1. Abstract manifolds and the Whitney embedding theorem

Theorem 1.1. *Let X be a compact Hausdorff space, and let $\{M_i\}$ be a set of n-manifolds. Suppose we are given $f_i : M_i \to X$ which are open maps and homeomorphisms onto their image, so that $X = \bigcup f_i(M_i)$. Finally, suppose that, for every pair i, j, the homeomorphism $f_i^{-1} \circ f_j : f_j^{-1}(f_i(M_i) \cap f_j(M_j)) \to f_i^{-1}(f_i(M_i) \cap f_j(M_j))$ is smooth. Then X is an n-manifold.*

Note. If X is not compact, but second countable, the theorem remains true. Usually, the hypotheses of the theorem are taken as the definition of an *abstract n-manifold*. Then the Whitney embedding theorem, stated in the traditional way, is "any (abstract) manifold can be realized as a submanifold of $\mathbb{R}^N$ for some N".

Lemma 1.2. *Let M be a manifold. Then there is a locally finite open covering $\{U_j\}$ of M, so that $\overline{U}_j$ is compact, and smooth functions $\varphi_j : M \to [0, \infty)$ so that $\varphi_j(x) > 0$ if and only if $x \in U_j$.*

Proof: Choose an arbitrary locally finite open covering of M by charts $\{V_j\}$, so that each V_j is diffeomorphic to an open subset of $\mathbb{R}^n$. Any open set $V_j \subseteq \mathbb{R}^n$ admits a locally finite open covering $\{U_{ij}\}$ so that $\overline{U_{ij}} \subseteq \mathbb{R}^n$ is diffeomorphic to the closed ball. The totality of these collections defines the necessary covering of M. To construct the functions φ_j, simply note that the open ball $B^n(1) \subseteq \mathbb{R}^n$ admits such a function:

$$\varphi(x) = \begin{cases} \exp(\frac{-1}{1-r}) & r \in [0,1) \\ 0 & r \geqslant 1 \end{cases}.$$

□

Proof of Whitney's embedding theorem: Since X is compact we can find finitely many $i = 1, \ldots, K$ so that $X = \bigcup f_i(M_i)$, we will only use these M_i. For each M_i, find a covering $\{U_{ij}\}$ as in the lemma, and corresponding functions φ_{ij}. We can extend the domain of definition of $\varphi_{ij} : X \to [0, \infty)$ by letting it be zero outside of $f_i(M_i)$; then this extension is continuous since X.

(Let $A \subseteq [0, \infty)$ be a closed set, and consider $\varphi_{ij}^{-1}(A)$. If $0 \in A$ then $\varphi^{-1}([0, \infty) \setminus A) \subseteq f_i(M_i)$ so this set is open since f_i is an open map. If $0 \notin A$ then $\varphi_{ij}^{-1}(A)$ is compact (since $\overline{U_{ij}}$ is compact), and since X is Hausdorff this set is closed.)

Define $\psi_i : X \to [0, \infty)$ by

$$\psi_i = \frac{\sum\limits_j \varphi_{ij}}{\sum\limits_i \sum\limits_j \varphi_{ij}}$$

then $\sum_i \psi_i = 1$ and if $\psi_i(x) > 0$ then $x \in f_i(M_i)$. Suppose that $M_i \subseteq \mathbb{R}^N$, let $e_i : M_i \to \mathbb{R}^N$ be the inclusion map. Then define $e : X \to \mathbb{R}^{NK+K}$ by $e(x) = \big(\psi_1(x)(e_1 \circ f_1^{-1})(x), \ldots, \psi_k(x)(e_K \circ f_K^{-1})(x), \psi_1(x), \ldots, \psi_K(x)\big)$. Notice that e is well defined since whenever $x \notin f_i(M_i)$ then $\psi_i(x) = 0$. e is obviously continuous, and it is injective because if $e(x) = e(y)$ then $\psi_i(x) = \psi_i(y)$ for all i. Choosing such an i so that $\psi_i(x) \neq 0$, we get $(e_i \circ f_i^{-1})(x) = (e_i \circ f_i^{-1})(y)$, and the map $e_i \circ f_i^{-1}$ is injective. Since X is compact it follows that e is a homeomorphism onto its image.

It remains to show that $e(X)$ is smooth. Given $x \in e(X)$, choose i so that $\psi_i(x) \neq 0$. Let $y = (f_i^{-1} \circ e^{-1})(x) \in M_i$. Since M_i is a manifold y has a neighborhood V

which is diffeomorphic to $\mathbb{R}^n$. But near x, $e(X)$ is a graph of the smooth function

$$z \mapsto \left(\frac{(\psi_1 \circ f_i)(z)}{(\psi_i \circ f_i)(z)} (e_1 \circ f_1^{-1} \circ f_i)(z), \dots, \frac{(\psi_K \circ f_i)(z)}{(\psi_i \circ f_i)(z)} (e_K \circ f_K^{-1} \circ f_i)(z), (\psi_1 \circ f_i)(z), \dots, (\psi_K \circ f_i)(z) \right)$$

for $z \in V$. □

Theorem 1.3. *Suppose M is a compact n-manifold. Then M is diffeomorphic to a submanifold of $\mathbb{R}^{2n+1}$.*

Note. This theorem also holds for non-compact M, but the proof is more difficult to write down (but not conceptually much more difficult).

Proof: Suppose $M \subseteq \mathbb{R}^N$, we show that M embeds into $\mathbb{R}^{N-1}$ as long as $N > 2n+1$. To do this, we simply project M onto a suitably chosen hyperplane (codimension 1 linear subspace).

Let $\mathbb{RP}^{N-1}$ denote the space of hyperplanes in $\mathbb{R}^N$. We claim that $\mathbb{RP}^{N-1}$ is an (abstract!) $N-1$-manifold. Indeed, given a hyperplane $H \in \mathbb{RP}^{N-1}$ with a normal vector v, every nearby hyperplane has a unique normal vector of the form $v + w$, where $w \in H$. This gives a diffeomorphism between the set of hyperplanes near H with an open subset of H.

For any $H \in \mathbb{RP}^{N-1}$, let $\pi_H : M \to H$ be the orthogonal projection. Given $x \neq y \in \mathbb{R}^N$ let $\overleftrightarrow{xy}$ be the line through them. Then $\pi_H(x) = \pi_H(y)$ if and only if $(\overleftrightarrow{xy})^\perp = H$ (or $x = y$). With this in mind, notice that $(x, y) \mapsto (\overleftrightarrow{xy})^\perp$ is a smooth map $(M \times M) \setminus \Delta \to \mathbb{RP}^{N-1}$ (here $\Delta \subseteq M \times M$ is the diagonal). By Sard's theorem the set of regular values is generic, but since $N - 1 > 2n$ the dimension of the domain is smaller than the dimension of the range, and therefore regular values are exactly those values in the complement of the image. Therefore, for a generic set in $\mathbb{RP}^{N-1}$, the map π_H is injective.

Next, notice that for $v \in TM$, $d\pi_H(v) = 0$ if and only if $(\mathbb{R}\cdot v)^\perp = H$. $v \mapsto (\mathbb{R}\cdot v)^\perp$ is a smooth map $TM \setminus Z \to \mathbb{RP}^{N-1}$ (here Z is the set of zero vectors in each TM_x; Z is called the *zero section* of TM). Again the complement of the image is a generic set in $\mathbb{RP}^{N-1}$ (for exactly the same reasons), and this set gives the hyperplanes H so that $d\pi_H$ is injective.

Since the intersection of two generic sets is generic (and therefore non-empty), we can find a hyperplane H so that π_H and $d\pi_H$ are both injective. Since M is compact it follows that π_H is a smooth embedding. □