Quantizing $G$-actions on Poisson manifolds

Victor Guillemin

MIT
One of Bert Kostant’s enduring contributions to mathematics: the theory of geometric quantization. More explicitly: let \((X^{2n}, \omega)\) be a symplectic manifold, let \(G\) be a Lie group and let \(G \times X \to X\) be a Hamiltonian action of \(G\) on \(X\).
Then $X$ is prequantizable if $[w] \in H^2(X, \mathbb{Z})$ in which case there exists a complex line $\mathbb{L} \to X$ and a connection on $\mathbb{L}$ with curvature form $w$.

The goal of geometric quantization: Associate with this data a Hilbert subspace $Q(X)$ of the space of $L^2$ sections of $\mathbb{L}$ and a representation of $G$ on $Q(X)$ with “nice” properties.
For instance, for co-adjoint orbits, $X$, of $G$ associate a subspace $\mathcal{Q}(X)$ of $L^2$ sections of $\mathbb{L}$ on which $G$ acts irreducibly.
Our goal in today’s lecture will be much more modest: To simplify we’ll assume $G$ us an $m$-torus and for $X$ compact we’ll take $Q(X)$ to be the virtual vector space

$$\text{Ker } \partial_L \oplus \text{Im } \partial_L$$

where $\partial_L$ is the $\mathbb{L}$-twisted spin $\mathbb{C}$ Dirac operator.
What about $X$ non-compact? The two main goals of today’s lecture

1. Describe a method for quantizing non-compact prequantizable Hamiltonian $G$-manifolds based upon the “quantization commutes with reduction” principle.

2. Apply this method to an interesting class of examples: $b$-symplectic manifolds
Remark 1
What I am about to report on is joint work with Jonathan Weitsman and Eva Miranda.

Remark 2
According to David Vogan, Bert was probably not the inventor of \([Q, R] = 0\), but it was certainly inspired by his ideas.
Let \((X, w)\) be a pre-quantizable symplectic manifold and \(G \times X \to X\) a Hamiltonian \(G\)-action with moment map \(\phi : X \to \mathfrak{g}^*\). In lieu of assuming \(X\) compact we’ll assume that \(\phi\) is proper.
Let $\mathbb{Z}_G \subseteq \mathfrak{g}^*$ be the weight lattice of $G$ and $\alpha \in \mathbb{Z}_G$ a regular value of $\phi$. For the moment we’ll also assume that $G$ acts freely on $\phi^{-1}(\alpha)$ in which case the reduced spree $X_\alpha = \phi^{-1}(\alpha)/G$ is a prequantizable symplectic manifold and $[Q, R] = 0$ asserts that $Q(X)_\alpha = Q(X_\alpha)$ where $Q(X)_\alpha$ is the $\alpha$-weight space of $Q(X)$. 
What if $\alpha$ is a regular value of $\phi$ but $G$ *doesn’t* act freely on $\phi^{-1}(\alpha)$? Then this assertion is true modulo some complications owing to the fact that $X_\alpha$ is, in this case, a symplectic orbifold, not a symplectic manifold.
What is $\alpha$ is a singular value of $\phi$?

The Meinrenken desingularization trick

Replace $\alpha$ by a nearby $\alpha'$ which is a regular value of $\phi$ then convert the reduced symplectic form on $X_{\alpha'}$ into a symplectic form on $X_{\alpha'}$ which would be, up to symplectomorphism, the reduced symplectic form on $X_\alpha$ if $\alpha$ were a regular value of $\phi$. 
Inspired by these comments we’ll define the formal quantization of $X$ to be the sum

$$\bigoplus_{\alpha} \mathcal{Q}(X_{\alpha})$$
These objects are easiest to define as Poisson manifolds. Namely, let $X = X^{2n}$ be a compact oriented Poisson manifold and let $\pi$ be its Poisson bi-vector field. Then $\pi^n$ is a section of the real line bundle $\Lambda^{2n}(TX) \to X$. 
The definition of $b$-Poisson

We will say that $\pi$ has this property if $\pi^n$ intersects the zero section of $\Lambda^{2n}(TX)$ transversally in a codimension one submanifold $Z$. 
Some implications of this assumption

1. Let's denote this intersection by $Z$. Then $\mathcal{U} = X - Z$ inherits from $\pi$ a symplectic structure whose symplectic form we'll denote by $\omega$.

2. The symplectic leaves of the restriction, $\pi \mid Z$ are $2n - 2$ dimensional symplectic submanifolds of $Z$ and these manifolds define a foliation of $Z$ by symplectic manifolds.
Let $Z_i$ be a connected component of $Z$. Then on a tubular neighborhood $Z_i \times (-\epsilon, \epsilon)$ the symplectic form $w$ has the form

$$w = \alpha \wedge \frac{dt}{t} + w_0, \quad -\epsilon < t < \epsilon$$

where $\alpha$ is a closed one-form on $Z_i$ and $w_0$ a closed two form on $Z_i$. 
Moreover, $\alpha$ and $w_0$ have the following properties:

I. For every symplectic leaf, $L$ of $Z_i$, $\iota^*_L \alpha = 0$, i.e. $\alpha$ defines the symplectic foliation of $Z$.

II. For every symplectic leaf, $L$, of $Z_i$, $\iota^*_L w_0$ is the symplectic form on this symplectic leaf.
We’ll say that the Poisson manifold \((X, \pi)\) is prequantizable if

1. \(w|\mathcal{U}\) is prequantizable: i.e. there exists a line bundle-connection pair \((L, \nabla)\) on \(\mathcal{U}\) with

\[ w = \text{curv}(\nabla) \]

2. On the tubular neighborhood \(\mathcal{U}_i = Z_i \times (-\epsilon, \epsilon)\) the two components. \(\alpha\) and \(w_0\) of the symplectic form

\[ w|\mathcal{U}_i = \alpha \wedge \frac{dt}{t} + w_0 \]

are pre-quantizable: For \(\alpha\) this means that there exists a fibration \(\gamma: Z_i \rightarrow S^1\) with

\[ \alpha = \gamma^* \frac{d\theta}{2\pi} \]
Remark

An implication of this condition is that the symplectic leaves of \( Z_i \), i.e. the level sets of \( \gamma \), are compact.
We will denote by $^bC^\infty(X)$ the space of functions $\varphi$ on $X$ having the properties

1. $\varphi|_{X - Z} \in C^\infty(X - Z)$

2. On the tubular neighborhood, $Z_i \times (-\epsilon, \epsilon)$ defined in item 2, $\varphi = c_i \log |t| + \psi$, with $c_i \in \mathbb{R}$ and $\psi \in C^\infty(Z_i \times (-\epsilon, \epsilon))$. 
Remark

Notice that if $t$ is the defining function for $Z \times \{0\}$ in the neighborhood $Z \times (-\epsilon, \epsilon)$ and $t' = ft$, $f \in C^\infty(Z_i \times (-\epsilon, \epsilon))$, is another defining function, i.e. $f \neq 0$, then

$$c \log |t'| = c \log |t| + c \log |f|$$

with $c \log |f| \in C^\infty(Z_i \times (-\epsilon, \epsilon))$
Hence this definition is an intrinsic definition: i.e. doesn’t depend on the choice of defining function.
Now let $G \times X \to X$ be a Poisson action of $G$ on $X$ and let $w$ be the $b$-symplectic form that we defined in item 3.

**Definition**
This action is Hamiltonian if for every $v \in g$

$$\iota(v_X)w = d\varphi_v, \quad \varphi_v \in bC^\infty(X)$$
Thus $\varphi_v = u_i(v) \log |t| + \psi_v$ where $\psi_v$ is in $C^\infty(Z_i \times (-\epsilon, \epsilon))$ and $u_i$ is an element of $\mathfrak{g}^*$.

**Definition**

$u_i$ is the **modular weight** of the hypersurface $Z_i$. 
Note that since

\[ \iota(v_X)w = \iota(v_X)\alpha \frac{dt}{t} + \iota(v_X)w_0 \]

(1) \[ u_i(v) = \iota(v_X)\alpha = \alpha(v_X) \]
Thus since the symplectic leaves of $Z_i$ are defined by the condition, $\iota_L^* \alpha = 0$, we conclude that $u_i(v) = 0$ implies that $v_X$ is everywhere tangent to the symplectic leaves of $Z$. 
In particular if $K$ is the subtorus of $G$ having

$$k = \{ v \in \mathfrak{g}, \ u_i(v) = 0 \}$$

as Lie subalgebra then the action of $K$ on $Z_i$ preserves the symplectic leaves of $Z_i$. 
An important property of modular weights

**Theorem**

*If* $X$ *is connected and* $u_i \neq 0$ *for some* $Z_i$ *then* $u_i \neq 0$ *for all* $Z_i$'s.

We’ll henceforth assume that this is the case.
Another assumption we will make:
The map $\gamma : Z_i \to S^1$ defined by the prequantizability hypothesis

$$\alpha = \gamma^* \frac{d\theta}{2\pi}$$

intertwines the action of $G$ on $Z$ with the action of $S^1 = G/K$ on itself.
Therefore if $S^1$ is a complementary circle to $K$ in $G$, $S^1$ acts freely on $Z_i$ and interchanges the leaves of the symplectic foliation of $Z_i$. 
In fact one can assume a bit more. Let $v \in \mathfrak{g}$ be the generator of this complementary circle and let $f = \iota(v_X)\alpha$. Then by the Moser trick one can prove
Lemma

The $b$-symplectic forms $w$ and $w' = w - \alpha \wedge df$ are $G$-symplectomorphic.

Hence replacing $w$ by $w'$ we get as a corollary the following result:
Theorem

If $v$ is the generator of the $S^1$ action above

$$\iota(v)w = d \log |t|$$

and

$$\iota(v)w_0 = 0$$

In particular if $u_i$ is the modular weight associated with $Z_i$, $u_i(v) = 1$. 
Moreover the condition

\[ \iota(v_x)w_0 = 0 \]

implies the following: Let's denote by \( Z_i/S^1 \) the symplectic reduction by \( S^1 \) of \( Z_i \times (-\epsilon, \epsilon) \) at some moment level \( \lambda = \log |t|, |t| \neq 0 \).
Then the reduced symplectic form on $Z_i/S^1$ has the defining property

$$\pi^* w_{\text{red}} = w_0$$

where $\pi$ is the projection

$$Z_i \to Z_i/S^1$$
Next let $k_i$ be the Lie subalgebra of $\mathfrak{g}$ defined by

$$v \in k_i \iff u_i(v) = 0$$

Since $u_i$ is a weight of $\mathfrak{g}$, (i.e., the modular weight) $k_i$ is the Lie algebra of a subtorus $K_i$ of $G$. 
Moreover, \( u_i(v) = 0 \) implies \( \iota(v_Z)\alpha = 0 \) which in turn implies

\[
\iota(v_Z)\omega = \iota(v_Z)\omega_0
\]

Thus the projection, \( \pi \), intertwines the actions of \( K \) on \( Z \) with a reduced symplectic action of \( K \) on \( Z/S^1 \).
Theorem 1

The moment map associated with the action of $G$ on $Z_i \times (-\epsilon, \epsilon)$ is the map

$$\phi(z, t) = -\log |t|u_i + \phi_K \circ \pi(z)$$

where $\phi_K$ is the moment map associated with the $K$-action on the symplectic quotient $Z_i/S^1$ and $\pi$ is the projection of $Z_i$ onto $Z_i/S^1$. 
Corollary

The moment image of $Z_i \times (-\epsilon, \epsilon)$ is the polytope in the figure below

The vertical slice at $t = 0$ being the moment polytope, $\Delta_K$, for the action of $K$ on $Z_i/S^1$. 
Note that as the moment image of $Z_i \times (-\epsilon, \epsilon)$ this vertical slice lies on the hyperplane $\text{Log} |t| = -\infty$ and its edges are the lines

$$p + tu_i, \quad -\infty \leq \text{Log} |t| < \log \epsilon$$

the $p$’s being the vertices of $\Delta_K$. 
In particular the weights figuring in the formal quantization of $\mathbb{Z}_i \times (-\epsilon, \epsilon)$ all lie along the lines

$$\beta + \log |t|u, \quad -\infty < \log |t| < \log \epsilon$$

where $\beta$ is a weight of $K$ lying in the moment polytope.
We’ll now describe our second main result

**Theorem 2**

*The formal geometric quantization of $\mathcal{U} = X - Z$ is a finite dimensional virtual vector space.*
Remark 1
The proof of this will be based on the following well-known fact.

Theorem 3
The index of the $\mathbb{L}$-twisted spin $\mathbb{C}$ Dirac operator changes by a factor of $(-1)$ if one changes the orientation of the underlying manifold.
Proof of Theorem 2.
If we delete the moment images of the regions $Z_i \times (-\epsilon, \epsilon)$ from the moment image of $\mathcal{U}$ the remaining image is compact. Hence the contributions to the formal quantization of $\mathcal{U}$ from weights in this region is a finite dimensional vector space. \qed
What about contributions coming from the moment image on slide 40 of $Z_i \times (-\epsilon, \epsilon)$?
We’ll prove these contributions add up to \{0\}. In fact we’ll prove a much stronger result.
Let $S^1$ be the circle group in the factorization

$$G = S^1 \times K$$

We will prove
For every weight, $-n$, $e^{-n} < \epsilon$ of the weight lattice, $\mathbb{Z}$, of $S^1$ let $X_{i,n}$ be the symplectic reduction of $Z_i \times (-\epsilon, \epsilon)$ with respect to the Hamiltonian action of $S^1$. Then

$$Q(X_{i,n}) = \{0\}$$
Proof.
This reduction consists of two copies of the symplectic manifold, $Z_i/S^1$, hence if we quantize them by assigning to them the orientations defined by their symplectic volumes we get two copies of $Q(Z_i/S^1)$.
But these orientations correspond, upstairs on $Z_i \times (-\epsilon, 0)$ and $Z_i \times (0, \epsilon)$, to the orientations defined by the symplectic volume form associated with the $b$-symplectic form

$$\alpha \wedge \frac{dt}{t} + w_0$$

If instead we fix a global volume form $v$ on $X$ and assign to these two spaces the orientations defined by $v$ then we get the same orientation as before on one of these two copies of $Z_i/S^1$ and the opposite orientation on the other.
Hence by theorem 3 the quantization of this symplectic reduction is zero.