Introduction to Proofs
IAP 2015
In-class problems for day 2

Problem 2. *Let X and Y be sets, and let $f : X \to Y$ be a given function. Show that for all $A, B \subset Y$,*

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

and for all $U, V \subset X$,

$$f(U \cup V) = f(U) \cup f(V).$$

Hint: The result can be divided into four separate implications.

Proof. Let $A, B \subset Y$ be given. We first show that

$$f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B).$$

For this, let $x \in f^{-1}(A \cap B)$ be given, so that $f(x) \in A \cap B$, and therefore both of the conditions $f(x) \in A$ and $f(x) \in B$ hold. Rewriting these conditions, we obtain $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$, so that $x \in f^{-1}(A) \cap f^{-1}(B)$ as desired. Since $x \in f^{-1}(A \cap B)$ was arbitrary, this shows the desired implication.

Please work with your team to complete the proof below:

We now show

$$f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B).$$

□

Problem 3. *In this problem, we examine a second proof of the fact that there is no rational $x \in \mathbb{Q}$ such that $x^2 = 2$.*

(1) Suppose that the claim failed, i.e. that there exists $x \in \mathbb{Q}$ with $x^2 = 2$. Choose $\tilde{x} = |x|$ (so that $\tilde{x}^2 = 2$). Show that there exists $q \in \mathbb{N} \setminus \{0\}$ such that

$$(\tilde{x} - 1)q \text{ is a non-negative integer.} \tag{1}$$

Proof.

□

(2) Choose $q_* \in \mathbb{N} \setminus \{0\}$ as the smallest positive integer such that (1) holds. Set $q' = (\tilde{x} - 1)q_*$. Show that:

(a) The inequalities $0 < q' < q_*$ hold (that is, show each of the inequalities $q' > 0$ and $q' < q_*$).

Proof.

□

(b) The quantity $(\tilde{x} - 1)q'$ is a non-negative integer.

Proof.

□

This contradicts the minimality of q_*, so that no such $x \in \mathbb{Q}$ exists.