I$.

Second, symplectic: $T_x X$ has symp form ω_x.

Third, structures compatible: $\omega_x(j_x u, j_x v) = \omega_x(u, v)$.

These structures define indefinite Riemannian structure $g_x(u, v) = \omega_x(u, j_x v)$.

Kähler structure is positive if all g_x are positive; signature (p, q) if all g_x have signature (p, q)

Example: X = complex structures on $\mathbb{R}^{2n}$ has signature $\left(\binom{n}{2}, \binom{n+1}{2}\right)$ or $\left(\binom{n+1}{2}, \binom{n}{2}\right)$.

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Dealing with indefinite Kähler

Example: $U(n)/U(1)^n$ has n! equivariant Kähler structures. Here's how...

1. Distinct reals $\ell = (\ell_1, \ldots, \ell_n) \rightsquigarrow U(n)$ coadjt orbit

$$\mathcal{O}_e(\ell) = U(n) \cdot \mathrm{diag}(i\ell_1, \ldots, i\ell_n);$$

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2. Isomorphic to complex X = complete flags in $\mathbb{C}^n$ by

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Kostant-Auslander idea:

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ℓ_j distinct real; now assume $\ell_j \equiv (n-1)/2 \pmod{\mathbb{Z}}$. Put

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Get $\mathcal{L}_{\ell-\rho}$ hol line bdle on X = flags in $\mathbb{C}^n$.
Recall $\sigma \cdot \ell$ decr; so $\mu = \sigma\ell - \rho$ = dom wt for $U(n)$.
Write E_μ = irr rep of $U(n)$ of highest weight μ.

Theorem (Borel-Weil-Bott-Kostant)

Write $\mathcal{O}_{\ell-\rho}$ = sheaf of germs of hol secs of $\mathcal{L}_{\ell-\rho}$. Then

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Recall $\sigma \cdot \ell$ decr; so $\mu = \sigma\ell - \rho$ = dom wt for $U(n)$.
Write E_μ = irr rep of $U(n)$ of highest weight μ.

Theorem (Borel-Weil-Bott-Kostant)

Write $O_{\ell-\rho}$ = sheaf of germs of hol secs of $\mathcal{L}_{\ell-\rho}$. Then

$$H^p(X, O_{\ell-\rho}) = \begin{cases} E_\mu & p = \ell(\sigma) \\ 0 & \text{otherwise.} \end{cases}$$

Moral of the story: look for representations not in holomorphic sections, but in cohomological degree given by signature of Kähler metric.

Quantizing $U(n)\cdot \mathrm{diag}(i\ell_1,\ldots,i\ell_n) \subset \mathfrak{u}(n)^*$

ℓ_j distinct real; now assume $\ell_j \equiv (n-1)/2 \pmod{\mathbb{Z}}$. Put

$$\rho = ((n-1)/2, (n-3)/2, \ldots, (-n+1)/2) \in \mathbb{R}^n$$

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