Definition 0.1. An *affine connection* ∇ on a manifold M is a method of assigning a vector field $\nabla_X Y$ to every pair of vector fields X and Y on M, so that:

- ∇ is bilinear
- ∇ is C^∞ linear in X, that is, $\nabla_{fX} Y = f\nabla_X Y$ for any $f \in C^\infty M$
- ∇ is Leibnitzian in Y, that is, $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

An affine connection should be thought of as a way to differentiate Y with respect to the vector field X. Notice that the vector $(\nabla_X Y)_p$ only depends on X_p, but it depends on Y in a neighborhood of p.

Suppose we are given local coordinates, inducing vector fields $\{\partial_i\}$. We can then define n^3 smooth functions Γ^k_{ij} by the equation

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k.$$

Conversely, given *any* choice of n^3 smooth functions Γ^k_{ij}, we can define a connection using the above formula, and extending to all vector fields by C^∞-linearity and Leibnitz's rule. Thus if $X = \sum_i x_i\partial_i$ and $Y = \sum_j y_j\partial_j$, then

$$\nabla_X Y = \sum_k \left(\sum_i x_i \left(\partial_i y_k + \sum_j \Gamma^k_{ij} y_j \right) \right) \partial_k.$$

Definition 0.2. A connection ∇ is *symmetric* if $\nabla_X Y - \nabla_Y X = [X,Y]$ for all vector fields X and Y.

Given a connection ∇ and a function $f : M \to \mathbb{R}$, define the *Hessian* of f to be $H^f(X,Y) = X(Y(f)) - (\nabla_X Y)(f)$. Notice that, at a critical point of f, the term $(\nabla_X Y)(f) = df(\nabla_X Y) = 0$ and thus this agrees with our previous definition. Also notice that H^f is tensorial: obviously $H^f(gX,Y) = gH^f(X,Y)$, but also $H^f(X,gY) = X(gY(f)) - (\nabla_X gY)(f) = X(g)Y(f) + gX(Y(f)) - (g\nabla_X Y + X(g)Y)(f) = gH^f(X,Y)$.

Proposition 0.3. ∇ *is symmetric if an only if, for every function* $f \in C^\infty M$, $H^f(X,Y) = H^f(Y,X)$. *In local coordinates,* ∇ *is symmetric if and only if* $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Proof: By definition, $H^f(X,Y) - H^f(Y,X) = ([X,Y] - \nabla_X Y + \nabla_Y X)(f)$. For the second claim, note that $[\partial_i, \partial_j] = 0$ since they are coordinate vector fields, and $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$ if and only if $\Gamma^k_{ij} = \Gamma^k_{ij}$. $\square$

Definition 0.4. A connection ∇ is *compatible* with a metric $\langle\cdot,\cdot\rangle$ if it is Leibnitzian with respect to the metric, that is, $X(\langle Y,Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

Proposition 0.5. *A metric admits a unique compatible symmetric connection (which we call the* Levi-Civita *connection of the metric).*

Proof: This is Lemma 8.6 in Milnor. $\square$

Second fundamental form and applications to Morse theory

Proposition 0.6. *Let M be a manifold with a metric $\langle\cdot,\cdot\rangle$, and let $Q \subseteq M$ be a submanifold. Let ∇ be the Levi-Civita connection, and let ∇^Q be the Levi-Civita*

connection of the restricted metric. Then, for any vector fields X and Y tangent to Q, $\nabla^Q_X Y$ is the projection of $\nabla_X Y$ to TQ.

Proof: Let $\pi : TM \to TQ$ be the projection. Since the Levi-Civita connection is unique, it suffices to show that $\pi \circ \nabla$ is a symmetric connection which is compatible with $\langle \cdot, \cdot \rangle|_Q$. Immediately it is a symmetric connection. To see that it is compatible with $\langle \cdot, \cdot \rangle|_Q$, note that for any vector fields U and V on M, so that V is tangent to Q, then $\langle U, V \rangle = \langle \pi(U), V \rangle$. Then use the fact that ∇ is compatible with $\langle \cdot, \cdot \rangle$. $\square$

Definition 0.7. Let M be a manifold with a metric $\langle \cdot, \cdot \rangle$, and let Q be a submanifold. Define the *second fundamental form of Q* by $\mathbb{I}(X, Y) = (\nabla_X Y)^{\perp TQ}$ (where $V^{\perp TQ}$ denotes the projection of V onto the othogonal complement of TQ). It is a function with inputs X and Y, vector fields tangent to Q, whose output is a vector field orthogonal to Q.

Proposition 0.8. $\mathbb{I}$ *is a symmetric tensor.*

Proof: $\mathbb{I}$ is clearly tensorial in X, therefore symmetry implies tensoriality. But $\mathbb{I}(X, Y) - \mathbb{I}(Y, X) = [X, Y]^{\perp TQ}$, and $[X, Y]$ is tangent to Q whenever X and Y are. $\square$

In some sense, $\mathbb{I}$ measures the failure of Q to be flat inside of M. For example, if $M = \mathbb{R}^k \times \mathbb{R}^n$ and Q is the graph of a function $f : \mathbb{R}^k \to \mathbb{R}^n$, then we can represent $\mathbb{I}$ be nk^2 smooth functions. (k^2 inputs for $\dim Q$ and n-dimensional output for the orthogonal directions. These functions are exactly the second partial derivatives of f. In particular, $\mathbb{I} = 0$ if and only if f is an affine linear function.

Let M be a manifold with a metric, and $Q \subseteq M$ a submanifold. Let $TM|_Q = \pi^{-1}(Q)$, where $\pi : TM \to M$ is the natural projection. Then $d\pi$ is surjective everywhere, so it is transverse to every manifold, and thus $TM|_Q$ is a submanifold. Let $\nu(Q) = \{(q, v) \in TM|_Q, v \perp TQ_q\}$. Since the map $\langle \cdot, \cdot \rangle : TM|_Q \to T^*Q$ has surjective differential everywhere, it follows that $\nu(Q)$ is a manifold (the pre-image of the zero section in T^*Q).

Intuitively, $\nu(Q)$ is the set of all vectors at Q which are tangent to M but orthogonal to Q. Now, suppose that $M = \mathbb{R}^N$ with the standard metric. There is a natural map $e : \nu(Q) \to \mathbb{R}^N$ defined by $e(q, v) = q + v$ (the "endpoint" map).

Proposition 0.9. *Let $Q \subseteq \mathbb{R}^N$ be a submanifold, and let $d_p^2 : Q \to (0, \infty)$ be the distance squared function to a point $p \in \mathbb{R}^N \setminus Q$. Then q is a critical point of d_p^2 if and only if $p = e(q, v)$ for some $v \in TQ_q^\perp$. Furthermore, q is a degenerate critical point if and only if (q, v) is a critical point of e.*

Lemma 0.10. *For $\partial_r = \sum_{i=1}^N (x_i - p_i) \partial_i$ being the radial vector field to the point p, $H^{d_p^2}(X, Y) = 2\langle X, Y \rangle + 2\langle \mathbb{I}(X, Y), \partial_r \rangle$.*

Proof: By translation in $\mathbb{R}^N$, we can assume that $p = 0$. Let $D : \mathbb{R}^N \to \mathbb{R}$ be the function $(x_1, \ldots, x_N) \mapsto x_1^2 + \ldots + x_N^2$, so $d_0^2 = D|_Q$. Since we are using the standard metric we see $H^D(X, Y) = 2\langle X, Y \rangle$. But by definition $H(X, Y) = X(Y(D)) - (\nabla_X Y)(D)$, which is equal to $X(Y(D)) - (\nabla^Q_X Y)(D) - (\nabla_X Y)^{\perp TQ}(D)$ (simply by splitting $\nabla_X Y$ into its tangential and normal coordinates). Now, assuming that X and Y are tangent to Q, we get $2\langle X, Y \rangle = H^{d_0^2}(X, Y) - \mathbb{I}(X, Y)(D)$. And $V(D) = 2\langle V, \partial_r \rangle$ for any vector V, since $\nabla D = 2\partial_r$. $\square$

Proof of Proposition 0.9: Again we can assume $p = 0$. q is a critical point if and only if TQ_q is tangent to a sphere centered at the origin. But this happens if and only if the radius is orthogonal to TQ_q. This proves the first claim.

Given a vector $w \in T\nu(Q)_{(q,v)}$, we realize $w = (q'(0), v'(0))$ for some path $(q(t), v(t)) \in \nu(Q)$, $t \in (-\varepsilon, \varepsilon)$ with $q(0) = q, v(0) = v$. The chain rule says that $v'(0) = \nabla_{q'(0)} V$, where V is any vector field orthogonal to Q satisfying $V(q(t)) = v(t)$. Let X be any vector field tangent to Q, then $\langle X, V \rangle = 0$ everywhere. Therefore $0 = q'(0)(\langle X, V \rangle) = \langle \nabla_{q'(0)} X, V \rangle + \langle X, \nabla_{q'(0)} V \rangle = \rangle \mathbb{II}(q'(0), X), v'(0)\rangle + \langle X, v'(0) \rangle$. Since $p = 0 = e(q(0), v(0)) = q(0) + v(0)$, we see that $v(0) = -\partial_r(q(0))$, and therefore
$$\langle X, v'(0) \rangle = \langle \mathbb{II}(q'(0), X), \partial_r \rangle$$
for every vector X which is tangent to Q.

Suppose (q, v) is a critical point of e, which means that there is a w with $de(w) = 0$. By definition $de(w) = q'(0) + v'(0)$, so $0 = \langle de(w), X \rangle = \langle q'(0), X \rangle + \langle v'(0), X \rangle = \langle q'(0), X \rangle + \langle \mathbb{II}(q'(0), X), \partial_r \rangle$. Thus by Lemma 0.10 we see that $H^{d_0^2}(q'(0), X) = 0$ for every vector X, showing that q is a degenerate critical point of d_0^2.

Conversely, suppose that q is a degenerate critical point of d_0^2, so we have a vector $q'(0)$ satisfying $H^{d_0^2}(q'(0), X) = 0$ for all vectors X which are tangent to Q. Choose an arbitrary path of vectors $\widetilde{v}(t)$ which is orthogonal to $TQ_{q(t)}$, satisfying $\widetilde{v}(0) = -\partial_r$. Then let $(\widetilde{v}'(t))^\perp$ be the component of $\widetilde{v}'(t)$ which is orthogonal to Q, and let $v(t) = \widetilde{v}(t) - t(\widetilde{v}'(t))^\perp$. $v(t)$ is orthogonal to Q, and $v(0) = -\partial_r$, and additionally we see that $v'(0) = \widetilde{v}'(0) - (\widetilde{v}'(0))^\perp$ so $v'(0)$ is tangent to $TQ_{q(0)}$.

By the above formula for $\langle X, v'(0) \rangle$ and Lemma 0.10, we see that $\langle X, v'(0) \rangle = -\langle X, q'(0) \rangle$. Since $v'(0)$ is tangent to $TQ_{q(0)}$ and X is an arbitrary vector X which is tangent to $TQ_{q(0)}$, this implies that $v'(0) = -q'(0)$. $\square$

Corollary 0.11. *For a generic choice of $p \in \mathbb{R}^N$, d_p^2 is a Morse function on Q, which is bounded below. Assuming that Q is properly embedded, all level sets of d_p^2 are compact.*

Proof: Simply apply Sard's theorem to e and use Proposition 0.9. $\square$

COMPLEX GEOMETRY AND THE LEFSCHETZ HYPERPLANE THEOREM

Definition 0.12. Let $(p, v) \in T\mathbb{C}^N$ be a vector in $\mathbb{C}^N$. Define $J : T\mathbb{C}^N \to T\mathbb{C}^N$ by $J(p, v) = (p, iv)$. J is called the *standard almost complex structure* on $\mathbb{C}^N$.

Definition 0.13. Let $Q \subseteq \mathbb{C}^N$ be a submanifold. Say that Q is a *holomorphic submanifold* (or simply *complex submanifold*) if $J(TQ) = TQ$.

Notice that, if $f_1, \ldots f_k : U \to \mathbb{C}$ are holomorphic functions for some open $U \subseteq \mathbb{C}^N$, then $Q = \{f_1(z) = \ldots = f_k(z) = 0\}$ is a holomorphic submanifold.

Theorem 0.14. *Let $Q \subseteq \mathbb{C}^N$ be any properly embedded holomorphic submanifold. Then Q admits a Morse function so that the index of any critical point is no greater than $\frac{1}{2} \dim_\mathbb{R} Q$.*

Lemma 0.15. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a symmetric quadratic form, that is a function of the form $f(z) = \sum_j c_j z_j^2$ for constants $c_j \in \mathbb{C}$. Then there is a decomposition of $\mathbb{C}^n = V_+ \oplus V_- \oplus V_0$ into the sum of real vector spaces, so that $\Re(f)$ is positive definite on V_+, negative definite on V_-, and identically zero on V_0. Furthermore, $\dim V_+ = \dim V_-$.*

Proof: Since $\Re(f)$ is a real symmetric quadratic form we can write $\mathbb{C}^n = V_+ \oplus V_- \oplus V_0$ as above, we only need to show that $\dim V_+ = \dim V_-$. Notice $f(iz) = -f(z)$, therefore $\Re(f)$ is negative definite on the space iV_+. Therefore $\dim V_+ = \dim iV_+ \leq \dim V_-$. The opposite inequality comes from considering iV_-. $\qquad \square$

Lemma 0.16. *Let ∇ be the Levi-Civita connection for the standard euclidean metric on $\mathbb{C}^N$. Then for any vector fields X and Y, $\nabla_X(JY) = J(\nabla_X Y)$.*

Proof: Just calculate in coordinates. If $Y = \sum_{j=1}^N f_j(x,y)\partial_{x_j} + g_j(x,y)\partial_{y_j}$, then $JY = \sum_{j=1}^N f_j(x,y)\partial_{y_j} - g_j(x,y)\partial_{x_j}$, and

$$\nabla_X(JY) = \sum_{j=1}^N X(f_j)\partial_{y_j} - X(g_j)\partial_{x_j}$$

which is the same as $J(\nabla_X Y)$. $\qquad \square$

Remark 0.17. Be careful! If we use a non-standard metric, or a non-standard almost complex structure, the lemma is false.

Proof of Theorem 0.14: Let $n = \frac{1}{2}\dim_{\mathbb{R}} Q$, and choose a Morse function of the form d_p^2. Since translation is holomorphic we can assume $p = 0$. Let $q \in Q$ be a critical point of index k, so there is a subspace $V_- \subseteq TQ_q$ of dimension k so that $H^{d_0^2}$ is negative definite on V_-. By Lemma 0.10 $H^{d_0^2}(v,v) = \langle v,v \rangle + \langle \mathbb{II}(v,v), \partial_r \rangle$, and since $\langle v,v \rangle \geq 0$, it follows that $\langle \mathbb{II}(\cdot,\cdot), \partial_r \rangle$ is negative definite on V_- as well.

Consider the map $f : TQ_q \to \mathbb{C}$ given by $f(v) = (\mathbb{II}(v,v), \partial_r)$, where $(\cdot,\cdot)$ is the standard Hermitian inner product on $\mathbb{C}^N$. Using Lemma 0.16 and the symmetry of $\mathbb{II}$, we see that $f(Jv) = -f(v)$. Therefore by diagonalizing f we can find a linear isomorphism $(TQ_q, J) \to (\mathbb{C}^n, i)$ so that f is of the form prescribed in Lemma 0.15. Thus, since $\Re(f(\cdot)) = \langle \mathbb{II}(\cdot,\cdot), \partial_r \rangle$ is negative definite on $V_- \subseteq TQ_q$, it follows Lemma 0.15 that $k = \dim V_- \leq n$. $\qquad \square$

Corollary 0.18. *A holomorphic submanifold of $\mathbb{C}^N$ is never compact (unless it is zero dimensional).*

Corollary 0.19. *Suppose Q is a holomorphic submanifold of $\mathbb{C}^N$. Then there is a collection of closed disks $D_j^{k_j} \subseteq Q$ of dimension $k_j \leq \frac{1}{2}\dim_{\mathbb{R}} Q$, so that Q deformation retracts onto $\bigcup_k D_j^{k_j}$.*

Corollary 0.20. *Let $Q \subseteq \mathbb{CP}^N$ be a holomorphic submanifold. Suppose Q is transverse to $\mathbb{CP}^{N-1} \subseteq \mathbb{CP}^N$, and let $M = Q \cap \mathbb{CP}^{N-1}$. Then there are finitely many disks $D_j^{k_j} \subseteq Q \setminus M$ of dimension $k_j < \frac{1}{2}\dim_{\mathbb{R}} Q$, so that $Q \setminus \bigcup_j D_j^{k_j}$ deformation retracts onto M.*

Proof: $\mathbb{CP}^N \setminus \mathbb{CP}^{N-1} = \mathbb{C}^N$, therefore $Q \setminus M$ is a properly embedded holomorphic submanifold of $\mathbb{C}^N$. Let $d_p^2 : Q \setminus M \to \mathbb{R}$ be a Morse function of type guaranteed by Theorem 0.14. If $B_p^{2N}(R) \subseteq \mathbb{C}^N$ is the closed ball of radius R around p, notice that $U(R) = \mathbb{CP}^N \setminus B_p^{2N}(R)$ is an open neighborhood of $\mathbb{CP}^{N-1}$ which deformation retracts onto $\mathbb{CP}^{N-1}$. Since $Q \pitchfork \mathbb{CP}^{N-1}$, we see that for sufficiently large R d_p^2 has no critical points inside $U(R) \cap Q$, and $U(R) \cap Q$ deformation retracts onto M.

Since all critical points of d_p^2 are inside $B^{2N}(R) \cap Q$, in particular we know it has finitely many critical points. Choose standard coordinates around every critical point, and choose a metric on Q which agrees with the Euclidean metric

in the chosen coordinates near the critical points (outside of those neighborhoods we let our metric be arbitrary). Let $V = \nabla d_p^2$, and let $\varphi_t : Q \setminus M \to Q \setminus M$ be the flow of V. For every point $q \in Q \setminus M$, we claim that either $\lim_{t \to \infty} \varphi_t(q)$ exists and is a critical point of d_p^2, or $d_p^2(\varphi_t(q)) > R$ for large t. Indeed, if $\varphi_t(q)$ never enters the neighborhood of a critical point then $||V||^2$ is bounded below, but $||V||^2 = \frac{d}{dt} d_p^2(\varphi_t(q))$. But if $\varphi_t(q)$ enters one of these neighborhoods then we understand the dynamics: $\varphi_t(q)$ either leaves the neighborhood eventually or converges to the critical point.

Let $\{q_j\}$ be a list of all the critical points of d_p^2 and define $D_j^{k_j} = \{q \in Q \setminus M$ so that $\lim_{t \to \infty} \varphi_t(q) = q_j\}$. Then $D_j^{k_j}$ is an open disk of dimension $k_j = \mathrm{ind}(q_j)$. Indeed, $D_j^{k_j}$ is the union of arcs $\{\varphi_t(q) : t \in (\infty, \infty)\}$ where q can be chosen inside our standard neighborhood. Therefore φ_t defines a deformation retraction of $Q \setminus \bigcup_j D_j^{k_j}$ onto $U(R) \cap Q$, which in turn deformation retracts onto M. $\qquad\square$

Corollary 0.21. *Let $Q \subseteq \mathbb{CP}^N$ be a holomorphic submanifold. Suppose Q is transverse to $\mathbb{CP}^{N-1} \subseteq \mathbb{CP}^N$, and let $M = Q \cap \mathbb{CP}^{N-1}$. Suppose $\dim_\mathbb{R} Q > 2$. Then every map $f : S^1 \to Q$ is homotopic to a map $\widetilde{f} : S^1 \to M$. (That is, $\pi_1 M \to \pi_1 Q$ is injective.)*

Proof: By the previous corollary, $Q \setminus \bigcup_j D_j^{k_j}$ deformation retracts onto M. Here $\dim_\mathbb{R} Q - k_j > \frac{1}{2} \dim_\mathbb{R} Q > 1$. Therefore by transversality f is homotopic to a map which is disjoint from all $D_j^{k_j}$. Composing with the deformation retraction defines $\widetilde{f}$. $\qquad\square$