1. The Frobenius integrability theorem

The goal of the next section is the following theorem which allows us to construct coordinate charts. It shows the why the Lie bracket is significant.

Theorem 1.1. *Let M be an n-manifold, and suppose we are given vector fields $X_1, \dots, X_n$ on M, so that at each point $q \in M$, $\{X_i(q)\}$ is a basis of TM_q. Furthermore, suppose that $[X_i, X_j] = 0$ for all i and j. Then around every point we can find a neighborhood $U \subseteq M$ and a coordinate chart $f : V \to U$ so that $df(\partial_i) = X_i$ (where $\{\partial_i\}$ are the standard coordinate vector fields on $V \subseteq \mathbb{R}^n$).*

First we prove a lemma

Lemma 1.2. *Let X and Y be two vector fields on M, and let $\varphi_t^X : M \to M$ be the flow of the vector field X. Then $\varphi_t^X \circ \varphi_s^Y = \varphi_s^Y \circ \varphi_t^X$ for all s, t, if and only if $[X, Y] = 0$ everywhere.*

Proof: If f is any diffeomorphism, then by the chain lemma and uniqueness of flows $\varphi_t^{df(X)} = f \circ \varphi_t^X \circ f^{-1}$. Therefore to prove the lemma it suffices to show that $X = d\varphi_s^Y(X)$ for all s. If $g \in C^\infty M$, then $d\varphi_s^Y(X)(g) = X(g \circ \varphi_s^Y) \circ (\varphi_s^Y)^{-1}$. We differentiate this expression with respect to s and evaluate at $s = 0$. This gives:

$$\frac{d}{ds} X(g \circ \varphi_s^Y) \circ (\varphi_s^Y)^{-1}\Big|_{s=0} = X(\frac{d}{ds} g \circ \varphi_s^Y) \circ (\varphi_0^Y)^{-1} + \frac{d}{ds} X(g \circ \varphi_0^Y) \circ (\varphi_s^Y)^{-1}\Big|_{s=0}$$

$$= X(\frac{d}{ds} g \circ \varphi_s^Y) + \frac{d}{ds} X(g) \circ \varphi_s^{-Y}\Big|_{s=0} = X(Y(g)) - Y(X(g)) = [X, Y](g) = 0.$$

Since $\varphi_{s_0+s}^Y = \varphi_{s_0}^Y \circ \varphi_s^Y$, if we instead evaluate the derivative at $s = s_0$ we get $d\varphi_{s_0}^Y([X, Y])(g) = 0$. Therefore we conclude that $d\varphi_s^Y(X)$ is a constant vector field, with respect to s. Since it equals X at $s = 0$, we conlude that it equals X always and this completes the lemma. □

Proof of the Frobenius integrability theorem: Let $\{X_i\}$ be as given, fix a point $q \in M$, and let $B(\varepsilon)$ be a small ball in $\mathbb{R}^n$. Define a map $f : B(\varepsilon) \to M$ by $f(x_1, \dots, x_n) = \varphi_{x_1}^{X_1} \circ \dots \circ \varphi_{x_n}^{X_n}(q)$. By definition of a flow, we have that $df(\partial_1) = X_1$. But since $[X_i, X_j] = 0$ all of these flows commute, and therefore $df(\partial_i) = X_i$ for all i. In particular df is a linear isomorphism and therefore it is locally a diffeomorphism, so it is a coordinate chart. □

Corollary 1.3. *Let M be a Riemannian n-manifold, and suppose that there are n vector fields $X_1, \dots, X_n$ which are orthonormal and satisfy $[X_i, X_j] = 0$ for all i, j. Then M is locally isometric to $\mathbb{R}^n$ with the Euclidean metric.*

Proof: Apply the Frobenius integrability theorem to M, then we get a diffeomorphism $f : V \subseteq \mathbb{R}^n \to M$ so that $\{df(\partial_i)\}$ is an orthonormal basis. But the definition of the Euclidean metric is that $\{\partial_i\}$ is orthonormal on $\mathbb{R}^n$, therefore f is an isometry. □

2. Riemannian curvature

Definition 2.1. Let ∇ be an affine connection on a manifold M. Given three vector fields X, Y and Z on M, we define a new vector field $R(X, Y)Z$ by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z.$$

We call $R(\cdot, \cdot)\cdot$ the *Riemannian curvature tensor* of ∇.

Proposition 2.2. *R is a tensor, that is, it is C^∞-linear in each slot.*

Proof: Simply check. $\square$

Proposition 2.3. *Suppose ∇ is the Levi-Civita connection of a metric. Then:*

(i) $R(X,Y)Z = -R(Y,X)Z$
(ii) $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$
(iii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$
(iv) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$.

Proof: *(i)* is obvious from simply plugging in. The proof of *(ii)* comes from the identity $[X,Y]\langle Z,W\rangle = (XY - YX)\langle Z,W\rangle$; use the compatibility of ∇ with the metric and expand the right hand side out into eight terms.

To prove *(iii)*, note that we can assume $[X,Y] = [X,Z] = [Y,Z] = 0$; since R is a tensor it only depends on vectors pointwise, and any set of vectors can be given *pointwise* as coordinate vector fields (this also makes the proof of *(ii)* simpler). Then $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y$ has six terms, three positive and three negative. Using symmetry of the connection we see that they all cancel.

(iv) follows formally from *(i)*, *(ii)*, and *(iii)*, see Milnor's very pretty octohedron argument. $\square$

We saw above in Lemma 1.2 that the Lie bracket is a measurement of non-commutativity of flows. Similarly, Riemannian curvature measures non-commutativity of parallel transport.

Proposition 2.4. *Let $s : \mathbb{R}^2 \to M$ be an embedded surface, and let V be a vector field along the surface. Then*

$$\frac{D}{dx}\frac{D}{dy}Z - \frac{D}{dy}\frac{D}{dx}Z = R(\partial_x, \partial_y)Z.$$

Proof: This follows immediately from the definition of $\frac{D}{dx}$ and $R(X,Y)Z$, together with the fact that $[\partial_x, \partial_y] = 0$ since they are coordinates. $\square$

Proposition 2.5. *Let $U \subseteq M$ be a chart on a Riemannian manifold. Let $c : [a,b] \to U$ be a smooth embedded curve satisfying $c(a) = c(b) = q$, but not necessarily $\dot{c}(a) = \dot{c}(b)$. Let $v \in TM_q$, and let $\widetilde{v} \in TM_q$ be the vector obtained by parallel translating v along the path c. Then $R(\cdot,\cdot)\cdot$ is zero everywhere in U, if and only if $v = \widetilde{v}$ for all $q \in U$ and for all curves c.*

Proof: We can find a surface $s : \mathbb{R}^2 \to U$ containing the image of c so that $s(0,0) = q$. Starting with $v \in TM_q$, define a vector field $v(y)$ along the curve $y \mapsto s(0,y)$, by parallel transport. Further extend this to a vector field along s, by parallel translating the vector $v(y)$ along the curve $x \mapsto s(x,y)$. Call the vector field defined in this way $V(x,y)$. By definition $\frac{D}{dx}V = 0$, therefore Proposition 2.4 implies

$$\frac{D}{dx}\frac{D}{dy}V = R(\partial_x, \partial_y)V.$$

If $R(\partial_x, \partial_y)V = 0$ everywhere then the vector field $\frac{D}{dy}V$ is parallel along the curves $x \mapsto s(x,y)$. But $\frac{D}{dy}V = 0$ when $x = 0$, since $V(0,y) = v(y)$, therefore we see that the vector field is $\frac{D}{dy}V = 0$ everywhere. Therefore for any curve c with image inside $s(\mathbb{R}^2)$ we see that $V(c(t))$ is parallel along c, since $\frac{D}{dt}$ is a linear combination of $\frac{D}{dx}$ and $\frac{D}{dy}$ at each point. Therefore $\widetilde{v} = V(0,0) = v$.

Conversely, if $R(\partial_x, \partial_y)V \neq 0$ somewhere, then $\frac{D}{dy}V \neq 0$ somewhere, and we can find some curve $c : [a, b] \to U$ so that $c(a) = v$ and the parallel transport of v along c is not $V(c(b))$. But we can find another curve $\widetilde{c}$ with the same endpoints so that the parallel translate of v is $V(c(b))$ (let $\widetilde{c}$ be the curve that first moves from $(0, 0)$ along the y-direction, then moves along the x-direction). By concatenating c with the reverse of $\widetilde{c}$ we get a loop with $v \neq \widetilde{v}$. $\square$

Theorem 2.6. *Let M be a Riemannian n-manifold satisfying $R(\cdot, \cdot)\cdot = 0$ everywhere. Then M is locally isometric to the Euclidean metric on $\mathbb{R}^n$.*

Proof: Choose a coordinate chart $U \subseteq M$, and let $v_1, \ldots, v_n$ be an orthonormal basis of $TU_{(0,0)}$. Since $R(\cdot, \cdot)\cdot = 0$ everywhere, the previous proposition implies that the parallel transport of v_i does not depend on the choice of path. Therefore we get a canonical vector fields X_i on U which are parallel along all curves. Parallel transport always preserves the inner product, therefore $\{X_i\}$ is an orthonormal basis at every point. Also since $\nabla_{\partial_j} X_i = \frac{D}{d_j} X_i = 0$ for all i and j, we see that $\nabla_V X_i = 0$ for any vector field V. In particular, $[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0$. Now apply Corollary 1.3. $\square$