Introduction to Proofs
IAP 2015
In-class problems for day 6

**Problem 9.** *Let* $f:(0,1)\to\mathbb{R}$ *be given and suppose that* $x\mapsto f(x)$ *has a limit as* $x\to c$ *for some* $c\in(0,1)$. *Show that the limit is unique – that is, suppose that* $a$, $b$ *are limits of* $f$ *as* $x\to c$ *and show that* $a=b$.

*Proof.* Let $a,b\in\mathbb{R}$ be given such that both $a$ and $b$ are limits of $f$ as $x\to c$. Then for each $\epsilon>0$, we may choose $\delta>0$ and $\delta'>0$ such that, for every $y,y'\in(0,1)$ with $0<|y-c|<\delta$ and $0<|y'-c|<\delta$, we have

$$|f(y)-a|<\epsilon$$

and

$$|f(y')-b|<\epsilon.$$

Fix $\epsilon>0$ to be determined later in the argument, and let $\delta,\delta'$ be chosen as described above. Then for every $y''\in(0,1)$ with

$$0<|y''-c|<\min\{\delta,\delta'\}$$

we have

$$|a-b|\le|a-f(y'')|+|f(y'')-b|<2\epsilon. \tag{1}$$

In particular, since the set of $y''$ satisfying these conditions is nonempty, we have

$$|a-b|<2\epsilon$$

for every $\epsilon>0$.

Letting $\epsilon$ tend to zero, we obtain $|a-b|\le 0$; since $|a-b|\ge 0$ holds trivially, we conclude $a=b$ as desired. □

*Remark* 0.1. The first inequality in (1) follows from the *triangle inequality*,

$$|a+b|\le|a|+|b|,\quad a,b\in\mathbb{R}.$$

Another useful form of this inequality often appears as

$$|a|=|(a-b)+b|\le|a-b|+|b|,$$

for $a,b\in\mathbb{R}$, which can be rewritten as

$$|a|-|b|\le|a-b|.$$

Since $a$ and $b$ are interchangable, we in fact have

$$\Big||a|-|b|\Big|\le|a-b|,\quad a,b\in\mathbb{R}.$$