Introduction to Proofs
IAP 2015
In-class problems for day 5

Problem 8. *Let* $(A_i : i \geq 1)$ *be a sequence of countable sets. Show that*

$$\bigcup_{i \in \mathbb{N} \setminus \{0\}} A_i$$

is also countable.

Proof.

$\square$

Problem 9. *Let $f : (0,1) \to \mathbb{R}$ be given and suppose that $x \mapsto f(x)$ has a limit as $x \to c$ for some $c \in (0,1)$. Show that the limit is unique – that is, suppose that a, b are limits of f as $x \to c$ and show that $a = b$.*

Proof. $\square$

Problem 10. *Define $f : \mathbb{R} \to \mathbb{R}$ by the conditions*

$$f(x) = 1 \text{ for } x \in \mathbb{Q}$$

and

$$f(x) = -1 \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Q}.$$

Show that, for every $x_0 \in \mathbb{R}$, $f(x)$ does not have a limit as $x \to x_0$. (In other words, given $x_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, show that the condition

$$\lim_{x \to x_0} f(x) = \lambda$$

is no t valid. You may use the fact that for any $z \in \mathbb{R}$ and $\epsilon > 0$ there exists $q \in \mathbb{Q}$ with $|z - q| < \epsilon$.)

Proof. $\square$

Problem 11. *Let* $f : (0,1) \to (0,1)$ *be a surjective function which is strictly increasing in the sense that for every* $x, y \in (0,1)$ *with* $x < y$ *we have* $f(x) < f(y)$. *Show that* f *is continuous.*

(Recall that f surjective (alternatively, onto) means that for every $y \in (0,1)$ there exists $x \in (0,1)$ with $f(x) = y$).

Proof.

$\square$