Math 966: Homework 2

1: Consider the three symplectic surfaces: $S^+(S^1)$, T^*S^1, and $(\mathbb{R}^2\backslash\{(0,0)\}, dx\wedge dy)$. All three are diffeomorphic, and have infinite volume. Which of these are symplectomorphic? (Give symplectomorphisms between them, and/or prove that there can be no symplectomorphisms between the distinct ones.)

2: Let $(M, \ker\alpha)$ be a contact manifold, and let X_h be a contact vector field on M defined by $\alpha(X_h) = h$. We mentioned in class that X lifts to a Hamiltonian vector field X_H on $S^+(M, \ker\alpha)$, meaning that $X_H = X_h + f\partial_r$. But the function f is not the zero function!

2a: In terms of h and α, calculate f, and also calculate H, so that $dH = X_H \lrcorner\, d\lambda_{\text{std}}$.

2b: Let $\varphi : S^1 \to S^1$ be the diffeomorphism which is the long-time flow of the gradient of the height function. That is, $\pi, 0 \in \mathbb{R}/2\pi\mathbb{Z}$ are the fixed points of φ, and $\varphi([-\pi+\epsilon, \pi-\epsilon]) \subseteq (-\epsilon, \epsilon)$ for a small value ϵ. Then φ lifts to a Hamiltonian symplectomorphism ψ of $S^+(S^1, \ker d\theta)$. Draw a picture of ψ, that is, draw a picture of $\psi(\{r = 0\})$.

3: A *Weinstein structure* on an (open!) manifold W is a pair (λ, φ). Here $d\lambda$ is symplectic, and $\varphi : W \to [0, \infty)$ is a proper Morse function, that is, a proper function with non-degenerate critical points. We require that the Liouville vector field Y is *gradient-like* for φ, which is to say that $K_1||Y||^2 < d\varphi(Y) < K_2||Y||^2$ for some metric $||\cdot||$ on W and some constants K_1, K_2. It immediately follow that every non-critical level set $\varphi^{-1}(c) \subseteq W$ is a contact hypersurface, since they are transverse to Y. Let $\psi_t : W \to W$ be the flow of Y.

3a: Let $p \in W$ be a critical point of p, and let $D^k_p \subseteq W$ be the *descending manifold* of p, that is, $D^k_p = \{q \in W; \lim_{t\to\infty} \psi_t(q) = p\}$. Prove that $\lambda|_{D^k_p} = 0$. (Hint: you may assume the standard result from Morse theory that $(D^k_p, Y|_{D^k_p})$ is diffeomorphic to $(D^k \subseteq \mathbb{R}^k, -\rho\partial_\rho)$, where ρ is the radial coordinate. Consider the function $\lambda(\psi_{t*}Y)$).

3b: Let $W \subseteq \mathbb{C}^N$ be holomorphically embedded, that is, $\sqrt{-1}TW_q = TW_q$ at every point $q \in W$. Let φ be the function $||\cdot||^2|_W$, $\lambda = \lambda_{\text{std}}|_W$, and $\omega = d\lambda$. We proved that ω is a symplectic form on W, and furthermore that $g(\cdot,\cdot) = \omega_{\text{std}}(\cdot, \sqrt{-1}\cdot)$ is the Riemannian metric on W (called the Kähler metric). Let $Y \in \Gamma(TW)$ be the Liouville vector field associated to λ, that is $Y \lrcorner\, \omega = \lambda$. Show that Y is the gradient of φ with respect to the Kähler metric.

(It follows that whenever W is embedded so that $||\cdot||^2|_W$ has non-degenerate critical points, that (W, λ, φ) is Weinstein. It is a remarkable fact that this is the essentially unique example: *every* Weinstein manifold is isotopic to a Weinstein structure induced by a holomorphic embedding into $\mathbb{C}^N$.)

3c: For any closed manifold Q, Prove that $(T^*Q, \omega_{\text{std}})$ admits a Weinstein structure. (You cannot choose $\lambda = \lambda_{\text{std}}$, since λ_{std} its zeros are non-isolated.)

3d: Let (W, λ, φ) be a Weinstein manifold, and let $c \in \mathbb{R}$ be a number so that all critical points of φ are less than c. Show that $W_{\text{end}} = \varphi^{-1}(c, \infty)$ is a "half symplectization", that is, give a Lioville form preserving symplectomorphism from W_{end} to $(M \times (a, \infty), r\alpha)$ for some contact manifold $(M, \ker\alpha)$ and $a > 0$.

4a: Define an embedding $f : S^2 \hookrightarrow (\mathbb{R}^3, \ker(dz + r^2 d\theta))$, so that there exists a orientation reversing diffomorphism $\psi : S^2 \to S^2$ preserving $\mathcal{F}_\xi$. (Note: for the standard round embedding no such diffeomorphism exists, because the "swirly elliptic" singularities (ie those with complex eigenvalues) define an orientation by which direction they rotate).

4b: Let $(M_1, \xi_1), (M_2, \xi_2)$ be two contact manifolds. Choose Darboux charts $U_j \subseteq M_j$, and then choose embeddings of the closed ball $B^3_j \subseteq U_j$, so that the characteristic foliation on ∂B^3_j is diffeomorphic to $\mathcal{F}_\xi$ from Problem 4a. Define $(M_1, \xi_1)\#(M_2, \xi_2) = (M_1 \setminus \operatorname{Int} B^3_1) \cup_\psi (M_2 \setminus \operatorname{Int} B^3_2)$, where the contact structure is defined by the gluing map φ from Theorem 7.4.

Show that $(M_1, \xi_1)\#(M_2, \xi_2)$ is well defined; that it doesn't depend on the choices B^3_j or U_j (after fixing a specific choice of $\mathcal{F}_\xi$ on S^2). (Hint: $(x, y, z) \mapsto (e^t x, e^t y, e^{2t} z)$ is a contact isotopy of $\mathbb{R}^3_{\text{std}}$. Combine this with a Mazur swindle argument.)

Due March 25th