18.S097 Introduction to Proofs
IAP 2015
Solution to Homework 2

Problem 1. *In this problem, we examine a second proof of the fact that there is no rational $x \in \mathbb{Q}$ such that $x^2 = 2$.*

(1) Suppose that the claim failed, i.e. that there exists $x \in \mathbb{Q}$ with $x^2 = 2$. Choose $\tilde{x} = |x|$ (so that $\tilde{x}^2 = 2$). Show that there exists $q \in \mathbb{N} \setminus \{0\}$ such that

$$(\tilde{x} - 1)q \text{ is a non-negative integer.} \tag{1}$$

Proof. Since $\tilde{x} \in \mathbb{Q}$, we may find $m, n \in \mathbb{Q}$ with $m, n > 0$ such that $\tilde{x} = \frac{m}{n}$. We may then write $\tilde{x} - 1 = \frac{m}{n} - 1$, so that after multiplying both sides by n we obtain

$$n(\tilde{x} - 1) = m - n.$$

Taking $q = n$, we conclude that $(\tilde{x} - 1)q$ is an integer.

It remains to check that $(\tilde{x} - 1)q = m - n$ is non-negative − that is, that the inequality $m - n \geq 0$ holds. For this, it suffices to observe that $(m/n)^2 = 2 > 1$ impleis $m/n > 1$ (since, for instance, the function $x \mapsto x^2$ is increasing on the interval $[0, \infty)$). The desired inequality now follows from the observation that $n > 0$ holds by construction. □

(2) Choose $q_* \in \mathbb{N} \setminus \{0\}$ as the smallest positive integer such that (1) holds. Set $q' = (\tilde{x} - 1)q_*$. Show that:

(a) The inequalities $0 < q' < q_*$ hold (that is, show each of the inequalities $q' > 0$ and $q' < q_*$).

Proof. Recall that $q_* > 0$ holds by definition, and observe that $\tilde{x} - 1 > 0$ (since $\tilde{x}^2 = 2 > 1$ implies $\tilde{x} > 1$). The inequality

$$q' = (\tilde{x} - 1)q_* > 0$$

now follows immediately.

On the other hand, noting that $\tilde{x}^2 = 2 < 4$, we have $\tilde{x} < 2$, and thus $\tilde{x} - 1 < 1$. Since $q_* > 0$ holds by definition, this implies

$$q' = (\tilde{x} - 1)q_* < q_*$$

as desired. □

(b) The quantity $(\tilde{x} - 1)q'$ is a non-negative integer.

Proof. Note that $(\tilde{x} - 1)q' > 0$ follows immediately from $\tilde{x} - 1 > 0$ (which we showed above) and inequality $q' > 0$ (shown in (2a) above). It remains to see that $(\tilde{x} - 1)q'$ is an integer. To see this, we write

$$\begin{aligned}(\tilde{x} - 1)q' &= (\tilde{x} - 1)^2 q_* \\ &= (\tilde{x}^2 - 2\tilde{x} + 1)q_* \\ &= 3q_* - 2\tilde{x}q_*. \end{aligned} \tag{2}$$

Noting that q_* satisfies $(\tilde{x} - 1)q_* \in \mathbb{Z}$, we can find $k \in \mathbb{Z}$ such that

$$\tilde{x}q_* = k + q_*.$$

Since $q_* \in \mathbb{Z}$ by construction, it now follows from (2) that

$$(\tilde{x} - 1)q' = 3q_* - 2(k + q_*)$$

is an integer as desired. $\square$

This contradicts the minimality of q_*, so that no such $x \in \mathbb{Q}$ exists.