Introduction to Proofs
IAP 2015
In-class problems for day 7

Problem 10. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by the conditions*

$$f(x) = 1 \text{ for } x \in \mathbb{Q}$$

and

$$f(x) = -1 \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Q}.$$

Show that, for every $x_0 \in \mathbb{R}$, $f(x)$ *does not have a limit as* $x \to x_0$. *(In other words, given* $x_0 \in \mathbb{R}$ *and* $\lambda \in \mathbb{R}$, *show that the condition*

$$\lim_{x \to x_0} f(x) = \lambda$$

is not valid. You may use the fact that for any $z \in \mathbb{R}$ *and* $\epsilon > 0$ *there exists* $q \in \mathbb{Q}$ *with* $|z - q| < \epsilon$.*)*

Proof.

□

Problem 11. *Let* $f : (0,1) \to (0,1)$ *be a surjective function which is strictly increasing in the sense that for every* $x, y \in (0,1)$ *with* $x < y$ *we have* $f(x) < f(y)$. *Show that* f *is continuous.*

(Recall that f surjective (alternatively, onto) means that for every $y \in (0,1)$ there exists $x \in (0,1)$ with $f(x) = y$).

Proof.

□

Problem 12. *We say that a set* $A \subset \mathbb{R}$ *is open if for every* $x \in A$ *there exists* $\delta > 0$ *such that if we set*

$$B(x;\delta) := \{y \in \mathbb{R} : |y - x| < \delta\}$$

then we have the inclusion

$$B(x;\delta) \subset A.$$

(1) Suppose A_1 and A_2 are two open subsets of $\mathbb{R}$. Show that $A_1 \cap A_2$ is also open.

Proof.

□

(2) The above claim can be extended by induction to show that if A_1, A_2, ..., A_n is a finite collection of open sets, then the intersection

$$A_1 \cap A_2 \cap \cdots \cap A_n$$

is also open. Construct an example to show that the same is not true for an *infinite* sequence of open sets. (*Hint: Consider the open sets* $(0, \frac{1}{n})$ *for* $n \in \mathbb{N} \setminus \{0\}$*).)

Proof.

□

(3) Show that for an arbitrary collection of open sets $(A_x)_{x\in X}$, the union

$$\bigcup_{x\in X} A_x$$

is open. (Note: the index set X is arbitrary, and may even be taken to be uncountable.)

Proof.

□