1. Parametric transversality

In the first homework, we defined the concept of transversality: a function $f : N \to M$ is transverse to $Q \subseteq M$ if, at every point $x \in f^{-1}(Q)$, $df_x(TN_x)+TQ_{f(x)} = TM_{f(x)}$. If $\partial N \neq \varnothing$, we also require that $f|_{\partial N}$ is transverse to Q. (We will always assume that Q is a closed manifold.) The principal reason that transversality is a useful concept is that, if f is transverse to Q (denoted $f \pitchfork Q$), then $f^{-1}(Q)$ is a submanifold of N (of dimension $\dim N + \dim Q - \dim M$. Notice that $f \pitchfork \{p\}$ simply means that p is a regular value of f.

Notice also, that if N, M, and Q all have orientations, then $f^{-1}(Q)$ comes equipped with an orientation. Let $\{v_i\}$ be a basis of $Tf^{-1}(Q)_x = df_x^{-1}(TQ)$. Then we can choose an oriented basis for TN_x of the form $\{v_i\} \cup \{u_i\}$, an oriented basis of $TQ_{f(x)}$ $\{w_i\}$. We say that $\{v_i\}$ is an oriented basis if $\{w_i\} \cup \{df_x(u_i)\}$ is an oriented basis for $TM_{f(x)}$.

Sard's theorem tells us that regular values are generic, and this fact gave us the power to define topological invariants, such as the degree of a map. To get more robust invariants, we would like to say that transversality is also a generic phenomenon. But what should generic mean in this context? A generic smooth map f? A generic smooth submanifold Q? But the space of all smooth maps or smooth submanifolds is not a manifold (neither space is locally compact, with any natural topology). There is such a thing as a theory of infinite dimensional manifolds, but to avoid dealing with this we will instead take a finite dimensional approach, by looking at spaces of maps which are parametrized by manifolds.

Theorem 1.1. *Let $F : N \times S \to M$ be a smooth map, so that N is a compact manifold (possibly with boundary), and S is a manifold without boundary. Let $Q \subseteq M$ be a closed submanifold inside M. Suppose that $F \pitchfork Q$. Let $f^s : N \to M$ be the map $f^s(x) = F(x, s)$, for a fixed $s \in S$. Then for a generic set in S, $f^s \pitchfork Q$.*

Proof: Let $\pi : F^{-1}(Q) \to S$ be the restriction of the projection map $N \times S \to S$. We want to show that, if s is a regular value of π, then $f^s \pitchfork Q$. The theorem then follows from Sard's theorem.

Let $x \in (f^s)^{-1}(Q)$, and let $v \in TM_{f^s(x)}$. Since $F \pitchfork S$ we can write $v = u + dF_{(x,s)}(w)$ for $u \in TQ_{f^s(x)}$ and $w \in T(N \times S)_{(x,s)}$. Write $w = w_1 + w_2$, where $w_1 \in TN_x$ and $w_2 \in TS_s$.

Assume that s is a regular value of π, which means that $d\pi_{(x,s)}(T(F^{-1}Q)_{(x,s)}) = TS_s$. Here, $T(F^{-1}Q)_{(x,s)} = (dF_{(x,s)})^{-1}(TQ_{F(x,s)})$, therefore there is a vector $\widetilde{w_2} \in T(N \times S)_{(x,s)}$ so that $d\pi_{(x,s)}(\widetilde{w_2}) = w_2$ and $dF_{(x,s)}(\widetilde{w_2}) \in TQ_{f^s(x)}$. But $T(N \times S)_{(x,s)} = TN_x \oplus TS_s$ and $d\pi_{(x,s)}$ is just the projection, therefore $w_2 - \widetilde{w_2} \in TN_x$. Since $dF_{(x,s)}|_{TN^s_x} = df^s_x$, we can write $v = u+dF_{(x,s)}(w_1+w_2) = (u-dF_{(x,s)}(\widetilde{w_2}))+ df^s_x(w_1 + w_2 - \widetilde{w_2})$. Therefore $f^s \pitchfork Q$.

In the case where $\partial N \neq \varnothing$, simply apply Sard's theorem to π and $\pi|_{F^{-1}(Q)\cap\partial N\times S}$ separately, and the intersection of two generic sets is generic. □

Definition 1.2. Let $f : N \to M$ be a smooth map. If $df_x : TN_x \to TM_{f(x)}$ is surjective for all $x \in N$, we say that f is a *submersion.*

A submersion has the nice property that it is transverse to every submanifold in M. We would like to say that, for any smooth map $f : N \to M$, we can find a submersion $F : N \times B^K \to M$ so that $F(x, 0) = f(x)$ (here $B^K \subseteq \mathbb{R}^K$ is an open ball of some (probably large) dimension K). The previous theorem then implies that,

for any submanifold $Q \subseteq M$, we can find $s \in B^K$ so that $f^s \pitchfork Q$. By construction f^s is homotopic to f, and since we can choose s to be arbitrarily close to $0 \in B^K$ it follows that we can take f^s to be arbitrarily C^∞ close to f. Furthermore, given some finite (or even countable) collection of submanifolds Q_i, we can find f^s which is transverse to all Q_i simultaneously.

Euler characteristic and Poincaré-Hopf

Before proving this in the general case, we will show this for the case of $M = TN$, which is nice because of its vector space properties. We use this to define the *Euler characteristic* of a manifold.

Recall that a *vector field* on a manifold M is a map $\sigma : M \to TM$ so that $\sigma(x) \in TM_x$, or equivalently $\pi \circ \sigma = \mathrm{id}$, where $\pi : TM \to M$ is the projection.

Lemma 1.3. *For any compact manifold M there is a finite set of vector fields $\{\sigma_i\}$ on M, so that at every point $x \in M$, the vectors $\{\sigma_i(x)\}$ span TM_x.*

Proof: Cover M by charts $\{U_j\}$, equipped with maps $f_j : M \to [0, \infty)$ so that $f_j(x) > 0$ if and only if $x \in U_j$. U_j are equipped with diffeomorphisms $\varphi_j : U_j \to \mathbb{R}^n$. Recall that TM is covered by charts TU_j, with maps $d\varphi_j : TU_j \to \mathbb{R}^n \times \mathbb{R}^n$. Choose a basis $\{v_i\}$ of $\mathbb{R}^n$, and define σ_{ji} by $\sigma_{ji}(x) = f_j(x)(d\varphi_j)_x^{-1}(v_{ji})$. Then $\{\sigma_{ji}\}$ satisfies the proposition. □

Let $\{\sigma_i\}_{i=1}^K$ be a spanning set of vector fields as in the lemma, then the map $F : M \times \mathbb{R}^K \to TM$ defined by $F(x, s_1, \ldots, s_k) = \sum_{i=1}^K s_i \sigma_i(x)$. Since $\frac{\partial F}{\partial s_i} = \sigma_i$ it follows that F is a submersion. For any fixed $s \in \mathbb{R}^K$, , notice that $F(\cdot, s) : M \to TM$ is a vector field, therefore Theorem 1.1 implies: given any submanifold $Q \subseteq TM$, there is a vector field transverse to Q.

Definition 1.4. Let M be a closed oriented manifold, and let $Z \subseteq TM$ be the *zero section*, the submanifold which is the image of the zero vector field. Choose another vector field $\sigma : M \to TM$ which is transverse to Z. Then $f^{-1}(Z)$ is a closed, oriented 0-manifold. Taking the signed count of $f^{-1}(Z)$ gives an integer $\chi(M)$, which we call the *Euler characteristic* of M.

To justify the notation:

Proposition 1.5. *Let σ_0, σ_1 be two vector fields on M which are transverse to Z. Then $\chi(M, \sigma_0) = \chi(M, \sigma_1)$.*

Proof: Let $h : M \times [0, 1] \to TM$ be defined by $h(x, t) = t\sigma_1(x) + (1 - t)\sigma_0(x)$. Letting $F : M \times \mathbb{R}^K \to TM$ be a submersive family of vector fields as above, define $H : M \times [0, 1] \times \mathbb{R}^K$ by $H(x, t, s) = h(x, t) + F(x, s)$. Then H is a submersion, and $H(x, t, 0) = h(x, t)$. Choose $s_0 \in \mathbb{R}^K$ so that $H(x, t, s_0) \pitchfork Z$. Denote $H(x, t, s_0)$ by $\widetilde{h}(x, t)$ and $\widetilde{h}(x, i)$ by $\widetilde{\sigma}_i$, for $i = 0, 1$.

Then $\widetilde{h}^{-1}(Z)$ is an oriented 1-manifold, and $\partial \widetilde{h}^{-1}(Z) = \widetilde{\sigma_1}^{-1}(Z) \amalg -\widetilde{\sigma_0}^{-1}(Z)$. (Here $\amalg$ denotes disjoint union and the minus sign denotes "opposite orientation".) Since the algebraic count of the boundary of any 1-manifold is zero, it follows that $\chi(M, \widetilde{\sigma}_0) = \chi(M, \widetilde{\sigma}_1)$.

Transversality is open in the C^∞ topology of maps (in fact even in the C^1 topology, which is weaker). Therefore we can find some $B^K(\varepsilon) \subseteq \mathbb{R}^K$ so that $H(\cdot, 0, s) \pitchfork Z$ and $H(\cdot, 1, s) \pitchfork Z$ for all $s \in B^K(\varepsilon)$. Thus if we choose $s_0 \in B^K(\varepsilon)$

it follows that $\widetilde{\sigma}_0$ is homotopic to σ_0 through transverse vector fields. Therefore $\chi(M, \widetilde{\sigma}_0) = \chi(M, \sigma)$ and the result follows. □

Since $\chi(M)$ does not depend on the choice of vector field, we are free to choose any transverse vector field in order to calculate it. For the simplest example, any manifold which has a non-vanishing vector field satisfies $\chi(M) = 0$. If M is a surface which has a smooth triangulation, we can construct the vector field on each vertex, edge, and face separately. We then see that $\chi(M) = f - e + v$ for closed oriented surfaces. In particular, the number $f - e + v$ does not depend on the choice of smooth triangulation, only diffeomorphism type. If M is the surface of genus g, we see that $\chi(M) = 2 - 2g$. Therefore any surface with genus $g > 1$ does not admit a non-vanishing vector field.

Remark 1.6. In fact, $\chi(M) = \sum_{i=0}^{\dim M} (-1)^i \dim H_i(M)$ (for any field coefficients, or more generally any ring coefficients if we replace dim with rk). Using simplicial homology, this follows from the fact that $\sum_{i=0}^{\dim M} (-1)^i \dim H_i(M) = \sum_{i=0}^{\dim M} (-1)^i \dim C_i(M)$, and then constructing a vector field on each simplex, as in the above case for surfaces. Usually this is taken as the definition of $\chi(M)$; it is more general since it is defined for any topological space with $\dim H_*(M) < \infty$. The fact that these definitions are equivalent for smooth manifolds is called the Poincaré-Hopf theorem.

Generic transversality

The Euler characteristic gives an algebraic count of zeros of a generic vector field. We can see this as a count of intersections of transverse submanifolds: the image of our vector field and the zero section, sitting in TM. In general, if we are given two submanifolds $N, Q \subseteq M$ satisfying $\dim N + \dim Q = \dim M$, we would like to be able to sensibly talk about how many times N and Q intersect in the generic situation. To give it a topological flavor we would like this count to be the same if we move N or Q around inside on M, because of this our count will have to be signed.

Theorem 1.7. *Let $f : N \to M$ be a smooth map, where N is a compact manifold. Then there exists a submersion $F : N \times B^K \to M$, so that $F(x, 0) = f(x)$. If f is an embedding of a submanifold, then we can choose F so that $F(\cdot, s)$ is also an embedding of a submanifold for all $s \in B^K$.*

In the previous section the construction of F was made easier by the vector properties of TM, but in general we have no way to add or scalar multiply functions. Instead, the main tool we will use is the flow of a vector field. Recall that a *smooth isotopy* $\varphi_t : M \to M$ is a smooth family of diffeomorphisms depending on a parameter $t \in I \subseteq \mathbb{R}$, so that $\varphi_0 = \mathrm{id}$.

Proposition 1.8. *Let M be a closed manifold, and let $V : M \to TM$ be a vector field on M. Then there is a smooth isotopy $\varphi_t : M \to M$, $t \in \mathbb{R}$, which satisfies $\frac{d}{dt}\varphi_t(x) = V \circ \varphi_t(x)$. φ_t is uniquely defined by this condition, and it furthermore satisfies $\varphi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2}$. φ_t is called the* flow *of the vector field V.*

More generally, if M is any manifold without boundary and $A \subseteq M$ is a compact set, then there is a unique isotopy $\varphi_t : A \to M$ satisfying the same equation, but only for $t \in (-\varepsilon, \varepsilon)$ (where $\varepsilon > 0$ depends on both V and A).

Proof: Fix a point $x \in M$, and think of $\varphi_t(x)$ as being a function of t, parametrized by x. Then, in any chart around x, the equation $\frac{d}{dt}\varphi_t(x) = V \circ \varphi_t(x)$ is a vector valued first order ordinary differential equation, subject to the initial condition $\varphi_0(x) = x$. Therefore there is a solution for $t \in (-\delta_x, \delta_x)$, and this solution is unique. Since $\varphi_{t+t_0} \circ \varphi_{t_0}^{-1}$ satisfies the same differential equation and initial condition, it follows that $\varphi_{t+t_0} = \varphi_t \circ \varphi_{t_0}$ for all t, t_0 satisfying $|t+t_0| < \delta_x$. In particular $\varphi_{-t}(x) = \varphi_t^{-1}(x)$.

Consider the case where $A \subseteq M$ is compact. Since δ_x depends on x continuously, we can choose $\varepsilon = \min_{x \in A} \delta_x$, and then $\varphi_t : A \to M$ is defined for all $T \in (-\varepsilon, \varepsilon)$. φ_t is a smooth map because solutions to first order ODE's depend smoothly on initial conditions, and since $\varphi_{-t}(x) = \varphi_t^{-1}(x)$, φ_t is a diffeomorphism onto its image for all t.

If M is compact, we can take $A = M$ and get an isotopy $\varphi_t : M \to M$ for $t \in (-\varepsilon, \varepsilon)$. Suppose φ_t cannot be defined for all $t \in \mathbb{R}$, and let $t_0 > \frac{1}{2} \sup \varepsilon$, where the supremum is taken over all ε so that φ_t can be defined for $t \in (-\varepsilon, \varepsilon)$. But then $\varphi_{t \pm t_0} := \varphi_t \circ \varphi_{\pm t_0}$ defines φ_t on a larger interval. □

Proof of Theorem 1.7: Choose a spanning set of vector fields on M, $\{\sigma_i\}_{i=1}^K$. Since N is compact we can define $\varphi_t^1 : f(N) \to M$ for $t \in (-\varepsilon_1, \varepsilon_1)$, where φ_t^1 is the flow of the vector field σ_1. We define $F^1 : N \times [-\frac{\varepsilon_1}{2}, \frac{\varepsilon_1}{2}] \to M$ to simply be the restriction of $\varphi^1 \circ f$, so $F^1(x, s_1) = \varphi_{s_1}(f(x))$.

We can then define $\varphi_t^2 : F^1(N \times [-\frac{\varepsilon_1}{2}, \frac{\varepsilon_1}{2}]) \to M$ for $t \in (-\varepsilon_2, \varepsilon_2)$, to be the flow of σ_2. We then restrict $\varphi^2 \circ F^1$ to the map $F^2 : [-\frac{\varepsilon_1}{2}, \frac{\varepsilon_1}{2}] \times [-\frac{\varepsilon_2}{2}, \frac{\varepsilon_2}{2}] \to M$. Thus $F^2(x, s_1, s_2) = \varphi_{s_2}^2(F^1(x, s_1)) = (\varphi_{s_2}^2 \circ \varphi_{s_1}^1)(f(x))$. Continuing in this manner we arrive at $F := F^K : N \times [-\frac{\varepsilon_1}{2}, \frac{\varepsilon_1}{2}] \times \ldots \times [-\frac{\varepsilon_K}{2}, \frac{\varepsilon_K}{2}] \to M$ defined by $F(x, s_1, \ldots, s_K) = (\varphi_{s_K}^K \circ \ldots \circ \varphi_{s_1}^1 \circ f)(x)$.

By the definition of φ_t^i, notice that $\left.\frac{\partial F}{\partial s_i}\right|_{s=0} = \sigma_i(x)$. So $dF_{(x,0)}$ is surjective for all $x \in N$. Since surjectivity is an open property and since N is compact, we can find a ball $B^K \subseteq [-\frac{\varepsilon_1}{2}, \frac{\varepsilon_1}{2}] \times \ldots \times [-\frac{\varepsilon_K}{2}, \frac{\varepsilon_K}{2}]$ so that $F : N \times B^K \to M$ is a submersion. □

Note. Though it is true that $\frac{\partial F}{\partial s_K} = \sigma_K(F(x, s))$, it is false in general that $\frac{\partial F}{\partial s_i} = \sigma_i(F(x, s))$ because flows do not commute: $\varphi_{s_1}^1 \circ \varphi_{s_2}^2 \neq \varphi_{s_2}^2 \circ \varphi_{s_1}^1$. For this reason it will not necessarily hold that F is a submersion on the entire domain $N \times [-\frac{\varepsilon_1}{2}, \frac{\varepsilon_1}{2}] \times \ldots \times [-\frac{\varepsilon_K}{2}, \frac{\varepsilon_K}{2}]$.

Intersection product

Definition 1.9. Let $f : N \to M$ be a smooth map and $Q \subseteq M$ a submanifold with $f \pitchfork Q$. Suppose that N, M, and Q are all boundaryless and oriented, and also that Q and N are closed. Assume that $\dim N + \dim Q = \dim M$. Then we define the *intersection product* of f and Q, denoted by $[f] \cdot [Q] \in \mathbb{Z}$, to be the signed count of points in $f^{-1}(Q)$.

Proposition 1.10. *Suppose there is a smooth map $F : X \to M$, where X is a compact oriented manifold with $\partial X = N$ and $F|_N = f$. Then $[f] \cdot [Q] = 0$.*

By applying this proposition to $N \times [0, 1]$ we get:

Corollary 1.11. *Suppose $f_0, f_1 : N \to M$ are homotopic maps, so that both are transverse to Q. Then $[f_0] \cdot [Q] = [f_1] \cdot [Q]$.*

This partially explains the notation, here $[f]$ could be thought of as the equivalence class of all maps homotopic to f. This allows us to extend the definition to arbitrary maps.

Definition 1.12. Let $f : N \to M$ be a smooth map which is not necessarily transverse to Q. We define $[f] \cdot [Q] = [\widetilde{f}] \cdot [Q]$ for any $\widetilde{f} : N \to M$ which is transverse to Q.

Theorem 1.7 tells us that the map $\widetilde{f}$ exists, and Corollary 1.11 tells us that $[f] \cdot [Q]$ doesn't depend on the choice.

Proof of Proposition 1.10: By this point the argument is standard. Using Theorems 1.7 and 1.1 we can find a map $\widetilde{F} : X \to M$ which is arbitrarily C^∞ close to F and transverse to Q. Then $\widetilde{F}^{-1}(Q)$ is an oriented 1-manifold with boundary $\widetilde{F}^{-1}(Q) \cap N$. Therefore the signed count of points in $\widetilde{F}^{-1}(Q) \cap N$ is zero. But $\widetilde{F}|_N$ can be taken arbitrarily close to $F|_N = f$, and transversality is an open condition, therefore $[f] \cdot [Q] = 0$. □

The most common application this can be used for is when N and Q are both closed oriented submanifolds of an oriented manifold M, satisfying $\dim N + \dim Q = \dim M$. If we let $e : N \to M$ be the inclusion map, we then define $[N] \cdot [Q] = [e] \cdot [Q]$.

Example 1.13. We claim that $S^1 \times S^2$ is not diffeomorphic to S^3. Both are three dimensional closed oriented 3-manifolds, and both admit non-vanishing vector fields so $\chi(S^3) = \chi(S^1 \times S^2) = 0$. If we let $N = S^1 \times \{\text{point}\} \subseteq S^1 \times S^2$ and $Q = \{\text{point}\} \times S^2 \subseteq S^1 \times S^2$, then is clear that $[N] \cdot [Q] = \pm 1$ (depending on how we orient everything). In particular this shows that there is no map $F : B^2 \to S^1 \times S^2$ so that $f(\partial B^2) = N$. We show there exists no such submanifold of S^3.

Let $f : S^1 \to S^3$ be any map. By Sard's theorem there is a point $p \in S^3$ so that $p \notin f(S^1)$. $S^3 \setminus \{p\}$ is diffeomorphic to $\mathbb{R}^3$ by stereographic projection. Choose a bump function $h : [0,1] \to [0,1]$ so that $h(1) = 1$ and $h(t) = 0$ in an open neighborhood of $t = 0$. Putting polar coordinates (r, θ) on the disk B^2, define a map $F : B^2 \to \mathbb{R}^3$ by $F(r, \theta) = h(r) f(\theta)$. Then F is a smooth map and $F|_\partial B^2 = f$.