Defining equations for some nilpotent varieties

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The Mathematical Legacy of Bertram Kostant
MIT
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Kostant's interest in the Buckyball
- Kostant’s interest in the Buckyball

- He didn’t like the Brooklyn Dodgers
Kostant’s most highly cited paper in Math Reviews

Let $\mathcal{N}$ be nilpotent cone in $\mathfrak{g}$. Kostant showed

- $\mathcal{N}$ is a normal variety
- The defining ideal of $\mathcal{N}$ is generated by $u_1, \ldots, u_\ell$,

a set of basic invariants in $S_{\mathfrak{g}^*}$. Assume $\deg(u_\ell) = h$.

- As a module for $G$, write

$$\mathbb{C}^\bullet[\mathcal{N}] \cong \bigoplus p_\lambda V_\lambda$$

where $p_\lambda = q^{m_1^\lambda} + \cdots + q^{m_\ell_\lambda}$. Then

$$\ell_\lambda = \dim V_\lambda^T.$$

These exponents are called the *generalized exponents* of $\lambda$. 
Let $e$ be a principal nilpotent and $e, h, f$ basis of $\mathfrak{sl}_2$-triple.

Form slice

$$v := f + g^e.$$ 

The $\mathbb{C}^*$-action on $v$ coming from $h$ and scaling so that $f$ is in degree 0.

Then $g^e$ is graded in degrees $2m_1 + 2, \ldots, 2m_\ell + 2$, where $m_1, \ldots, m_\ell$ are the usual exponents.

Restrict $u_i$ to $v$. Then

**Theorem (Kostant)**

$u_i$ has linear term when expressed in the graded basis of $g^e$.

Jacobian matrix of $u_i$'s is rank $\ell$ everywhere on $v$, including at $f$. 

Explicit list of basic invariants

- Can take a possible list and restrict to a Cartan subalgebra \( \mathfrak{h} \). Look for a point where determinant of Jacobian is nonzero.

Theorem
The invariants
\[
\text{tr}((\text{ad } X)^2), \ldots, \text{tr}((\text{ad } X)^d) \ldots, \text{tr}((\text{ad } X)^{30})
\]
is a list of basic invariants for \( E_8 \).

Can use smaller representations for other types if adjoint representation doesn’t work (e.g., if there is an odd fundamental degree).
Explicit list of basic invariants

- Can take a possible list and restrict to a Cartan subalgebra $\mathfrak{h}$.
  Look for a point where determinant of Jacobian is nonzero.
- Take a possible single basic invariant and restrict to $v$ and see that it has a linear term.

**Theorem**

*The invariants*

$$tr((ad X)^2), \ldots tr((ad X)^{d_i}), \ldots tr((ad X)^{30})$$

*is a list of basic invariants for $E_8$.*

Can use smaller representations for other types if adjoint representation doesn’t work (e.g., if there is an odd fundamental degree).
Example in MAGMA

Adjoint representation of the slice $v$ for $F_4$, using 4 variables: $m[1], m[2], m[3], m[4]$.

```
> M;
[0  m[1]  0  0  0  0  m[2]  -m[2]  0  0  0  0  -m[3]  -m[3]  0  0  0  0  0  0  0  0  m[4]  0  0
[22  0  m[1]  0  0  0  0  m[2]  0  2*m[2]  0  0  0  -m[3]  -m[3]  0  0  0  0  0  0  0  0  0  0]
[0  42  0  m[1]  0  0  0  0  m[2]  0  2*m[2]  0  0  0  -m[3]  0  0  0  0  0  0  0  0  0  0  0]
[0  0  60  0  2*m[1]  m[1]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]
[0  0  16  0  0  0  m[1]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]
[0  0  0  42  0  0  0  m[1]  m[1]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]
[0  0  42  0  0  0  m[1]  m[1]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]
[0  0  0  30  30  0  0  0  m[1]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]
[0  0  0  32  0  0  0  m[1]  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]
[0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]
[0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]
```

Linear term in each case.
Usual exponents

When $\lambda = \theta$ is the highest root, the generalized exponents are the usual exponents.

Two ways to see this:

- Generalized exponents come from grading of

$$V^g_{\lambda} = g^g \subset g^e.$$ 

Have equality since $g^e$ is abelian.

- Fix $i$. Then

$$\left\{ \frac{\partial u_i}{\partial x_j} \right\}$$

is a basis of a copy of the adjoint representation in $Sg^*$. Non-zero on $\mathcal{N}$; in fact, the copies are linearly independent.

So $\{d_i - 1\}$ are generalized exponents for $\lambda = \theta$. 
Applications to intersections

- \( \mathcal{N} \cap \mathfrak{h} = \{0\} \). Functions at 0 are

\[
\mathcal{S}_{\mathfrak{h}}^* / (\text{positive invariants}) \cong H^* (G/B).
\]

Starting point to look at \( \overline{\mathcal{O}} \cap \mathfrak{h} \).

We obtain cohomology of Springer fiber for dual orbit in type A.

Kraft, De Concini-Procesi, Tanisaki, Carrell

- \( \mathcal{N} \cap S_{f'} \) for smaller nilpotent, where

\[
S_{f'} = f' + g^{e'}
\]

Also can do this by replacing \( \mathcal{N} \) by smaller nilpotent orbit \( \mathcal{O} \):

\[
\overline{\mathcal{O}} \cap S_{f'}.
\]

Brieskorn, Slodowy, Kraft-Procesi, Fu-Juteau-Levy-S.
Let $f'$ be in the subregular orbit. Take slice to $f'$:

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Then $u_1, \ldots, u_{\ell-1}$ have (linearly independent) linear terms on $S_{f'}$.

While $u_\ell$ of highest degree, the Coxeter number, is exactly the defining equation in the remaining three dimensions of an ADE-singularity.

Carried out by Slodowy
Example in $E_6$

The latter is the equation for the $E_6$ singularity in $\mathbb{C}^3$: $x^2 + y^3 + z^4 = 0$. 
Singularities in $E_6$
Find equations for other orbits

- Weyman for $GL_n(\mathbb{C})$.

\[ O = O_\lambda \]

$\lambda = (\lambda_1, \lambda_2, \ldots)$ partition of $n$

Let $k_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i - i + 1$

Equations come from subspace of $k_i \times k_i$-minors isomorphic to representation of highest weight $\omega_i + \omega_{n-i}$, plus the basic invariants.
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  Equations come from subspace of \( k_i \times k_i \)-minors isomorphic to representation of highest weight \( \varpi_i + \varpi_{n-i} \), plus the basic invariants.

- Hook \( \lambda = (a, 1, \ldots 1) \).
  Minimal generators: all \( a \times a \) minors. Rank conditions plus basic invariants up to degree \( a \).
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  \[ O = O_\lambda \]
  \[ \lambda = (\lambda_1, \lambda_2, \ldots) \text{ partition of } n \]
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  Equations come from subspace of $k_i \times k_i$-minors isomorphic to representation of highest weight $\varpi_i + \varpi_{n-i}$, plus the basic invariants.

- Hook $\lambda = (a, 1, \ldots 1)$.
  Minimal generators: all $a \times a$ minors. Rank conditions plus basic invariants up to degree $a$.

- Almost rectangular: $\lambda = (a, a, \ldots a, b)$.
  Minimal generators: Just need a copy of the adjoint in degree $a$ and basic invariants up to degree $a$.
  Take entries $X^a$ where $X = (x_{ij})$ is a generic matrix for a copy of the adjoint rep.
Let $\phi$ be the dominant short root.

The highest generalized exponent for $V_\phi$ occurs in $ht(\phi)$ degree, the dual Coxeter number. This is true for any representation $V_{\lambda}$.

The ideal for the subregular nilpotent variety is given by a copy of $V_\phi$ in this top degree together with $u_1, \ldots, u_{\ell-1}$.

These are minimal generators.
Main result

- Let $\Omega$ be a set of orthogonal, short, simple roots. Let $s = |\Omega|$.
  Let $n_\Omega$ be nilradical of the parabolic subalgebra attached to $\Omega$.
- Let $O_\Omega$ be the Richardson orbit in $n_\Omega$.
  These orbits were considered by Broer in Kostant 65th volume.
  For $s = 1$, we get the subregular orbit. For $s = 0$, we get principal nilpotent orbit.
- Let $r$ be dimension of zero weight space of $V_\phi$, which is the number of short simple roots, and order the generalized exponents for $V_\phi$ by $m_1^\phi \leq \cdots \leq m_r^\phi$. 

Theorem (Johnson, S-)

The ideal for $O_\Omega$ is minimally generated by:

\[ a \text{ copy of } V_\phi \text{ in either degree } m_{r + s - 1}^\phi \text{ or } m_{\lfloor r/2 \rfloor}^\phi. \]

\[ (\text{sometimes}) \text{ a copy of } V_\phi \text{ is degree } m_{r + s - 2}^\phi. \]

\[ r - s \text{ of the basic invariants } \]
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**Theorem (Johnson, S-)**

The ideal for $O_\Omega$ is minimally generated by:

- a copy of $V_\phi$ in either degree $m_{r-s+1}^\phi$ or $m_{\lfloor \frac{r}{2} \rfloor}^\phi$.
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Main result

**Theorem (Johnson, S-, arXiv:1706.04820)**

The ideal for $O_{\Omega}$ is minimally generated by:

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- $r - s$ of the basic invariants

![Diagram of studied nilpotent varieties](image)
Pick a basis \( \{ x_i \} \) of \( \mathfrak{g} \) and a dual basis \( \{ y_i \} \) with respect to the Killing form \((\cdot, \cdot)\).

Let \( p \) and \( q \) be two homogeneous invariants of degree \( a + 1 \) and \( b + 1 \), respectively. Then

\[
p \circ q := \sum_i \frac{\partial p}{\partial x_i} \frac{\partial q}{\partial y_i}
\]

is again an invariant.

Homogeneous of degree \( a + b \).

Saito’s flat basis, first considered in a paper by Saito, Yano, Sekiguchi: unique basis (up to scalars) with

\( u_i \circ u_j \in \mathbb{C}[u_1, \ldots, u_{\ell-1}] + cu_\ell \), where \( c \) is a constant.

De Concini, Papi, Procesi:

\( u_i \circ u_j \) a generator of the invariants when \( u_i \circ u_j \) is the degree of some \( u_k \).

A weaker statement is true in type \( D_{2k} \).
Containment of ideals

- Consider the copy of the adjoint representation $V_{u_i}$ determined by $u_i$ by taking derivatives.
- Take its ideal $(V_{u_i})$ in $\mathbb{C}[\mathcal{N}]$.

Theorem (Johnson, S-)

The following are equivalent:

- **Containment**: $V_{u_j} \subset (V_{u_i})$
- There exists an invariant $p$ such that $p \circ u_i = u_j$ modulo expressions in lower degree invariants.

Hence, by DPP result, containment question, outside of $D_{2k}$, is equivalent to $(m_j + 1) - m_i$ is an exponent. This helps us find minimal generators.

For example, in $E_7$, adjoint rep in degree 13 is not in the ideal generated by copy in degree 11, but it is in ideal generated by copy in degree 9, since 3 is not an exponent, but 5 is.
Next we search for a set of generators. Let $P = P_\Omega$. Consider the Springer type map:

$$G \times^P n_\Omega \to O_\Omega$$

If this is a resolution,

$$\mathbb{C}[O_\Omega] = \mathbb{C}[G \times^P n_\Omega] = H^0(G/P, S^\bullet n^*_\Omega)$$

For $\Omega = \emptyset$, $O_\Omega$ is regular nilpotent orbit.

$$\mathbb{C}[\mathcal{N}] = \mathbb{C}[O_{\text{reg}}] = \mathbb{C}[G \times^B n] = H^0(G/B, S^\bullet n^*)$$

Paper by R. Brylinski (Twisted Ideals paper):

- thinking about ideals in $\mathbb{C}[\mathcal{N}]$ coming from cohomology
- subregular ideal
Twisted ideals

Higher vanishing

\[ H^i(G/B, S^\bullet n^* \otimes \mathbb{C}_\mu) = 0 \text{ for } i > 0 \text{ when} \]

- \( \mu = 0 \) (Borho-Kraft, Hesselink)
- \( \mu \text{ dominant} \) (Broer)
- \( \mu \text{ slightly not dominant} \) (Broer)

To compute the occurrences of \( V_\lambda \) in \( H^0 \), compute Euler characteristic \( \sum (-1)^i H^i \), and thus replace \( S^\bullet n^* \) by a sum of one-dimensional representations and then use Bott-Borel-Weil:

Bott-Borel-Weil

\[ H^i(G/B, \mathbb{C}_\lambda) = 0 \text{ except if } w \cdot \lambda \text{ is dominant and } i = \ell(w), \text{ in which case it is } V_{w \cdot \lambda}. \text{ Here, } w \in W, \text{ the Weyl group, is unique.} \]

Conclude: \( H^0(G/B, S^\bullet n^* \otimes \mathbb{C}_\mu) \) is computable in terms of Lusztig’s \( q \)-analog of Kostant weight multiplicity.
Hence, multiplicity of $V_\lambda$ in $H^0(G/B, S^\bullet u^* \otimes \mathbb{C}_\mu)$ is the dimension of the $\mu$-weight space in $V_\lambda$.

The graded version is an affine Kazhdan-Lusztig polynomial (Lusztig).

If $\mu = 0$, get a formula for generalized exponents and also get another way of seeing that their number is dimension of the zero weight space of $V_\lambda$.

For $\mu$ dominant:

$$H^0(G/B, S^\bullet u^* \otimes \mathbb{C}_\mu)$$

will identify with an ideal in

$$\mathbb{C}[\mathcal{N}]$$

and the unique copy of $V_\mu$ in lowest degree generates the ideal.
Let $\Omega = \{\alpha_1, \alpha_3\}$. Consider parabolic and nilradical for subregular: $n_{\alpha_3}$. Take Koszul resolution:

$$0 \to S^{n-1} n_{\alpha_3}^* \otimes \mathbb{C}_{\alpha_1} \to S^n n_{\alpha_3}^* \to S^n n_{\alpha_1, \alpha_3}^* \to 0.$$ 

Take cohomology over $G/B$:

$$0 \to H^0(S^{n-1} n_{\alpha_3}^* \otimes \mathbb{C}_{\alpha_1}) \to H^0(S^n n_{\alpha_3}^*)$$

$$\to H^0(S^n n_{\alpha_1, \alpha_3}^*) \to H^1(S^{n-1} n_{\alpha_3}^* \otimes \mathbb{C}_{\alpha_1}) \to \ldots.$$

Key facts are that $H^1$ vanishes and

$$H^0(S^{n-1} n_{\alpha_3}^* \otimes \mathbb{C}_{\alpha_1}) \simeq H^0(S^{n-2} n_{\alpha_2}^* \otimes \mathbb{C}_\phi).$$

This is a generalization of Broer's result for $n$ when weights are slightly not dominant (see next slide).

Hence, the ideal of the orbit is cut out by a copy of $V_\phi$ in degree 2 in the closure of the subregular orbit. General case uses this kind of induction.
Let $n_m$ be the nilradical for the maximal parabolic in type $A_l$ with simple root $\alpha_m$ not in the Levi subalgebra.

**Theorem (S-)**

Let $r$ be in the range $-|l + 1 - 2m| - 1 \leq r \leq 0$. Then there is a $G$-module isomorphism:

$$H^i(S^n n_m^* \otimes r \varpi_m) \cong H^i(S^{n+rm} n_{l+1-m}^* \otimes -r \varpi_{l+1-m})$$

for all $i, n \geq 0$.

This is always an isomorphism for $H^0$ when $r < 0$.

**Type $A_2$**

$$H^i(S^n n_1^* \otimes -\varpi_1) \cong H^i(S^{n-1} n_2^* \otimes \varpi_2)$$

Can use this $A_2$ result in any bigger Lie algebra. In $A_3$ it says

$$H^i(S^n n_3^* \otimes \alpha_1) \cong H^i(S^{n-1} n_2^* \otimes (\alpha_1 + \alpha_2 + \alpha_3))$$

since $\alpha_1$ has inner product $-1$ with $\alpha_2$ and 0 with $\alpha_3$. This was the key fact on the previous slide.