$$\varphi^* f_i = \sum \frac{\partial \varphi_i}{\partial x_j} v_j$$

where $f_1, \ldots, f_N$ are the coordinates of the map, f_v, and $\varphi_1, \ldots, \varphi_N$ the coordinates of $\iota \circ \varphi$.

11. Let v be a vector field on X and $\varphi : X \to \mathbb{R}$, a $\mathcal{C}^\infty$ function. Show that if the function

$$L_v \varphi = \iota(v)\, d\varphi \tag{4.3.13}$$

is zero φ is constant along integral curves of v.

12. Suppose that $\varphi : X \to \mathbb{R}$ is proper. Show that if $L_v \varphi = 0$, v is complete.

Hint: For $p \in X$ let $a = \varphi(p)$. By assumption, $\varphi^{-1}(a)$ is compact. Let $\rho \in \mathcal{C}_0^\infty(X)$ be a "bump" function which is one on $\varphi^{-1}(a)$ and let w be the vector field, ρv. By Theorem 4.3.14, w is complete and since

$$L_w \varphi = \iota(\rho v)\, d\varphi = \rho \iota(v)\, d\varphi = 0$$

φ is constant along integral curves of w. Let $\gamma(t)$, $-\infty < t < \infty$, be the integral curve of w with initial point, $\gamma(0) = p$. Show that γ is an integral curve of v.

4.4 Orientations

The last part of Chapter 4 will be devoted to the "integral calculus" of forms on manifolds. In particular we will prove manifold versions of two basic theorems of integral calculus on $\mathbb{R}^n$, Stokes theorem and the divergence theorem, and also develop a manifold version of degree theory. However, to extend the integral calculus to manifolds without getting involved in horrendously technical "orientation" issues we will confine ourselves to a special class of manifolds: orientable manifolds. The goal of this section will be to explain what this term means.

Definition 4.4.1. *Let X be an n-dimensional manifold. An* orientation *of X is a rule for assigning to each $p \in X$ an orientation of T_pX.*

Thus by definition 1.9.1 one can think of an orientation as a "labeling" rule which, for every $p \in X$, labels one of the two components of the set, $\Lambda^n(T_p^*X) - \{0\}$, by $\Lambda^n(T_p^*X)_+$, which we'll henceforth call the "plus" part of $\Lambda^n(T_p^*X)$, and the other component by $\Lambda^n(T_p^*X)_-$, which we'll henceforth call the "minus" part of $\Lambda^n(T_p^*X)$.

Definition 4.4.2. *An orientation of X is* smooth *if, for every $p \in X$, there exists a neighborhood, U, of p and a non-vanishing n-form, $\omega \in \Omega^n(U)$ with the property*

$$\omega_q = \Lambda^n(T_q^*X)_+ \tag{4.4.1}$$

for every $q \in U$.

Remark 4.4.3. *If we're given an orientation of X we can define another orientation by assigning to each $p \in X$ the opposite orientation to the orientation we already assigned, i.e., by switching the labels on $\Lambda^n(T_p^*)_+$ and $\Lambda^n(T_p^*)_-$. We will call this the* reversed *orientation of X. We will leave for you to check as an exercise that if X is connected and equipped with a smooth orientation, the only smooth orientations of X are the given orientation and its reversed orientation.*

Hint: *Given any smooth orientation of X the set of points where it agrees with the given orientation is open, and the set of points where it doesn't is also open. Therefore one of these two sets has to be empty.*

Note that if $\omega \in \Omega^n(X)$ is a non-vanishing n-form one gets from ω a smooth orientation of X by requiring that the "labeling rule" above satisfy

$$\omega_p \in \Lambda^n(T_p^*X)_+ \tag{4.4.2}$$

for every $p \in X$. If ω has this property we will call ω a *volume form*. It's clear from this definition that if ω_1 and ω_2 are volume forms on X then $\omega_2 = f_{2,1}\omega_1$ where $f_{2,1}$ is an everywhere positive $\mathcal{C}^\infty$ function.

Example 1.

Open subsets, U of $\mathbb{R}^n$. We will usually assign to U its *standard orientation*, by which we will mean the orientation defined by the n-form, $dx_1 \wedge \cdots \wedge dx_n$.

Example 2.

Let $f : \mathbb{R}^N \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map. If zero is a regular value of f, the set $X = f^{-1}(0)$ is a submanifold of $\mathbb{R}^N$ of dimension, $n = N - k$, by Theorem 4.2.5. Moreover, for $p \in X$, T_pX is the kernel of the surjective map

$$df_p : T_p\mathbb{R}^N \to T_o\mathbb{R}^k$$

so we get from df_p a bijective linear map

$$T_p\mathbb{R}^N / T_pX \to T_o\mathbb{R}^k \,. \tag{4.4.3}$$

As explained in example 1, $T_p\mathbb{R}^N$ and $T_o\mathbb{R}^k$ have "standard" orientations, hence if we require that the map (4.4.3) be orientation preserving, this gives $T_p\mathbb{R}^N/T_pX$ an orientation and, by Theorem 1.9.4, gives T_pX an orientation. It's intuitively clear that since df_p varies smoothly with respect to p this orientation does as well; however, this fact requires a proof, and we'll supply a sketch of such a proof in the exercises.

Example 3.

A special case of example 2 is the n-sphere

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,,\, x_1^2 + \cdots + x_{n+1}^2 = 1\} \,,$$

which acquires an orientation from its defining map, $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x) = x_1^2 + \cdots + x_{n+1}^2 - 1$.

Example 4.

Let X be an oriented submanifold of $\mathbb{R}^N$. For every $p \in X$, T_pX sits inside $T_p\mathbb{R}^N$ as a vector subspace, hence, via the identification, $T_p\mathbb{R}^N \leftrightarrow \mathbb{R}^N$ one can think of T_pX as a vector subspace of $\mathbb{R}^N$. In particular from the standard Euclidean inner product on $\mathbb{R}^N$ one gets, by restricting this inner product to vectors in T_pX, an inner product,

$$B_p : T_pX \times T_pX \to \mathbb{R}$$

on T_pX. Let σ_p be the volume element in $\Lambda^n(T_p^*X)$ associated with B_p (see §1.9, exercise 10) and let $\sigma = \sigma_X$ be the non-vanishing n-form on X defined by the assignment

$$p \in X \to \sigma_p\,.$$

In the exercises at the end of this section we'll sketch a proof of the following.

Theorem 4.4.4. *The form, σ_X, is $\mathcal{C}^\infty$ and hence, in particular, is a volume form. (We will call this form the* Riemannian *volume form.)*

Example 5. The Möbius strip. The Möbius strip is a surface in $\mathbb{R}^3$ which is *not* orientable. It is obtained from the rectangle

$$R = \{(x,y)\,;\, 0 \le x \le 1\,,\, -1 < y < 1\}$$

by gluing the ends together in the wrong way, i.e., by gluing $(1,y)$ to $(0,-y)$. It is easy to see that the Möbius strip can't be oriented by taking the standard orientation at $p = (1,0)$ and moving it along the line, $(t,0)$, $0 \le t \le 1$ to the point, $(0,0)$ (which is *also* the point, p, after we've glued the ends of the rectangle together).

We'll next investigate the "compatibility" question for diffeomorphisms between oriented manifolds. Let X and Y be n-dimensional manifolds and $f : X \to Y$ a diffeomorphism. Suppose both of these manifolds are equipped with orientations. We will say that f is *orientation preserving* if, for all $p \in X$ and $q = f(p)$ the linear map

$$df_p : T_pX \to T_qY$$

is orientation preserving. It's clear that if ω is a volume form on Y then f is orientation preserving if and only if $f^*\omega$ is a volume form on X, and from (1.9.5) and the chain rule one easily deduces

Theorem 4.4.5. *If Z is an oriented n-dimensional manifold and $g : Y \to Z$ a diffeomorphism, then if both f and g are orientation preserving, so is $g \circ f$.*

If $f : X \to Y$ is a diffeomorphism then the set of points, $p \in X$, at which the linear map,

$$df_p : T_pX \to T_qY\,, \quad q = f(p)\,,$$

is orientation preserving is open, and the set of points at which its orientation reversing is open as well. Hence if X is connected, df_p has to be orientation preserving at all points or orientation reversing at all points. In the latter case we'll say that f is *orientation reversing.*

If U is a parametrizable open subset of X and $\varphi : U_0 \to U$ a parametrization of U we'll say that this parametrization is an *oriented* parametrization if φ is orientation preserving with respect to the standard orientation of U_0 and the given orientation on U. Notice that if this parametrization isn't oriented we can convert it into one that is by replacing every connected component, V_0, of U_0 on which φ isn't orientation preserving by the open set

$$V_0^\sharp = \{(x_1, \dots, x_n) \in \mathbb{R}^n\,, \quad (x_1, \dots, x_{n1}, -x_n) \in V_0\} \tag{4.4.4}$$

and replacing φ by the map

$$\psi(x_1, \dots, x_n) = \varphi(x_1, \dots, x_{n-1}, -x_n)\,. \tag{4.4.5}$$

If $\varphi_i : U_i \to U$, $i = 0, 1$, are oriented parametrizations of U and $\psi : U_0 \to U_1$ is the diffeomorphism, $\varphi_1^{-1} \circ \varphi_0$, then by the theorem above ψ is orientation preserving or in other words

$$\det \left[\frac{\partial \psi_i}{\partial x_j}\right] > 0 \tag{4.4.6}$$

at every point on U_0.

We'll conclude this section by discussing some orientation issues which will come up when we discuss Stokes theorem and the divergence theorem in §4.6. First a definition.

Definition 4.4.6. *An open subset, D, of X is a* smooth domain *if*

(a) its boundary is an $(n-1)$-dimensional submanifold of X and

(b) the boundary of D coincides with the boundary of the closure of D.

Examples.

1. The n-ball, $x_1^2 + \cdots + x_n^2 < 1$, whose boundary is the sphere, $x_1^2 + \cdots + x_n^2 = 1$.

2. The n-dimensional annulus,

$$1 < x_1^2 + \cdots + x_n^2 < 2$$

whose boundary consists of the spheres,

$$x_1^2 + \cdots + x_n^2 = 1 \text{ and } x_1^2 + \cdots + x_n^2 = 2\,.$$

3. Let S^{n-1} be the unit sphere, $x_1^2 + \cdots + x_2^2 = 1$ and let $D = \mathbb{R}^n - S^{n-1}$. Then the boundary of D is S^{n-1} but D is not a smooth domain since the boundary of its closure is empty.

4. The simplest example of a smooth domain is the half-space

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n\,, \quad x_1 < 0\} \tag{4.4.7}$$

whose boundary

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n\,, \quad x_1 = 0\} \tag{4.4.8}$$

we can identify with $\mathbb{R}^{n-1}$ via the map,

$$(x_2, \ldots, x_n) \in \mathbb{R}^{n-1} \to (0, x_2, \ldots, x_n)\,.$$

We will show that every bounded domain looks locally like this example.

Theorem 4.4.7. *Let D be a smooth domain and p a boundary point of D. Then there exists a neighborhood, U, of p in X, an open set, U_0, in $\mathbb{R}^n$ and a diffeomorphism, $\psi : U_0 \to U$ such that ψ maps $U_0 \cap \mathbb{H}^n$ onto $U \cap D$.*

Proof. Let Z be the boundary of D. First we will prove:

Lemma 4.4.8. *For every $p \in Z$ there exists an open set, U, in X containing p and a parametrization*

$$\psi : U_0 \to U \tag{4.4.9}$$

of U with the property

$$\psi(U_0 \cap Bd\mathbb{H}^n) = U \cap Z\,. \tag{4.4.10}$$

Proof. X is locally diffeomorphic at p to an open subset of $\mathbb{R}^n$ so it suffices to prove this assertion for X equal to $\mathbb{R}^n$. However, if Z is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ then by 4.2.7 there exists, for every $p \in Z$ a neighborhood, U, of p in $\mathbb{R}^n$ and a function, $\varphi \in \mathcal{C}^\infty(U)$ with the properties

$$x \in U \cap Z \Leftrightarrow \varphi(x) = 0 \tag{4.4.11}$$

and

$$d\varphi_p \neq 0\,. \tag{4.4.12}$$

Without loss of generality we can assume by (4.4.12) that

$$\frac{\partial \varphi}{\partial x_1}(p) \neq 0\,. \tag{4.4.13}$$

Hence if $\rho : U \to \mathbb{R}^n$ is the map

$$\rho(x_1, \ldots, x_n) = (\varphi(x), x_2, \ldots, x_n) \tag{4.4.14}$$

$(d\rho)_p$ is bijective, and hence ρ is locally a diffeomorphism at p. Shrinking U we can assume that ρ is a diffeomorphism of U onto an open set, U_0. By (4.4.11) and (4.4.14) ρ maps $U \cap Z$ onto $U_0 \cap Bd\mathbb{H}^n$ hence if we take ψ to be ρ^{-1}, it will have the property (4.4.10).

□

We will now prove Theorem 4.4.4. Without loss of generality we can assume that the open set, U_0, in Lemma 4.4.8 is an open ball with center at $q \in Bd\mathbb{H}^n$ and that the diffeomorphism, ψ maps q to p. Thus for $\psi^{-1}(U \cap D)$ there are three possibilities.

i. $\psi^{-1}(U \cap D) = (\mathbb{R}^n - Bd\mathbb{H}^n) \cap U_0$.

ii. $\psi^{-1}(U \cap D) = (\mathbb{R}^n - \overline{\mathbb{H}}^n) \cap U_0$.

or

iii. $\psi^{-1}(U \cap D) = \mathbb{H}^n \cap U_0$.

However, i. is excluded by the second hypothesis in Definition 4.4.6 and if ii. occurs we can rectify the situation by composing φ with the map, $(x_1, \ldots, x_n) \to (-x_1, x_2, \ldots, x_n)$.

□

Definition 4.4.9. *We will call an open set, U, with the properties above a D-adapted parametrizable open set.*

We will now show that if X is oriented and $D \subseteq X$ is a smooth domain then the boundary, Z, of D acquires from X a natural orientation. To see this we first observe

Lemma 4.4.10. *The diffeomorphism, $\psi : U_0 \to U$ in Theorem 4.4.7 can be chosen to be orientation preserving.*

Proof. If it is not, then by replacing ψ with the diffeomorphism, $\psi^\sharp(x_1, \dots, x_n) = \psi(x_1, \dots, x_{n-1}, -x_n)$, we get a D-adapted parametrization of U which *is* orientation preserving. (See (4.4.4)–(4.4.5).) □

Let $V_0 = U_0 \cap \mathbb{R}^{n-1}$ be the boundary of $U_0 \cap \mathbb{H}^n$. The restriction of ψ to V_0 is a diffeomorphism of V_0 onto $U \cap Z$, and we will orient $U \cap Z$ by requiring that this map be an oriented parametrization. To show that this is an "intrinsic" definition, i.e., doesn't depend on the choice of ψ, we'll prove

Theorem 4.4.11. *If $\psi_i : U_i \to U$, $i = 0, 1$, are oriented parametrizations of U with the property*

$$\psi_i : U_i \cap \mathbb{H}^n \to U \cap D$$

the restrictions of ψ_i to $U_i \cap \mathbb{R}^{n-1}$ induce compatible orientations on $U \cap X$.

Proof. To prove this we have to prove that the map, $\varphi_1^{-1} \circ \varphi_0$, restricted to $U \cap Bd\mathbb{H}^n$ is an orientation preserving diffeomorphism of $U_0 \cap \mathbb{R}^{n-1}$ onto $U_1 \cap \mathbb{R}^{n-1}$. Thus we have to prove the following:

Proposition 4.4.12. *Let U_0 and U_1 be open subsets of $\mathbb{R}^n$ and $f : U_0 \to U_1$ an orientation preserving diffeomorphism which maps $U_0 \cap \mathbb{H}^n$ onto $U_1 \cap \mathbb{H}^n$. Then the restriction, g, of f to the boundary, $U_0 \cap \mathbb{R}^{n-1}$, of $U_0 \cap \mathbb{H}^n$ is an orientation preserving diffeomorphism, $g : U_0 \cap \mathbb{R}^{n-1} \to U_1 \cap \mathbb{R}^{n-1}$.*

Let $f(x) = (f_1(x), \dots, f_n(x))$. By assumption $f_1(x_1, \dots, x_n)$ is less than zero if x_1 is less than zero and equal to zero if x_1 is equal to zero, hence

$$\frac{\partial f_1}{\partial x_1}(0, x_2, \dots, x_n) \geq 0 \tag{4.4.15}$$

and

$$\frac{\partial f_1}{\partial x_i}(0, x_2, \dots, x_n) = 0\,, \quad i > 1 \tag{4.4.16}$$

Moreover, since g is the restriction of f to the set $x_1 = 0$

$$\frac{\partial f_i}{\partial x_j}(0, x_2, \dots, x_n) = \frac{\partial g_i}{\partial x_j}(x_2, \dots, x_1) \tag{4.4.17}$$

for $i, j \geq 2$. Thus on the set, $x_1 = 0$

$$\det \left[\frac{\partial f_i}{\partial x_j}\right] = \frac{\partial f_1}{\partial x_1} \det \left[\frac{\partial g_i}{\partial x_j}\right]\,. \tag{4.4.18}$$

Since f is orientation preserving the left hand side of (4.4.18) is positive at all points $(0, x_2, \dots, x_n) \in U_0 \cap \mathbb{R}^{n-1}$ hence by (4.4.15) the same is true for $\frac{\partial f_1}{\partial x_1}$ and $\det \left[\frac{\partial g_i}{\partial x_j}\right]$. Thus g is orientation preserving.

Remark 4.4.13. *For an alternative proof of this result see exercise 8 in §3.2 and exercises 4 and 5 in §3.6.*

We will now orient the boundary of D by requiring that for every D-adapted parametrizable open set, U, the orientation of Z coincides with the orientation of $U \cap Z$ that we described above. We will conclude this discussion of orientations by proving a global version of Proposition 4.4.12.

Proposition 4.4.14. *Let X_i, $i = 1, 2$, be an oriented manifold, $D_i \subseteq X_i$ a smooth domain and Z_i its boundary. Then if f is an orientation preserving diffeomorphism of (X_1, D_1) onto (X_2, D_2) the restriction, g, of f to Z_1 is an orientation preserving diffeomorphism of Z_1 onto Z_2.*

Let U be an open subset of X_1 and $\varphi : U_0 \to U$ an oriented D_1-compatible parametrization of U. Then if $V = f(U)$ the map $f \circ \varphi : U \to V$ is an oriented D_2-compatible parametrization of V and hence $g : U \cap Z_1 \to V \cap Z_2$ is orientation preserving.

□

Exercises.

1. Let V be an oriented n-dimensional vector space, B an inner product on V and $e_i \in V$, $i = 1, \ldots, n$ an oriented orthonormal basis. Given vectors, $v_i \in V$, $i = 1, \ldots, n$ show that if

$$b_{i,j} = B(v_i, v_j) \tag{4.4.19}$$

and

$$v_i = \sum a_{j,i} e_j \,, \tag{4.4.20}$$

the matrices $\mathcal{A} = [a_{i,j}]$ and $\mathcal{B} = [b_{i,j}]$ satisfy the identity:

$$\mathcal{B} = \mathcal{A}^t \mathcal{A} \tag{4.4.21}$$

and conclude that $\det \mathcal{B} = (\det \mathcal{A})^2$. (In particular conclude that $\det \mathcal{B} > 0$.)

2. Let V and W be oriented n-dimensional vector spaces. Suppose that each of these spaces is equipped with an inner product, and let $e_i \in V$, $i = 1, \ldots, n$ and $f_i \in W$, $i = 1, \ldots, n$ be oriented orthonormal bases. Show that if $A : W \to V$ is an orientation preserving linear mapping and $Af_i = v_i$ then

$$A^* \operatorname{vol}_V = (\det[b_{i,j}])^{\frac{1}{2}} \operatorname{vol}_W \tag{4.4.22}$$

where $\operatorname{vol}_V = e_1^* \wedge \cdots \wedge e_n^*$, $\operatorname{vol}_W = f_1^* \wedge \cdots \wedge f_n^*$ and $[b_{i,j}]$ is the matrix (4.4.19).

3. Let X be an oriented n-dimensional submanifold of $\mathbb{R}^n$, U an open subset of X, U_0 an open subset of $\mathbb{R}^n$ and $\varphi : U_0 \to U$ an oriented parametrization. Let φ_i, $i = 1, \ldots, N$, be the coordinates of the map

$$U_0 \to U \hookrightarrow \mathbb{R}^N \,.$$

the second map being the inclusion map. Show that if σ is the Riemannian volume form on X then

$$\varphi^* \sigma = (\det[\varphi_{i,j}])^{\frac{1}{2}} \, dx_1 \wedge \cdots \wedge dx_n \tag{4.4.23}$$

where

$$\varphi_{i,j} = \sum_{k=1}^{N} \frac{\partial \varphi_k}{\partial x_i} \frac{\partial \varphi_k}{\partial x_j} \quad 1 \leq i, j \leq n \,. \tag{4.4.24}$$

(*Hint:* For $p \in U_0$ and $q = \varphi(p)$ apply exercise 2 with $V = T_q X$, $W = T_p \mathbb{R}^n$, $A = (d\varphi)_p$ and $v_i = (d\varphi)_p \left(\frac{\partial}{\partial x_i}\right)_p$.) Conclude that σ is a $\mathcal{C}^\infty$ infinity n-form and hence that it *is* a volume form.

4. Given a $\mathcal{C}^\infty$ function $f : \mathbb{R} \to \mathbb{R}$, its graph

$$X = \{(x, f(x))\,, \quad x \in \mathbb{R}\}$$

is a submanifold of $\mathbb{R}^2$ and

$$\varphi : \mathbb{R} \to X\,, \quad x \to (x, f(x))$$

is a diffeomorphism. Orient X by requiring that φ be orientation preserving and show that if σ is the Riemannian volume form on X then

$$\varphi^*\sigma = \left(1 + \left(\frac{df}{dx}\right)^2\right)^{\frac{1}{2}} dx\,. \tag{4.4.25}$$

Hint: Exercise 3.

5. Given a $\mathcal{C}^\infty$ function $f : \mathbb{R}^n \to \mathbb{R}$ its graph

$$X = \{(x, f(x))\,, \quad x \in \mathbb{R}^n\}$$

is a submanifold of $\mathbb{R}^{n+1}$ and

$$\varphi : \mathbb{R}^n \to X\,, \quad x \to (x, f(x)) \tag{4.4.26}$$

is a diffeomorphism. Orient X by requiring that φ is orientation preserving and show that if σ is the Riemannian volume form on X then

$$\varphi^*\sigma = \left(1 + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2\right)^{\frac{1}{2}} dx_1 \wedge \cdots \wedge dx_n\,. \tag{4.4.27}$$

Hints:

(a) Let $v = (c_1, \ldots, c_n) \in \mathbb{R}^n$. Show that if $C : \mathbb{R}^n \to \mathbb{R}$ is the linear mapping defined by the matrix $[c_i c_j]$ then $Cv = (\sum c_i^2)v$ and $Cw = 0$ if $w \cdot v = 0$.

(b) Conclude that the eigenvalues of C are $\lambda_1 = \sum c_i^2$ and $\lambda_2 = \cdots = \lambda_n = 0$.

(c) Show that the determinant of $I + C$ is $1 + \sum c_i^2$.

(d) Use (a)–(c) to compute the determinant of the matrix (4.4.24) where φ is the mapping (4.4.26).

6. Let V be an oriented N-dimensional vector space and $\ell_i \in V^*$, $i = 1, \ldots, k$, k linearly independent vectors in V^*. Define

$$L : V \to \mathbb{R}^k$$

to be the map $v \to (\ell_1(v), \ldots, \ell_k(v))$.

(a) Show that L is surjective and that the kernel, W, of L is of dimension $n = N - k$.

(b) Show that one gets from this mapping a bijective linear mapping

$$V/W \to \mathbb{R}^K \tag{4.4.28}$$

and hence from the standard orientation on $\mathbb{R}^k$ an induced orientation on V/W and on W. *Hint:* §1.2, exercise 8 and Theorem 1.9.4.

(c) Let ω be an element of $\Lambda^N(V^*)$. Show that there exists a $\mu \in \Lambda^n(V^*)$ with the property

$$\ell_1 \wedge \cdots \wedge \ell_k \wedge \mu = \omega\,. \tag{4.4.29}$$

Hint: Choose an oriented basis, $e_1, \ldots, e_N$ of V such that $\omega = e_1^* \wedge \cdots \wedge e_N^*$ and $\ell_i = e_i^*$ for $i = 1, \ldots, k$, and let $\mu = e_{i+1}^* \wedge \cdots \wedge e_N^*$.

(d) Show that if ν is an element of $\Lambda^n(V^*)$ with the property

$$\ell_1 \wedge \cdots \wedge \ell_k \wedge \nu = 0$$

then there exist elements, ν_i, of $\Lambda^{n-1}(V^*)$ such that

$$\nu = \sum \ell_i \wedge \nu_i\,. \tag{4.4.30}$$

Hint: Same hint as in part (c).

(e) Show that if $\mu = \mu_i$, $i = 1, 2$, are elements of $\Lambda^n(V^*)$ with the property (4.4.29) and $\iota : W \to V$ is the inclusion map then $\iota^*\mu_1 = \iota^*\mu_2$. *Hint:* Let $\nu = \mu_1 - \mu_2$. Conclude from part (d) that $\iota^*\nu = 0$.

(f) Conclude that if μ is an element of $\Lambda^n(V^*)$ satisfying (4.4.29) the element, $\sigma = \iota^*\mu$, of $\Lambda^n(W^*)$ is *intrinsically* defined independent of the choice of μ.

(g) Show that σ lies in $\Lambda^n(V^*)_+$.

7. Let U be an open subset of $\mathbb{R}^N$ and $f : U \to \mathbb{R}^k$ a $\mathcal{C}^\infty$ map. If zero is a regular value of f, the set, $X = f^{-1}(0)$ is a manifold of dimension $n = N - k$. Show that this manifold has a natural smooth orientation. Some suggestions:

(a) Let $f = (f_1, \ldots, f_k)$ and let

$$df_1 \wedge \cdots \wedge df_k = \sum f_I \, dx_I$$

summed over multi-indices which are strictly increasing. Show that for every $p \in X$ $f_I(p) \neq 0$ for some multi-index, $I = (i_1, \ldots, i_k)$, $1 \leq i_1 < \cdots < i_k \leq N$.

(b) Let $J = (j_1, \ldots, j_n)$, $1 \leq j_1 < \cdots < j_n \leq N$ be the complementary multi-index to I, i.e., $j_r \neq i_s$ for all r and s. Show that

$$df_1 \wedge \cdots \wedge df_k \wedge dx_J = \pm f_I \, dx_1 \wedge \cdots \wedge dx_N$$

and conclude that the n-form

$$\mu = \pm \frac{1}{f_I} \, dx_J$$

is a $\mathcal{C}^\infty$ n-form on a neighborhood of p in U and has the property:

$$df_1 \wedge \cdots \wedge df_k \wedge \mu = dx_1 \wedge \cdots \wedge dx_N \,. \tag{4.4.31}$$

(c) Let $\iota : X \to U$ be the inclusion map. Show that the assignment

$$p \in X \to (\iota^*\mu)_p$$

defines an *intrinsic* nowhere vanishing n-form

$$\sigma \in \Omega^n(X)$$

on X. *Hint:* Exercise 6.

(d) Show that the orientation of X defined by σ coincides with the orientation that we described earlier in this section. *Hint:* Same hint as above.

8. Let S^n be the n-sphere and $\iota : S^n \to \mathbb{R}^{n+1}$ the inclusion map. Show that if $\omega \in \Omega^n(\mathbb{R}^{n+1})$ is the n-form, $\omega = \sum(-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \ldots dx_{n+1}$, the n-form $\iota^*\omega \in \Omega^n(S^n)$ is the Riemannian volume form.

9. Let S^{n+1} be the $(n+1)$-sphere and let

$$S^{n+1}_{+} = \{(x_1, \ldots, x_{n+2}) \in S^{n+1}, \quad x_1 < 0\}$$

be the lower hemi-sphere in S^{n+1}.

(a) Prove that S^{n+1}_{+} is a smooth domain.

(b) Show that the boundary of S^{n+1}_{+} is S^n.

(c) Show that the boundary orientation of S^n agrees with the orientation of S^n in exercise 8.

4.5 Integration of forms over manifolds

In this section we will show how to integrate differential forms over manifolds. In what follows X will be an oriented n-dimensional manifold and W an open subset of X, and our goal will be to make sense of the integral

$$\int_W \omega \tag{4.5.1}$$

where ω is a compactly supported n-form. We'll begin by showing how to define this integral when the support of ω is contained in a parametrizable open set, U. Let U_0 be an open subset of $\mathbb{R}^n$ and $\varphi_0 : U_0 \to U$ a parametrization. As we noted in §4.4 we can assume without loss of generality that this parametrization is oriented. Making this assumption, we'll define

$$\int_W \omega = \int_{W_0} \varphi_0^* \omega \tag{4.5.2}$$

where $W_0 = \varphi_0^{-1}(U \cap W)$. Notice that if $\varphi^*\omega = f dx_1 \wedge \cdots \wedge dx_n$, then, by assumption, f is in $\mathcal{C}_0^\infty(U_0)$. Hence since

$$\int_{W_0} \varphi_0^*\omega = \int_{W_0} f dx_1 \dots dx_n$$

and since f is a bounded continuous function and is compactly supported the Riemann integral on the right is well-defined. (See Appendix B.) Moreover, if $\varphi_1 : U_1 \to U$ is another oriented parametrization of U and $\psi : U_0 \to U_1$ is the map, $\psi = \varphi_1^{-1} \circ \varphi_0$ then $\varphi_0 = \varphi_1 \circ \psi$, so by Proposition 4.3.3

$$\varphi_0^*\omega = \psi^*\varphi_1^*\omega\,.$$

Moreover, by (4.3.5) ψ is orientation preserving. Therefore since

$$W_1 = \psi(W_0) = \varphi_1^{-1}(U \cap W)$$

Theorem 3.5.2 tells us that

$$\int_{W_1} \varphi_1^*\omega = \int_{W_0} \varphi_0^*\omega\,. \tag{4.5.3}$$

Thus the definition (4.5.2) is a legitimate definition. It doesn't depend on the parametrization that we use to define the integral on the right. From the usual additivity properties of the Riemann integral one gets analogous properties for the integral (4.5.2). Namely for $\omega_i \in \Omega_c^n(U)$, $i = 1, 2$

$$\int_W \omega_1 + \omega_2 = \int_W \omega_1 + \int_W \omega_2 \tag{4.5.4}$$

and for $\omega \in \Omega_c^n(U)$ and $c \in \mathbb{R}$

$$\int_W c\omega = c\int_W \omega\,. \tag{4.5.5}$$

We will next show how to define the integral (4.5.1) for *any* compactly supported n-form. This we will do in more or less the same way that we defined improper Riemann integrals in Appendix B: by using partitions of unity. We'll begin by deriving from the partition of unity theorem in Appendix B a manifold version of this theorem.

Theorem 4.5.1. *Let*

$$\mathbb{U} = \{U_\alpha\,,\ \alpha \in \mathcal{I}\} \tag{4.5.6}$$

be a covering of X be open subsets. Then there exists a family of functions, $\rho_i \in \mathcal{C}_0^\infty(X)$, $i = 1, 2, 3, \ldots$, with the properties

> *(a)* $\rho_i \geq 0$.
>
> *(b) For every compact set, $C \subseteq X$ there exists a positive integer N such that if $i > N$, $\operatorname{supp} \rho_i \cap C = \emptyset$.*
>
> *(c)* $\sum \rho_i = 1$.
>
> *(d) For every i there exists an $\alpha \in \mathcal{I}$ such that* $\operatorname{supp} \rho_i \subseteq U_\alpha$.

Remark 4.5.2. *Conditions (a)–(c) say that the ρ_i's are a partition of unity and (d) says that this partition of unity is subordinate to the covering (4.5.6).*

Proof. To simplify the proof a bit we'll assume that X is a closed subset of $\mathbb{R}^N$. For each U_α choose an open subset, $\mathcal{O}_\alpha$ in $\mathbb{R}^N$ with

$$U_\alpha = \mathcal{O}_\alpha \cap X \tag{4.5.7}$$

and let $\mathcal{O}$ be the union of the $\mathcal{O}_\alpha$'s. By the theorem in Appendix B that we cited above there exists a partition of unity, $\widetilde{\rho}_i \in \mathcal{C}_0^\infty(\mathcal{O})$, $i = 1, 2, \ldots$, subordinate to the covering of $\mathcal{O}$ by the $\mathcal{O}_\alpha$'s. Let ρ_i be the restriction of $\widetilde{\rho}_i$ to X. Since the support of $\widetilde{\rho}_i$ is compact and X is closed, the support of ρ_i is compact, so $\rho_i \in \mathcal{C}_0^\infty(X)$ and it's clear that the ρ_i's inherit from the $\widetilde{\rho}_i$'s the properties (a)–(d).

□

Now let the covering (4.5.6) be any covering of X by parametrizable open sets and let $\rho_i \in \mathcal{C}_0^\infty(X)$, $i = 1, 2, \ldots$, be a partition of unity subordinate to this covering. Given $\omega \in \Omega_c^n(X)$ we will define the integral of ω over W by the sum

$$\sum_{i=1}^{\infty} \int_W \rho_i \omega\,. \tag{4.5.8}$$

Note that since each ρ_i is supported in some U_α the individual summands in this sum are well-defined and since the support of ω is compact all but finitely many of these summands are zero by part (b) of Theorem 4.5.1. Hence the sum itself is well-defined. Let's show that this sum doesn't depend on the choice of $\mathbb{U}$ and the ρ_i's. Let $\mathbb{U}'$ be another covering of X by parametrizable open sets and ρ'_j, $j = 1, 2, \ldots$, a partition of unity subordinate to $\mathbb{U}'$. Then

$$\begin{aligned} \sum_j \int_W \rho'_j \omega &= \sum_j \int_W \sum_i \rho'_j \rho_i \omega \\ &= \sum_j \left(\sum_i \int_W \rho'_j \rho_i \omega \right) \end{aligned} \tag{4.5.9}$$

by (4.5.4). Interchanging the orders of summation and resuming with respect to the j's this sum becomes

$$\sum_i \int_W \sum_j \rho'_j \rho_i \omega$$

or

$$\sum_i \int_W \rho_i \omega \,.$$

Hence

$$\sum_i \int_W \rho'_j \omega = \sum_i \int_W \rho_i \omega \,,$$

so the two sums are the same. □

From (4.5.8) and (4.5.4) one easily deduces

Proposition 4.5.3. *For $\omega_i \in \Omega^n_c(X)$, $i = 1, 2$*

$$\int_W \omega_1 + \omega_2 = \int_W \omega_1 + \int_W \omega_2 \tag{4.5.10}$$

and for $\omega \in \Omega^n_c(X)$ and $c \in \mathbb{R}$

$$\int_W c\omega = c \int_W \omega \,. \tag{4.5.11}$$

The definition of the integral (4.5.1) depends on the choice of an orientation of X, but it's easy to see *how* it depends on this choice. We pointed out in Section 4.4 that if X is connected, there is just one way to orient it smoothly other than by its given orientation, namely by *reversing* the orientation of T_p at each point, p, and it's clear from the definitions (4.5.2) and (4.5.8) that the effect of doing this is to change the sign of the integral, i.e., to change $\int_X \omega$ to $-\int_X \omega$.

In the definition of the integral (4.5.1) we've allowed W to be an arbitrary open subset of X but required ω to be compactly supported. This integral is also well-defined if we allow ω to be an arbitrary element of $\Omega^n(X)$ but require the closure of W in X to be compact. To see this, note that under this assumption the sum (4.5.7) is still a finite sum, so the definition of the integral still makes sense, and the double sum on the right side of (4.5.9) is still a finite sum so it's still true that the definition of the integral doesn't depend on the choice of partitions of unity. In particular if the closure of W in X is compact we will define the volume of W to be the integral,

$$\mathrm{vol}(W) = \int_W \sigma_{\mathrm{vol}}\,, \tag{4.5.12}$$

where σ_{vol} is the Riemannian volume form and if X itself is compact we'll define its volume to be the integral

$$\mathrm{vol}(X) = \int_X \sigma_{\mathrm{vol}}\,. \tag{4.5.13}$$

We'll next prove a manifold version of the change of variables formula (3.5.1).

Theorem 4.5.4. *Let X' and X be oriented n-dimensional manifolds and $f : X' \to X$ an orientation preserving diffeomorphism. If W is an open subset of X and $W' = f^{-1}(W)$*

$$\int_{W'} f^*\omega = \int_W \omega \tag{4.5.14}$$

for all $\omega \in \Omega^n_c(X)$.

Proof. By (4.5.8) the integrand of the integral above is a finite sum of $\mathcal{C}^\infty$ forms, each of which is supported on a parametrizable open subset, so we can assume that ω itself as this property. Let V be a parametrizable open set containing the support of ω and let $\varphi_0 : U \to V$ be an oriented parameterization of V. Since f is a diffeomorphism its inverse exists and is a diffeomorphism of X onto X_1. Let $V' = f^{-1}(V)$ and $\varphi_0' = f^{-1} \circ \varphi_0$. Then $\varphi_0' : U \to V'$ is an oriented parameterization of V'. Moreover, $f \circ \varphi_0' = \varphi_0$ so if $W_0 = \varphi_0^{-1}(W)$ we have

$$W_0 = (\varphi_0')^{-1}(f^{-1}(W)) = (\varphi_0')^{-1}(W')$$

and by the chain rule we have

$$\varphi_0^*\omega \quad = \quad (f \circ \varphi_0')^*\omega = (\varphi_0')^* f^*\omega$$

hence

$$\int_W \omega \quad = \quad \int_{W_0} \varphi_0^*\omega = \int_{W_0} (\varphi_0')^*(f^*\omega) = \int_{W'} f^*\omega\,.$$

□

Exercise.

Show that if $f : X' \to X$ is orientation reversing

$$\int_{W'} f^*\omega = -\int_W \omega\,. \tag{4.5.15}$$

We'll conclude this discussion of "integral calculus on manifolds" by proving a preliminary version of Stokes theorem.

Theorem 4.5.5. *If μ is in $\Omega_c^{n-1}(X)$ then*

$$\int_X d\mu = 0\,. \tag{4.5.16}$$

Proof. Let ρ_i, $i = 1, 2, \ldots$ be a partition of unity with the property that each ρ_i is supported in a parametrizable open set $U_i = U$. Replacing μ by $\rho_i\mu$ it suffices to prove the theorem for $\mu \in \Omega_c^{n-1}(U)$. Let $\varphi : U_0 \to U$ be an oriented parametrization of U. Then

$$\int_U d\mu = \int_{U_0} \varphi^*\, d\mu = \int_{U_0} d\varphi^*\mu = 0$$

by Theorem 3.3.1.

□

Exercises.

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function and let

$$X = \{(x, x_{n+1}) \in \mathbb{R}^{n+1}\,, \quad x_{n+1} = f(x)\}$$

be the graph of f. Let's orient X by requiring that the diffeomorphism

$$\varphi : \mathbb{R}^n \to X\,, \quad x \to (x, f(x))$$

be orientation preserving. Given a bounded open set U in $\mathbb{R}^n$ compute the Riemannian volume of the image

$$X_U = \varphi(U)$$

of U in X as an integral over U. *Hint:* §4.4, exercise 5.

2. Evaluate this integral for the open subset, X_U, of the paraboloid, $x_3 = x_1^2 + x_2^2$, U being the disk $x_1^2 + x_2^2 < 2$.

3. In exercise 1 let $\iota : X \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion map of X onto $\mathbb{R}^{n+1}$.

(a) If $\omega \in \Omega^n(\mathbb{R}^{n+1})$ is the n-form, $x_{n+1}\, dx_1 \wedge \cdots \wedge dx_n$, what is the integral of $\iota^*\omega$ over the set X_U? Express this integral as an integral over U.

(b) Same question for $\omega = x_{n+1}^2\, dx_1 \wedge \cdots \wedge dx_n$.

(c) Same question for $\omega = dx_1 \wedge \cdots \wedge dx_n$.

4. Let $f : \mathbb{R}^n \to (0, +\infty)$ be a positive $\mathcal{C}^\infty$ function, U a bounded open subset of $\mathbb{R}^n$, and W the open set of $\mathbb{R}^{n+1}$ defined by the inequalities

$$0 < x_{n+1} < f(x_1, \ldots, x_n)$$

and the condition $(x_1, \ldots, x_n) \in U$.

(a) Express the integral of the $(n+1)$-form $\omega = x_{n+1}\, dx_1 \wedge \cdots \wedge dx_{n+1}$ over W as an integral over U.

(b) Same question for $\omega = x_{n+1}^2\, dx_1 \wedge \cdots \wedge dx_{n+1}$.

(c) Same question for $\omega = dx_1 \wedge \cdots \wedge dx_n$

5. Integrate the "Riemannian area" form

$$x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2$$

over the unit 2-sphere S^2. (See §4.4, exercise 8.)

Hint: An easier problem: Using polar coordinates integrate $\omega = x_3\, dx_1 \wedge dx_2$ over the hemisphere, $x_3 = \sqrt{1 - x_1^2 - x_2^2}$, $x_1^2 + x_2^2 < 1$.

6. Let α be the one-form $\sum_{i=1}^n y_i\, dx_i$ in formula (2.7.2) and let $\gamma(t)$, $0 \le t \le 1$, be a trajectory of the Hamiltonian vector field (2.7.3). What is the integral of α over $\gamma(t)$?

4.6 Stokes theorem and the divergence theorem

Let X be an oriented n-dimensional manifold and $D \subseteq X$ a smooth domain. We showed in §4.4 that if Z is the boundary of D it acquires from D a natural orientation. Hence if $\iota : Z \to X$ is the inclusion map and μ is in $\Omega_c^{n-1}(X)$, the integral

$$\int_Z \iota^* \mu$$

is well-defined. We will prove:

Theorem 4.6.1 (Stokes theorem)**.** *For* $\mu \in \Omega_c^{k-1}(X)$

$$\int_Z \iota^* \mu = \int_D d\mu \,. \tag{4.6.1}$$

Proof. Let ρ_i, $i = 1, 2, \dots$, be a partition of unity such that for each i, the support of ρ_i is contained in a parametrizable open set, $U_i = U$, of one of the following three types:

(a) $U \subseteq \operatorname{Int} D$.

(b) $U \subseteq \operatorname{Ext} D$.

(c) There exists an open subset, U_0, of $\mathbb{R}^n$ and an oriented D-adapted parametrization

$$\varphi : U_0 \to U \,. \tag{4.6.2}$$

Replacing μ by the finite sum $\sum \rho_i \mu$ it suffices to prove (4.6.1) for each $\rho_i \mu$ separately. In other words we can assume that the support of μ itself is contained in a parametrizable open set, U, of type (a), (b) or (c). But if U is of type (a)

$$\int_D d\mu = \int_U d\mu = \int_X d\mu$$

and $\iota^* \mu = 0$. Hence the left hand side of (4.6.1) is zero and, by Theorem 4.5.5, the right hand side is as well. If U is of type (b) the situation is even simpler: $\iota^* \mu$ is zero and the restriction of μ to D is zero, so both sides of (4.6.1) are automatically zero. Thus one is reduced to proving (4.6.1) when U is an open subset of type (c).

In this case the restriction of the map (4.6.1) to $U_0 \cap Bd\mathbb{H}^n$ is an orientation preserving diffeomorphism

$$\psi : U_0 \cap Bd\mathbb{H}^n \to U \cap Z \tag{4.6.3}$$

and

$$\iota_Z \circ \psi = \varphi \circ \iota_{\mathbb{R}^{n-1}} \tag{4.6.4}$$

where the maps $\iota = \iota_Z$ and

$$\iota_{\mathbb{R}^{n-1}} : \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$$

are the inclusion maps of Z into X and $Bd\mathbb{H}^n$ into $\mathbb{R}^n$. (Here we're identifying $Bd\mathbb{H}^n$ with $\mathbb{R}^{n-1}$.) Thus

$$\int_D d\mu = \int_{\mathbb{H}^n} \varphi^* \, d\mu = \int_{\mathbb{H}^n} d\varphi^* \mu$$

and by (4.6.4)

$$\begin{aligned} \int_Z \iota_Z^* \mu &= \int_{\mathbb{R}^{n-1}} \psi^* \iota_Z^* \mu \\ &= \int_{\mathbb{R}^{n-1}} \iota_{\mathbb{R}^{n-1}}^* \varphi^* \mu \\ &= \int_{Bd\mathbb{H}^n} \iota_{\mathbb{R}^{n-1}}^* \varphi^* \mu \, . \end{aligned}$$

Thus it suffices to prove Stokes theorem with μ replaced by $\varphi^*\mu$, or, in other words, to prove Stokes theorem for $\mathbb{H}^n$; and this we will now do.

Stokes theorem for $\mathbb{H}^n$: Let

$$\mu = \sum (-1)^{i-1} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \, .$$

Then

$$d\mu = \sum \frac{\partial f_i}{\partial x_i} \, dx_1 \wedge \cdots \wedge dx_n$$

and

$$\int_{\mathbb{H}^n} d\mu = \sum_i \int_{\mathbb{H}^n} \frac{\partial f_i}{\partial x_i} \, dx_1 \cdots dx_n \, .$$

We will compute each of these summands as an iterated integral doing the integration with respect to dx_i first. For $i > 1$ the dx_i integration ranges over the interval, $-\infty < x_i < \infty$ and hence since f_i is compactly supported

$$\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i}\, dx_i = f_i(x_1, \ldots, x_i, \ldots, x_n)\Big|_{x_i=-\infty}^{x_i=+\infty} = 0\,.$$

On the other hand the dx_1 integration ranges over the integral, $-\infty < x_1 < 0$ and

$$\int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1}\, dx_1 = f(0, x_2, \ldots, x_n)\,.$$

Thus integrating with respect to the remaining variables we get

$$\int_{\mathbb{H}^n} d\mu = \int_{\mathbb{R}^{n-1}} f(0, x_2, \ldots, x_n)\, dx_2 \ldots\, dx_n\,. \tag{4.6.5}$$

On the other hand, since $\iota^*_{\mathbb{R}^{n-1}} x_1 = 0$ and $\iota^*_{\mathbb{R}^{n-1}}\, x_i = x_i$ for $i > 1$,

$$\iota^*_{\mathbb{R}^{n-1}} \mu = f_1(0, x_2, \ldots, x_n)\, dx_2 \wedge \cdots \wedge dx_n$$

so

$$\int \iota^*_{\mathbb{R}^{n-1}} \mu = \int f(0, x_2, \ldots, x_n)\, dx_2 \ldots\, dx_n\,. \tag{4.6.6}$$

Hence the two sides, (4.6.5) and (4.6.6), of Stokes theorem are equal. □

One important variant of Stokes theorem is the divergence theorem: Let ω be in $\Omega^n_c(X)$ and let v be a vector field on X. Then

$$L_v \omega = \iota(v)\, d\omega + d\iota(v)\omega = d\iota(v)\omega\,,$$

hence, denoting by ι_Z the inclusion map of Z into X we get from Stokes theorem, with $\mu = \iota(v)\omega$:

Theorem 4.6.2 (The manifold version of the divergence theorem)**.**

$$\int_D L_v \omega = \int_Z \iota^*_Z(\iota(v)\omega)\,. \tag{4.6.7}$$

If D is an open domain in $\mathbb{R}^n$ this reduces to the usual divergence theorem of multi-variable calculus. Namely if $\omega = dx_1 \wedge \cdots \wedge dx_n$ and $v = \sum v_i \frac{\partial}{\partial x_i}$ then by (2.4.14)

$$L_v \, dx_1 \wedge \cdots \wedge dx_n = \operatorname{div}(v) \, dx_1 \wedge \cdots \wedge dx_n$$

where

$$\operatorname{div}(v) = \sum \frac{\partial v_i}{\partial x_i} \,. \tag{4.6.8}$$

Thus if Z is the boundary of D and ι_Z the inclusion map of Z into $\mathbb{R}^n$

$$\int_D \operatorname{div}(v) \, dx = \int_Z \iota_Z^*(\iota_v \, dx_1 \wedge \cdots \wedge dx_n) \,. \tag{4.6.9}$$

The right hand side of this identity can be interpreted as the "flux" of the vector field, v, through the boundary of D. To see this let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^\infty$ defining function for D, i.e., a function with the properties

$$p \in D \Leftrightarrow f(p) < 0 \tag{4.6.10}$$

and

$$df_p \neq 0 \text{ if } p \in BdD \,. \tag{4.6.11}$$

This second condition says that zero is a regular value of f and hence that $Z = BdD$ is defined by the non-degenerate equation:

$$p \in Z \Leftrightarrow f(p) = 0 \,.$$

Let w be the vector field

$$\left(\sum \left(\frac{\partial f}{\partial x_i} \right)^2 \right)^{-1} \sum \frac{\partial f_i}{\partial x_i} \frac{\partial}{\partial x_i} \,.$$

In view of (4.6.11) this vector field is well-defined on a neighborhood, U, of Z and satisfies

$$\iota(w) \, df = 1 \,. \tag{4.6.12}$$

Now note that since $df \wedge dx_1 \wedge \cdots \wedge dx_n = 0$

$$\begin{aligned} 0 &= \iota(w)(df \wedge dx_1 \wedge \cdots \wedge dx_n) \\ &= (\iota(w)\, df)\, dx_1 \wedge \cdots \wedge dx_n - df \wedge \iota(w)\, dx_1 \wedge \cdots \wedge dx_n \\ &= dx_1 \wedge \cdots \wedge dx_n - df \wedge \iota(w)\, dx_1 \wedge \cdots \wedge dx_n\,, \end{aligned}$$

hence letting ν be the $(n-1)$-form $\iota(w)\, dx_1 \wedge \cdots \wedge dx_n$ we get the identity

$$dx_1 \wedge \cdots \wedge dx_n = df \wedge \nu \tag{4.6.13}$$

and by applying the operation, $\iota(v)$, to both sides of (4.6.13) the identity

$$\iota(v)\, dx_1 \wedge \cdots \wedge dx_n = (L_v f)\nu - df \wedge \iota(v)\nu\,. \tag{4.6.14}$$

Let $\nu_Z = \iota_Z^* \nu$ be the restriction of ν to Z. Since $\iota_Z^* = 0$, $\iota_Z^*\, df = 0$ and hence by (4.6.14)

$$\iota_Z^*(\iota(v)\, dx_1 \wedge \cdots \wedge dx_n) = \iota_Z^*(L_v f)\nu_Z\,,$$

and the formula (4.6.9) now takes the form

$$\int_D \operatorname{div}(v)\, dx = \int_Z L_v f \nu_Z \tag{4.6.15}$$

where the term on the right is by definition the flux of v through Z. In calculus books this is written in a slightly different form. Letting

$$\sigma_Z = \left(\sum \left(\frac{\partial f}{\partial x_i}\right)^2\right)^{\frac{1}{2}} \nu_Z$$

and letting

$$\vec{n} = \left(\sum \left(\frac{\partial f}{\partial x_i}\right)^2\right)^{-\frac{1}{2}} \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$$

and

$$\vec{v} = (v_1, \ldots, v_n)$$

we have

$$L_v \nu_Z = (\vec{n} \cdot \vec{v}) \sigma_Z$$

and hence

$$\int_D \operatorname{div}(v)\, dx = \int_Z (\vec{n} \cdot \vec{v}) \sigma_Z \,. \tag{4.6.16}$$

In three dimensions σ_Z is just the standard "infinitesimal element of area" on the surface Z and n_p the unit outward normal to Z at p, so this version of the divergence theorem is the version one finds in most calculus books.

As an application of Stokes theorem, we'll give a very short alternative proof of the Brouwer fixed point theorem. As we explained in §3.6 the proof of this theorem basically comes down to proving

Theorem 4.6.3. *Let B^n be the closed unit ball in $\mathbb{R}^n$ and S^{n-1} its boundary. Then the identity map*

$$\mathrm{id}_{S^{n-1}} : S^{n-1} \to S^{n-1}$$

can't be extended to a $\mathcal{C}^\infty$ map

$$f : B^n \to S^{n-1} \,.$$

Proof. Suppose that f is such a map. Then for every $n-1$-form, $\mu \in \Omega^{n-1}(S^{n-1})$,

$$\int_{B^n} df^* \mu = \int_{S^{n-1}} (\iota_{S^{n-1}})^* f^* \mu \,. \tag{4.6.17}$$

But $df^*\mu = f^*\, d\mu = 0$ since μ is an $(n-1)$-form and S^{n-1} is an $(n-1)$-dimensional manifold, and since f is the identity map on S^{n-1}, $(\iota_{S_{n-1}})^* f^* \mu = (f \circ \iota_{S^{n-1}})^* \mu = \mu$. Thus for *every* $\mu \in \Omega^{n-1}(S^{n-1})$, (4.6.17) says that the integral of μ over S^{n-1} is zero. Since there are lots of $(n-1)$-forms for which this is not true, this shows that a mapping, f, with the property above can't exist.

$\square$

Exercises.

1. Let B^n be the open unit ball in $\mathbb{R}^n$ and S^{n-1} the unit $(n-1)$-sphere, Show that volume $(S^{n-1}) = n\,\mathrm{volume}\,(B^n)$. *Hint:* Apply

Stokes theorem to the $(n-1)$-form $\mu = \sum(-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ and note (§4.4, exercise 9) that μ is the Riemannian volume form of S^{n-1}.

2. Let $D \subseteq \mathbb{R}^n$ be a smooth domain with boundary Z. Show that there exists a neighborhood, U, of Z in $\mathbb{R}^n$ and a $\mathcal{C}^\infty$ defining function, $g : U \to \mathbb{R}$ for D with the properties

(I) $p \in U \cap D \Leftrightarrow g(p) < 0$.

and

(II) $dg_p \neq 0$ if $p \in Z$

Hint: Deduce from Theorem 4.4.4 that a local version of this result is true. Show that you can cover Z by a family

$$\mathbb{U} = \{U_\alpha \, , \, \alpha \in \mathcal{I}\}$$

of open subsets of $\mathbb{R}^n$ such that for each there exists a function, $g_\alpha : U_\alpha \to \mathbb{R}$, with properties (I) and (II). Now let ρ_i, $i = 1, 2, \ldots,$ be a partition of unity and let $g = \sum \rho_i g_{\alpha_i}$ where $\operatorname{supp} \rho_i \subseteq U_{\alpha_i}$.

3. In exercise 2 suppose Z is compact. Show that there exists a global defining function, $f : \mathbb{R}^n \to \mathbb{R}$ for D with properties (I) and (II). *Hint:* Let $\rho \in \mathcal{C}_0^\infty(U)$, $0 \leq \rho \leq 1$, be a function which is one on a neighborhood of Z, and replace g by the function

$$f = \begin{cases} \rho g + (1-\rho) \text{ on } \operatorname{ext} D \\ g \text{ on } Z \\ \rho - g(1-\rho) \text{ on } \operatorname{int} D \, . \end{cases}$$

4. Show that the form $L_v f \nu_Z$ in formula (4.6.15) doesn't depend on what choice we make of a defining function, f, for D. *Hints:*

(a) Show that if g is another defining function then, at $p \in Z$, $df_p = \lambda \, dg_p$, where λ is a positive constant.

(b) Show that if one replaces df_p by $(dg)_p$ the first term in the product, $(L_v f)(p)(\nu_Z)_p$ changes by a factor, λ, and the second term by a factor $1/\lambda$.

5. Show that the form, ν_Z, is *intrinsically defined* in the sense that if ν is *any* $(n-1)$-form satisfying (4.6.13), ν_Z is equal to $\iota_Z^* \nu$. *Hint:* §4.5, exercise 7.

6. Show that the form, σ_Z, in the formula (4.6.16) is the Riemannian volume form on Z.

7. Show that the $(n-1)$-form

$$\mu = (x_1^2 + \cdots + x_n^2)^{-n} \sum (-1)^{r-1} x_r \, dx_1 \wedge \cdots \wedge \widehat{dx_r} \cdots dx_n$$

is closed and prove directly that Stokes theorem holds for the annulus $a < x_1^2 + \cdots + x_n^2 < b$ by showing that the integral of μ over the sphere, $x_1^2 + \cdots + x_n^2 = a$, is equal to the integral over the sphere, $x_1^2 + \cdots + x_n^2 = b$.

8. Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be an everywhere positive $\mathcal{C}^\infty$ function and let U be a bounded open subset of $\mathbb{R}^{n-1}$. Verify directly that Stokes theorem is true if D is the domain

$$0 < x_n < f(x_1, \ldots, x_{n-1}), \quad (x_1, \ldots, x_{n-1}) \in U$$

and μ an $(n-1)$-form of the form

$$\varphi(x_1, \ldots, x_n) \, dx_1 \wedge \cdots \wedge dx_{n-1}$$

where φ is in $\mathcal{C}_0^\infty(\mathbb{R}^n)$.

9. Let X be an oriented n-dimensional manifold and v a vector field on X which is complete. Verify that for $\omega \in \Omega_c^n(X)$

$$\int_X L_v \omega = 0,$$

(a) directly by using the divergence theorem,

(b) indirectly by showing that

$$\int_X f_t^* \omega = \int_X \omega$$

where $f_t : X \to X$, $-\infty < t < \infty$, is the one-parameter group of diffeomorphisms of X generated by v.

10. Let X be an oriented n-dimensional manifold and $D \subseteq X$ a smooth domain whose closure is compact. Show that if Z is the boundary of D and $g : Z \to Z$ a diffeomorphism, g *can't* be extended to a smooth map, $f : D \to Z$.

4.7 Degree theory on manifolds

In this section we'll show how to generalize to manifolds the results about the "degree" of a proper mapping that we discussed in Chapter 3. We'll begin by proving the manifold analogue of Theorem 3.3.1.

Theorem 4.7.1. *Let X be an oriented connected n-dimensional manifold and $\omega \in \Omega_c^n(X)$ a compactly supported n-form. Then the following are equivalent*

(a) $\int_X \omega = 0$.

(b) $\omega = d\mu$ *for some* $\mu \in \Omega_c^{n-1}(X)$.

We've already verified the assertion (b) $\Rightarrow$ (a) (see Theorem 4.5.5), so what is left to prove is the converse assertion. The proof of this is more or less identical with the proof of the "(a) $\Rightarrow$ (b)" part of Theorem 3.2.1:

Step 1. Let U be a connected parametrizable open subset of X. If $\omega \in \Omega_c^n(U)$ has property (a), then $\omega = d\mu$ for some $\mu \in \Omega_c^{n-1}(U)$.

Proof. Let $\varphi : U_0 \to U$ be an oriented parametrization of U. Then

$$\int_{U_0} \varphi^*\omega = \int_X \omega = 0$$

and since U_0 is a connected open subset of $\mathbb{R}^n$, $\varphi^*\omega = d\nu$ for some $\nu \in \Omega_c^{n-1}(U_0)$ by Theorem 3.3.1. Let $\mu = (\varphi^{-1})^*\nu$. Then $d\mu = (\varphi^{-1})^*\, d\nu = \omega$.

□

Step 2. Fix a base point, $p_0 \in X$ and let p be any point of X. Then there exists a collection of connected parametrizable open sets, W_i, $i = 1, \dots, N$ with $p_0 \in W_1$ and $p \in W_N$ such that, for $1 \le i \le N-1$, the intersection of W_i and W_{i+1} is non-empty.

Proof. The set of points, $p \in X$, for which this assertion is true is open and the set for which it is not true is open. Moreover, this assertion is true for $p = p_0$.

□

Step 3. We deduce Theorem 4.7.1 from a slightly stronger result. Introduce an equivalence relation on $\Omega_c^n(X)$ by declaring that two n-forms, ω_1 and ω_2, in $\Omega_c^n(X)$ are *equivalent* if $\omega_1 - \omega_2 \in d\Omega_x^{n-1}(X)$. Denote this equivalence relation by a wiggly arrow: $\omega_1 \sim \omega_2$. We will prove

Theorem 4.7.2. *For ω_1 and $\omega_2 \in \Omega_c^n(X)$ the following are equivalent*

(a) $\int_X \omega_1 = \int_X \omega_2$

(b) $\omega_1 \sim \omega_2$.

Applying this result to a form, $\omega \in \Omega_c^n(X)$, whose integral is zero, we conclude that $\omega \sim 0$, which means that $\omega = d\mu$ for some $\mu \in \Omega_c^{n-1}(X)$. Hence Theorem 4.7.2 implies Theorem 4.7.1. Conversely, if $\int_X \omega_1 = \int_X \omega_2$. Then $\int_X(\omega_1 - \omega_2) = 0$, so $\omega_1 - \omega_2 = d\mu$ for some $\mu \in \Omega_c^n(X)$. Hence Theorem 4.7.1 implies Theorem 4.7.2.

Step 4. By a partition of unity argument it suffices to prove Theorem 4.7.2 for $\omega_1 \in \Omega_c^n(U_1)$ and $\omega_2 \in \Omega_c^n(U_2)$ where U_1 and U_2 are connected parametrizable open sets. Moreover, if the integrals of ω_1 and ω_2 are zero then $\omega_i = d\mu_i$ for some $\mu_i \in \Omega_c^n(U_i)$ by step 1, so in this case, the theorem is true. Suppose on the other hand that

$$\int_X \omega_1 = \int_X \omega_2 = c \neq 0\,.$$

Then dividing by c, we can assume that the integrals of ω_1 and ω_2 are both equal to 1.

Step 5. Let W_i, $i = 1, \dots, N$ be, as in step 2, a sequence of connected parametrizable open sets with the property that the intersections, $W_1 \cap U_1$, $W_N \cap U_2$ and $W_i \cap W_{i+1}$, $i = 1, \dots, N-1$, are all non-empty. Select n-forms, $\alpha_0 \in \Omega_c^n(U_1 \cap W_1)$, $\alpha_N \in \Omega_c^n(W_N \cap U_2)$ and $\alpha_i \in \Omega_c^n(W_i \cap W_{i+1})$, $i = 1, \dots, N-1$ such that the integral of each α_i over X is equal to 1. By step 1 Theorem 4.7.1 is true for U_1, U_2 and the W_i's, hence Theorem 4.7.2 is true for U_1, U_2 and the W_i's, so

$$\omega_1 \sim \alpha_0 \sim \alpha_1 \sim \cdots \sim \alpha_N \sim \omega_2$$

and thus $\omega_1 \sim \omega_2$. $\square$

Just as in (3.4.1) we get as a corollary of the theorem above the following "definition–theorem" of the degree of a differentiable mapping:

Theorem 4.7.3. *Let X and Y be compact oriented n-dimensional manifolds and let Y be connected. Given a proper $\mathcal{C}^\infty$ mapping, $f : X \to Y$, there exists a topological invariant,* $\deg(f)$, *with the defining property:*

$$\int_X f^*\omega = \deg f \int_Y \omega \,. \tag{4.7.1}$$

Proof. As in the proof of Theorem 3.4.1 pick an n-form, $\omega_0 \in \Omega_c^n(Y)$, whose integral over Y is one and define the degree of f to be the integral over X of $f^*\omega_0$, i.e., set

$$\deg(f) = \int_X f^*\omega_0 \,. \tag{4.7.2}$$

Now let ω be any n-form in $\Omega_c^n(Y)$ and let

$$\int_Y \omega = c \,. \tag{4.7.3}$$

Then the integral of $\omega - c\omega_0$ over Y is zero so there exists an $(n-1)$-form, μ, in $\Omega_c^{n-1}(Y)$ for which $\omega - c\omega_0 = d\mu$. Hence $f^*\omega = cf^*\omega_0 + df^*\mu$, so

$$\int_X f^*\omega = c\int_X f^*\omega_0 = \deg(f)\int_Y \omega$$

by (4.7.2) and (4.7.3).

$\square$

It's clear from the formula (4.7.1) that the degree of f is independent of the choice of ω_0. (Just apply this formula to any $\omega \in \Omega_c^n(Y)$ having integral over Y equal to one.) It's also clear from (4.7.1) that "degree" behaves well with respect to composition of mappings:

Theorem 4.7.4. *Let Z be an oriented, connected n-dimensional manifold and $g : Y \to Z$ a proper $\mathcal{C}^\infty$ map. Then*

$$\deg g \circ f = (\deg f)(\deg g) \,. \tag{4.7.4}$$

Proof. Let ω be an element of $\Omega^n_c(Z)$ whose integral over Z is one. Then

$$\begin{aligned}\deg g\circ f = \int_X (g\circ f)^*\omega &= \int_X f^*\circ g^*\omega = \deg f \int_Y g^*\omega \\ &= (\deg f)(\deg g)\,.\end{aligned}$$

□

We will next show how to compute the degree of f by generalizing to manifolds the formula for $\deg(f)$ that we derived in §3.6.

Definition 4.7.5. *A point, $p \in X$ is a critical point of f if the map*

$$df_p : T_pX \to T_{f(p)}Y \tag{4.7.5}$$

is not bijective.

We'll denote by C_f the set of all critical points of f, and we'll call a point $q \in Y$ a *critical* value of f if it is in the image, $f(C_f)$, of C_f and a *regular* value if it's not. (Thus the set of regular values is the set, $Y - f(C_f)$.) If q is a regular value, then as we observed in §3.6, the map (4.7.5) is bijective for every $p \in f^{-1}(q)$ and hence by Theorem 4.2.5, f maps a neighborhood U_p of p diffeomorphically onto a neighborhood, V_p, of q. In particular, $U_p \cap f^{-1}(q) = p$. Since f is proper the set $f^{-1}(q)$ is compact, and since the sets, U_p, are a covering of $f^{-1}(q)$, this covering must be a finite covering. In particular the set $f^{-1}(q)$ itself has to be a finite set. As in §2.6 we can shrink the U_p's so as to insure that they have the following properties:

(i) Each U_p is a parametrizable open set.

(ii) $U_p \cap U_{p'}$ is empty for $p \neq p'$.

(iii) $f(U_p) = f(U_{p'}) = V$ for all p and p'.

(iv) V is a parametrizable open set.

(v) $f^{-1}(V) = \bigcup U_p$, $p \in f^{-1}(q)$.

To exploit these properties let ω be an n-form in $\Omega^n_c(V)$ with integral equal to 1. Then by (v):

$$\deg(f) = \int_X f^*\omega = \sum_p \int_{U_p} f^*\omega\,.$$

But $f : U_p \to V$ is a diffeomorphism, hence by (4.5.14) and (4.5.15)

$$\int_{U_p} f^*\omega = \int_V \omega$$

if $f : U_p \to V$ is orientation preserving and

$$\int_{U_p} f^*\omega = -\int_V \omega$$

if $f : U_p \to V$ is orientation reversing. Thus we've proved

Theorem 4.7.6. *The degree of f is equal to the sum*

$$\sum_{p \in f^{-1}(q)} \sigma_p \tag{4.7.6}$$

where $\sigma_p = +1$ if the map (4.7.5) is orientation preserving and $\sigma_p = -1$ if it is orientation reversing.

We will next show that Sard's Theorem is true for maps between manifolds and hence that there exist lots of regular values. We first observe that if U is a parametrizable open subset of X and V a parametrizable open neighborhood of $f(U)$ in Y, then Sard's Theorem is true for the map, $f : U \to V$ since, up to diffeomorphism, U and V are just open subsets of $\mathbb{R}^n$. Now let q be any point in Y, let B be a compact neighborhood of q, and let V be a parametrizable open set containing B. Then if $A = f^{-1}(B)$ it follows from Theorem 3.4.2 that A can be covered by a finite collection of parametrizable open sets, $U_1, \ldots, U_N$ such that $f(U_i) \subseteq V$. Hence since Sard's Theorem is true for each of the maps $f : U_i \to V$ and $f^{-1}(B)$ is contained in the union of the U_i's we conclude that *the set of regular values of f intersects the interior of B in an open dense set.* Thus, since q is an arbitrary point of Y, we've proved

Theorem 4.7.7. *If X and Y are n-dimensional manifolds and $f : X \to Y$ is a proper $\mathcal{C}^\infty$ map the set of regular values of f is an open dense subset of Y.*

Since there exist lots of regular values the formula (4.7.6) gives us an effective way of computing the degree of f. We'll next justify our assertion that $\deg(f)$ is a topological invariant of f. To do so, let's generalize to manifolds the Definition 2.5.1, of a homotopy between $\mathcal{C}^\infty$ maps.

Definition 4.7.8. *Let X and Y be manifolds and $f_i : X \to Y$, $i = 0, 1$, a $\mathcal{C}^\infty$ map. A $\mathcal{C}^\infty$ map*

$$F : X \times [0,1] \to Y \tag{4.7.7}$$

is a homotopy between f_0 and f_1 if, for all $x \in X$, $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Moreover, if f_0 and f_1 are proper maps, the homotopy, F, is a proper *homotopy if it is proper as a $\mathcal{C}^\infty$ map, i.e., for every compact set, C, of Y, $F^{-1}(C)$ is compact.*

Let's now prove the manifold analogue of Theorem 3.6.8.

Theorem 4.7.9. *Let X and Y be oriented n-dimensional manifolds and let Y be connected. Then if $f_i : X \to Y$, $i = 0, 1,$, is a proper map and the map (4.7.4) is a property homotopy, the degrees of these maps are the same.*

Proof. Let ω be an n-form in $\Omega_c^n(Y)$ whose integral over Y is equal to 1, and let C be the support of ω. Then if F is a proper homotopy between f_0 and f_1, the set, $F^{-1}(C)$, is compact and its projection on X

$$\{x \in X \,;\, (x,t) \in F^{-1}(C) \text{ for some } t \in [0,1]\} \tag{4.7.8}$$

is compact. Let

$$f_t : X \to Y$$

be the map: $f_t(x) = F(x,t)$. By our assumptions on F, f_t is a proper $\mathcal{C}^\infty$ map. Moreover, for all t the n-form, $f_t^*\omega$ is a $\mathcal{C}^\infty$ function of t and is supported on the fixed compact set (4.7.8). Hence it's clear from the Definition 4.5.8 that the integral

$$\int_X f_t^*\omega$$

is a $\mathcal{C}^\infty$ function of t. On the other hand this integral is by definition the degree of f_t and hence by Theorem 4.7.3 is an integer, so it doesn't depend on t. In particular, $\deg(f_0) = \deg(f_1)$. $\square$

Exercises.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the map, $x \to x^n$. Show that $\deg(f) = 0$ if n is even and 1 if n is odd.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the polynomial function,

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n\,,$$

where the a_i's are in $\mathbb{R}$. Show that if n is even, $\deg(f) = 0$ and if n is odd, $\deg(f) = 1$.

3. Let S^1 be the unit circle

$$\{e^{i\theta}\,, \quad 0 \leq \theta < 2\pi\}$$

in the complex plane and let $f : S^1 \to S^1$ be the map, $e^{i\theta} \to e^{iN\theta}$, N being a positive integer. What's the degree of f?

4. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and $\sigma : S^{n-1} \to S^{n-1}$ the antipodal map, $x \to -x$. What's the degree of σ?

5. Let A be an element of the group, $O(n)$ of orthogonal $n \times n$ matrices and let

$$f_A : S^{n-1} \to S^{n-1}$$

be the map, $x \to Ax$. What's the degree of f_A?

6. A manifold, Y, is *contractable* if for some point, $p_0 \in Y$, the identity map of Y onto itself is homotopic to the constant map, $f_{p_0} : Y \to Y$, $f_{p_0}(y) = p_0$. Show that if Y is an oriented contractable n-dimensional manifold and X an oriented connected n-dimensional manifold then for every proper mapping $f : X \to Y$ $\deg(f) = 0$. In particular show that if n is greater than zero and Y is compact then Y can't be contractable. *Hint:* Let f be the identity map of Y onto itself.

7. Let X and Y be oriented connected n-dimensional manifolds and $f : X \to Y$ a proper $\mathcal{C}^\infty$ map. Show that if $\deg(f) \neq 0$ f is surjective.

8. Using Sard's Theorem prove that if X and Y are manifolds of dimension k and ℓ, with $k < \ell$ and $f : X \to Y$ is a proper $\mathcal{C}^\infty$ map, then the complement of the image of X in Y is open and dense. *Hint:* Let $r = \ell - k$ and apply Sard's Theorem to the map

$$g : X \times S^r \to Y\,, \quad g(x,a) = f(x)\,.$$

9. Prove that the sphere, S^2, and the torus, $S^1 \times S^2$, are not diffeomorphic.

4.8 Applications of degree theory

The purpose of this section will be to describe a few typical applications of degree theory to problems in analysis, geometry and topology. The first of these applications will be yet another variant of the Brouwer fixed point theorem.

Application 1. Let X be an oriented $(n+1)$-dimensional manifold, $D \subseteq X$ a smooth domain and Z the boundary of D. Assume that the closure, $\bar{D} = Z \cup D$, of D is compact (and in particular that X is compact).

Theorem 4.8.1. *Let Y be an oriented connected n-dimensional manifold and $f : Z \to Y$ a $\mathcal{C}^\infty$ map. Suppose there exists a $\mathcal{C}^\infty$ map, $F : \bar{D} \to Y$ whose restriction to Z is f. Then the degree of f is zero.*

Proof. Let μ be an element of $\Omega_c^n(Y)$. Then $d\mu = 0$, so $dF^*\mu = F^*\,d\mu = 0$. On the other hand if $\iota : Z \to X$ is the inclusion map,

$$\int_D dF^*\mu = \int_Z \iota^* F^*\mu = \int_Z f^*\mu = \deg(f) \int_Y \mu$$

by Stokes theorem since $F \circ \iota = f$. Hence $\deg(f)$ has to be zero.

□

Application 2. (a non-linear eigenvalue problem)

This application is a non-linear generalization of a standard theorem in linear algebra. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. If n is even, A may not have *real* eigenvalues. (For instance for the map

$$A : \mathbb{R}^2 \to \mathbb{R}^2\,, \quad (x, y) \to (-y, x)$$

the eigenvalues of A are $\pm\sqrt{-1}$.) However, if n is odd it is a standard linear algebra fact that there exists a vector, $\mathrm{v} \in \mathbb{R}^n - \{0\}$, and a $\lambda \in \mathbb{R}$ such that $A\mathrm{v} = \lambda\mathrm{v}$. Moreover replacing v by $\frac{\mathrm{v}}{|\mathrm{v}|}$ one can assume that $|\mathrm{v}| = 1$. This result turns out to be a special case of a much more general result. Let S^{n-1} be the unit $(n-1)$-sphere in $\mathbb{R}^n$ and let $f : S^{n-1} \to \mathbb{R}^n$ be a $\mathcal{C}^\infty$ map.

Theorem 4.8.2. *There exists a vector, $\mathrm{v} \in S^{n-1}$ and a number $\lambda \in \mathbb{R}$ such that $f(\mathrm{v}) = \lambda\mathrm{v}$.*

Proof. The proof will be by contradiction. If the theorem isn't true the vectors, v and $f(\mathrm{v})$, are linearly independent and hence the vector

$$g(\mathrm{v}) = f(\mathrm{v}) - (f(\mathrm{v}) \cdot \mathrm{v})\mathrm{v} \tag{4.8.1}$$

is non-zero. Let

$$h(\mathrm{v}) = \frac{g(\mathrm{v})}{|g(\mathrm{v})|}\,. \tag{4.8.2}$$

By (4.8.1)–(4.8.2), $|\mathrm{v}| = |h(\mathrm{v})| = 1$ and $\mathrm{v} \cdot h(\mathrm{v}) = 0$, i.e., v and $h(\mathrm{v})$ are both unit vectors and are perpendicular to each other. Let

$$\gamma_t : S^{n-1} \to S^{n-1}\,, \quad 0 \le t \le 1 \tag{4.8.3}$$

be the map

$$\gamma_t(\mathrm{v}) = (\cos \pi t)\mathrm{v} + (\sin \pi t)h(\mathrm{v})\,. \tag{4.8.4}$$

For $t = 0$ this map is the identity map and for $t = 1$, it is the antipodal map, $\sigma(\mathrm{v}) = \mathrm{v}$, hence (4.8.3) asserts that the identity map and the antipodal map are homotopic and therefore that the degree of the antipodal map is one. On the other hand the antipodal map is the restriction to S^{n-1} of the map, $(x_1, \ldots, x_n) \to (-x_1, \ldots, -x_n)$ and the volume form, ω, on S^{n-1} is the restriction to S^{n-1} of the $(n-1)$-form

$$\sum (-1)^{i-1} x_i \, dx_i \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\,. \tag{4.8.5}$$

If we replace x_i by $-x_i$ in (4.8.5) the sign of this form changes by $(-1)^n$ hence $\sigma^*\omega = (-1)^n\omega$. Thus if n is odd, σ is an orientation reversing diffeomorphism of S^{n-1} onto S^{n-1}, so its degree is -1, and this contradicts what we just deduced from the existence of the homotopy (4.8.4).

□

From this argument we can deduce another interesting fact about the sphere, S^{n-1}, when $n-1$ is even. For $\mathrm{v} \in S^{n-1}$ the tangent space to S^{n-1} at v is just the space,

$$\{(\mathrm{v}, w)\,; \quad w \in \mathbb{R}^n\,,\ \mathrm{v} \cdot w = 0\}\,,$$

so a vector field on S^{n-1} can be viewed as a function, $g : S^{n-1} \to \mathbb{R}^n$ with the property

$$g(\mathrm{v}) \cdot \mathrm{v} = 0 \tag{4.8.6}$$

for all $\mathrm{v} \in S^{n-1}$. If this function is non-zero at all points, then, letting h be the function, (4.8.2), and arguing as above, we're led to a contradiction. Hence we conclude:

Theorem 4.8.3. *If $n-1$ is even and v is a vector field on the sphere, S^{n-1}, then there exists a point $p \in S^{n-1}$ at which $v(p) = 0$.*

Note that if $n-1$ is odd this statement is *not* true. The vector field

$$x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + \cdots + x_{2n-1} \frac{\partial}{\partial x_{2n}} - x_{2n} \frac{\partial}{\partial x_{2n-1}} \tag{4.8.7}$$

is a counterexample. It is nowhere vanishing and at $p \in S^{n-1}$ is tangent to S^{n-1}.

Application 3. (The Jordan–Brouwer separation theorem.) Let X be a compact oriented $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. In this subsection of §4.8 we'll outline a proof of the following theorem (leaving the details as a string of exercises).

Theorem 4.8.4. *If X is connected, the complement of $X : \mathbb{R}^n - X$ has exactly two connected components.*

This theorem is known as the Jordan-Brouwer separation theorem (and in two dimensions as the Jordan curve theorem). For simple, easy to visualize, submanifolds of $\mathbb{R}^n$ like the $(n-1)$-sphere this result is obvious, and for this reason it's easy to be misled into thinking of it as being a trivial (and not very interesting) result. However, for submanifolds of $\mathbb{R}^n$ like the curve in $\mathbb{R}^2$ depicted in the figure on page 214 it's much less obvious. (In ten seconds or less, is the point, p, in this figure inside this curve or outside?)

To determine whether a point, $p \in \mathbb{R}^n - X$ is inside X or outside X, one needs a topological invariant to detect the difference, and such an invariant is provided by the "winding number".

Definition 4.8.5. *For $p \in \mathbb{R}^n - X$ let*

$$\gamma_p : X \to S^{n-1} \tag{4.8.8}$$

be the map

$$\gamma_p(x) = \frac{x-p}{|x-p|} \,. \tag{4.8.9}$$

The *winding number* of X about p is the degree of this map.

Denoting this number by $W(X,p)$ we will show below that $W(X,p) = 0$ if p is outside X and $W(X,p) = \pm 1$ (depending on the orientation of X) if p is inside X, and hence that the winding number tells us which of the two components of $\mathbb{R}^n - X$, p is contained in.

Exercise 1.

Let U be a connected component of $\mathbb{R}^n - X$. Show that if p_0 and p_1 are in U, $W(X,p_0) = W(X,p_1)$.

Hints:

(a) First suppose that the line segment,

$$p_t = (1-t)p_0 + tp_1 \,, \quad 0 \le t \le 1$$

lies in U. Conclude from the homotopy invariance of degree that $W(X,p_0) = W(X,p_t) = W(X,p_1)$.

(b) Show that there exists a sequence of points

$$q_i \,, \quad i = 1, \dots, N \,, \quad q_i \in U \,,$$

with $q_1 = p_0$ and $q_N = p_1$, such that the line segment joining q_i to q_{i+1} is in U.

Exercise 2.

Show that $\mathbb{R}^n - X$ has at most two connected components.

Hints:

(a) Show that if q is in X there exists a small ϵ-ball, $B_\epsilon(q)$, centered at q such that $B_\epsilon(q) - X$ has two components. (See Theorem 4.2.7.

(b) Show that if p is in $\mathbb{R}^n - X$, there exists a sequence

$$q_i \,, \, i = 1, \dots, N \,, \quad q_i \in \mathbb{R}^n - X \,,$$

such that $q_1 = p$, $q_N \in B_\epsilon(q)$ and the line segments joining q_i to q_{i+1} are in $\mathbb{R}^n - X$.

Exercise 3.

For $\mathrm{v} \in S^{n-1}$, show that $x \in X$ is in $\gamma_p^{-1}(\mathrm{v})$ if and only if x lies on the ray

$$p + t\mathrm{v}\,, \quad 0 < t < \infty\,. \tag{4.8.10}$$

Exercise 4.

Let $x \in X$ be a point on this ray. Show that

$$(d\gamma_p)_x : T_pX \to T_{\mathrm{v}}S^{n-1} \tag{4.8.11}$$

is bijective if and only if $v \notin T_pX$, i.e., if and only if the ray (4.8.10) is *not* tangent to X at x. *Hint:* $\gamma_p : X \to S^{n-1}$ is the composition of the maps

$$\tau_p : X \to \mathbb{R}^n - \{0\}\,, \quad x \to x - p\,, \tag{4.8.12}$$

and

$$\pi : \mathbb{R}^n - \{0\} \to S^{n-1}\,, \quad y \to \frac{y}{|y|}\,. \tag{4.8.13}$$

Show that if $\pi(y) = \mathrm{v}$, then the kernel of $(d\pi)_g$, is the one-dimensional subspace of $\mathbb{R}^n$ spanned by v. Conclude that if $y = x - p$ and $\mathrm{v} = y/|y|$ the composite map

$$(d\gamma_p)_x = (d\pi)_y \circ (d\tau_p)_x$$

is bijective if and only if $\mathrm{v} \notin T_xX$.

Exercise 5.

From exercises 3 and 4 conclude that v is a regular value of γ_p if and only if the ray (4.8.10) intersects X in a finite number of points and at each point of intersection is *not* tangent to X at that point.

Exercise 6.

In exercise 5 show that the map (4.8.11) is orientation preserving if the orientations of T_xX and v are compatible with the standard orientation of $T_p\mathbb{R}^n$. (See §1.9, exercise 5.)

Exercise 7.

Conclude that $\deg(\gamma_p)$ counts (with orientations) the number of points where the ray (4.8.10) intersects X.

Exercise 8.

Let $p_1 \in \mathbb{R}^n - X$ be a point on the ray (4.8.10). Show that if $\mathrm{v} \in S^{n-1}$ is a regular value of γ_p, it is a regular value of γ_{p_1} and show that the number

$$\deg(\gamma_p) - \deg(\gamma_{p_1}) = W(X,p) - W(X,p_1)$$

counts (with orientations) the number of points on the ray lying between p and p_1. *Hint:* Exercises 5 and 7.

Exercise 8.

Let $x \in X$ be a point on the ray (4.8.10). Suppose $x = p + t\mathrm{v}$. Show that if ϵ is a small positive number and

$$p_\pm = p + (t \pm \epsilon)\mathrm{v}$$

then

$$W(X,p_+) = W(X,p_-) \pm 1\,,$$

and from exercise 1 conclude that p_+ and p_- lie in different components of $\mathbb{R}^n - X$. In particular conclude that $\mathbb{R}^n - X$ has exactly two components.

Exercise 9.

Finally show that if p is very large the difference

$$\gamma_p(x) - \frac{p}{|p|}\,, \quad x \in X\,,$$

is very small, i.e., γ_p is *not* surjective and hence the degree of γ_p is zero. Conclude that for $p \in \mathbb{R}^n - X$, p is in the unbounded component of $\mathbb{R}^n - X$ if $W(X,p) = 0$ and in the bounded component if $W(X,p) = \pm 1$ (the "$\pm$" depending on the orientation of X).

Notice, by the way, that the proof of Jordan–Brouwer sketched above gives us an effective way of deciding whether the point, p, in Figure 4.9.2, is inside X or outside X. Draw a non-tangential ray from p. If it intersects X in an even number of points, p is outside X and if it intersects X is an odd number of points p inside.

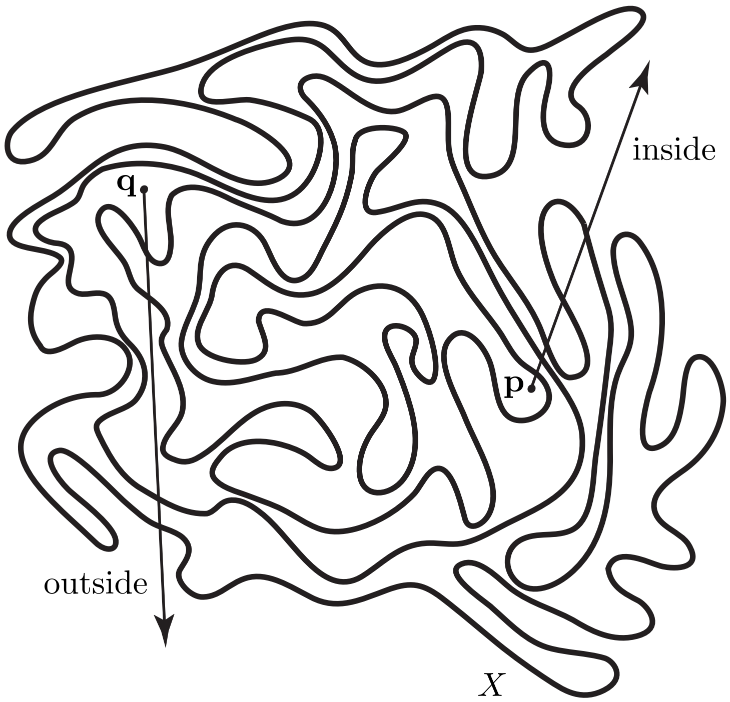

Figure 4.8.2.

Application 3. (The Gauss–Bonnet theorem.) Let $X \subseteq \mathbb{R}^n$ be a compact, connected, oriented $(n-1)$-dimensional submanifold. By the Jordan–Brouwer theorem X is the boundary of a bounded smooth domain, so for each $x \in X$ there exists a unique outward pointing unit normal vector, n_x. The Gauss map

$$\gamma : X \to S^{n-1}$$

is the map, $x \to n_x$. Let σ be the Riemannian volume form of S^{n-1}, or, in other words, the restriction to S^{n-1} of the form,

$$\sum (-1)^{i-1} x_i \, dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge \, dx_n \,,$$

and let σ_X be the Riemannian volume form of X. Then for each $p \in X$

$$(\gamma^* \sigma)_p = K(p)(\sigma_X)_q \tag{4.8.14}$$

where $K(p)$ is the *scalar curvature* of X at p. This number measures the extent to which "X is curved" at p. For instance, if X_a is the circle, $|x| = a$ in $\mathbb{R}^2$, the Gauss map is the map, $p \to p/a$, so for all p, $K_a(p) = 1/a$, reflecting the fact that, for $a < b$, X_a is more curved than X_b.

The scalar curvature can also be negative. For instance for surfaces, X in $\mathbb{R}^3$, $K(p)$ is positive at p if X is *convex* at p and negative if X is convex–concave at p. (See Figure 4.8.3 below. The surface in part (a) is convex at p, and the surface in part (b) is convex–concave.)

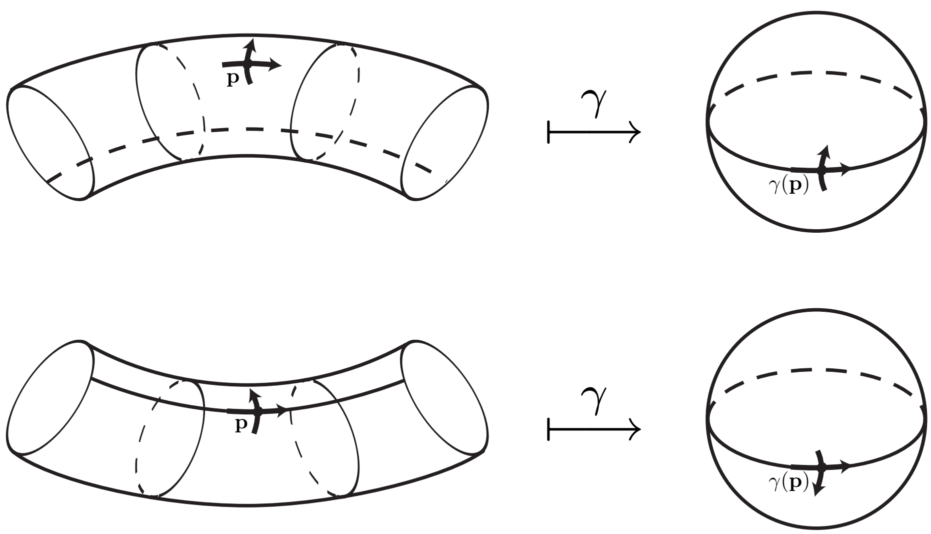

Figure 4.8.3.

Let $\mathrm{vol}\,(S^{n-1})$ be the Riemannian volume of the $(n-1)$-sphere, i.e., let

$$\mathrm{vol}\,(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

(where Γ is the gamma function). Then by (4.8.14) the quotient

$$\frac{\int K\sigma_X}{\mathrm{vol}\,(S^{n-1})} \tag{4.8.15}$$

is the degree of the Gauss map, and hence is a topological invariant of the surface of X. For $n = 3$ the *Gauss–Bonnet theorem* asserts that this topological invariant is just $1 - g$ where g is the *genus* of X or, in other words, the "number of holes". Figure 4.8.4 gives a pictorial proof of this result. (Notice that at the points, $p_1, \dots, p_g$ the surface,X is convex–concave so the scalar curvature at these points is negative, i.e., the Gauss map is orientation reversing. On the other hand, at the point, p_0, the surface is convex, so the Gauss map at this point us orientation preserving.)

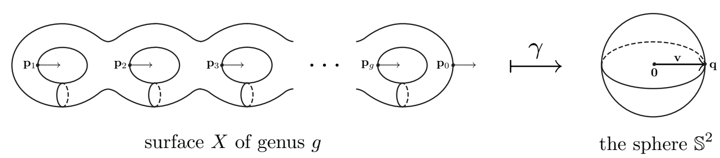

Figure 4.8.4.

4.9 Indices of Vector Fields

Let D be a bounded smooth domain in $\mathbb{R}^n$ and $v = \sum v_i \partial/\partial x_i$ a $\mathcal{C}^\infty$ vector field defined on the closure, $\bar{D}$, of D. We will define below a topological invariant which, for "generic" v's , counts (with appropriate $\pm$-signs) the number of zeroes of v. To avoid complications we'll assume v has *no* zeroes on the boundary of D.

Definition 4.9.1. *Let X be the boundary of D and let*

$$f_v : X \to S^{n-1} \tag{4.9.1}$$

be the mapping

$$p \to \frac{\mathrm{v}(p)}{|\mathrm{v}(p)|} ,$$

where $\mathrm{v}(p) = (v_1(p), \dots, v_n(p))$*. The* index *of* v with respect to D *is by definition the degree of this mapping.*

We'll denote this index by ind (v, D) and as a first step in investigating its properties we'll prove

Theorem 4.9.2. *Let D_1 be a smooth domain in $\mathbb{R}^n$ whose closure is contained in D. Then*

$$\operatorname{ind}(v, D) = \operatorname{ind}(v, D_1) \tag{4.9.2}$$

provided that v has no zeroes in the set $D - D_1$.

Proof. Let $W = D - \bar{D}_1$. Then the map (4.9.1) extends to a $\mathcal{C}^\infty$ map

$$F : W \to S^{n-1}, \quad p \to \frac{\mathrm{v}(p)}{|\mathrm{v}(p)|}. \tag{4.9.3}$$

Moreover,

$$Bd(W) = X \cup X_1^-$$

where X is the boundary of D with its natural boundary orientation and X_1^- is the boundary of D_1 with its boundary orientation reversed. Let ω be an element of $\Omega^{n-1}(S^{n-1})$ whose integral over S^{n-1} is 1. Then if $f = f_v$ and $f_1 = (f_1)_v$ are the restrictions of F to X and X_1 we get from Stokes theorem (and the fact that $d\omega = 0$) the identity

$$\begin{aligned} 0 &= \int_W F^* \, d\omega = \int_W dF^* \omega \\ &= \int_X f^* \omega - \int_{X_1} f_1^* \omega = \deg(f) - \deg(f_I). \end{aligned}$$

Hence

$$\mathrm{ind}(v, D) = \deg(f) = \deg(f_1) = \deg \mathrm{ind}(v, D).$$

□

Suppose now that v has a finite number of isolated zeroes, $p_1, \ldots, p_k$, in D. Let $B_\epsilon(p_i)$ be the open ball of radius ϵ with center at p_i. By making ϵ small enough we can assume that each of these balls is contained in D and that they're mutually disjoint. We will define the *local index*, $\mathrm{ind}(v, p_i)$ of v at p_i to be the index of v with respect to $B_\epsilon(p_i)$. By Theorem 4.9.2 these local indices are unchanged if we replace ϵ by a smaller ϵ, and by applying this theorem to the domain

$$D_1 = \bigcup_{i=1}^{k} B_{p_i}(\epsilon)$$

we get, as a corollary of the theorem, the formula

$$\mathrm{ind}(v, D) = \sum_{i=1}^{k} \mathrm{ind}(v, p_i) \tag{4.9.4}$$

which computes the global index of v with respect to D in terms of these local indices.

Let's now see how to compute these local indices. Let's first suppose that the point $p = p_i$ is at the origin of $\mathbb{R}^n$. Then near $p = 0$

$$v = v_{\text{lin}} + v'$$

where

$$v = v_{\text{lin}} = \sum a_{ij} x_i \frac{\partial}{\partial xy} \tag{4.9.5}$$

the a_{ij}'s being constants and

$$v_i = \sum f_{ij} \frac{\partial}{\partial x_j} \tag{4.9.6}$$

where the f_{ij}'s vanish to second order near zero, i.e., satisfy

$$|f_{i,j}(x)| \leq C|x|^2 \tag{4.9.7}$$

for some constant, $C > 0$.

Definition 4.9.3. *We'll say that the point, $p = 0$, is a non-degenerate zero of v is the matrix, $A = [a_{i,j}]$ is non-singular.*

This means in particular that

$$\sum_j |\sum a_{i,j} x_i| \geq C_1 |x| \tag{4.9.8}$$

form some constant, $C_1 > 0$. Thus by (4.9.7) and (4.9.8) the vector field

$$v_t = v_{\text{lin}} + tv' , \quad 0 \leq t \leq 1$$

has no zeroes in the ball, $B_\epsilon(p)$, $p = 0$, other than the point, $p = 0$, itself provided that ϵ is small enough. Therefore if we let X_ϵ be the boundary of $B_\epsilon(0)$ we get a homotopy

$$F : X_\epsilon \times [0,1] \to S^{n-1} , \quad (x,t) \to f v_t(x)$$

between the maps

$$f v_{\text{lin}} \quad : X_\epsilon \to S^{n-1}$$

and

$$f_v \quad : X_\epsilon \to S^{n-1}$$

and thus by Theorem 4.7.9, $\deg(f_v) = \deg(f v_{\text{lin}})$, and hence

$$\text{ind}(v, p) = \text{ind}(v_{\text{lin}\,,p})\,. \tag{4.9.9}$$

We've assumed in the above discussion that $p = 0$, but by introducing at p a translated coordinate system for which p is the origin, these comments are applicable to any zero, p of v. More explicitly if $c_1, \dots, c_n$ are the coordinates of p then as above

$$v = v_{\text{lin}} + v'$$

where

$$v_{\text{lin}} = \sum a_{ij}(x_i - c_i)\tfrac{\partial}{\partial x_j} \tag{4.9.10}$$

and v' vanishes to second order at p, and if p is a non-degenerate zero of q, i.e., if the matrix, $A = [a_{i,j}]$, is non-singular, then exactly the same argument as before shows that

$$\text{ind}(v, p) = \text{ind}(v_{\text{lin}}\,, p)\,.$$

We will next prove:

Theorem 4.9.4. *If p is a non-degenerate zero of v, the local index,* $\text{ind}(v, p)$*, is $+1$ or -1 depending on whether the determinant of the matrix, $A = [a_{i,j}]$ is positive or negative.*

Proof. As above we can. without loss of generality, assume $p = 0$, and by (4.9.9) we can assume $v = v_{\text{lin}}$. Let D be the domain

$$\sum_j \Big(\sum a_{i,j} x_i\Big)^2 < 1 \tag{4.9.11}$$

and let X be its boundary. Since the only zero of v_{lin} inside this domain is $p = 0$ we get from (4.9.4) and (4.9.7)

$$\text{ind}_p(v) = \text{ind}_p(v_{\text{lin}}) = \text{ind}(v_{\text{lin}}\,, D)\,.$$

Moreover, the map

$$f_{v_{\text{lin}}} : X \to S^{n-1}$$

is, in view of (4.9.11), just the linear map, v $\to A$v, restricted to X. In particular, since A is a diffeomorphism, this mapping is as well, sothe degree of this map is $+1$ or -1 depending on whether this map is orientation preserving or not. To decide which of these alternatives is true let $\omega = \sum (-1)^i x_i \, dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots dx_n$ be the Riemannian volume form on S^{n-1} then

$$\int_X f_{v_{\text{lin}}}^* \omega = \deg(f v_{\text{lim}}) \operatorname{vol}(S^{n-1}) . \tag{4.9.12}$$

Since v is the restriction of A to X this is equal, by Stokes theorem, to

$$\begin{aligned} \int_D A^* \, d\omega &= n \int_D A^* dx_1 \wedge \cdots \wedge dx_n \\ &= n \det(A) \int_D dx_1 \wedge \cdots \wedge dx_n \\ &= n \det(A) \operatorname{vol}(D) \end{aligned}$$

which gives us the formula

$$\deg(f v_{\text{lin}}, D) = \frac{n \det A \operatorname{vol}(D)}{\operatorname{vol}(S^{n-1})} \tag{4.9.13}$$

□

We'll briefly describe the various types of non-degenerate zeroes that can occur for vector fields in two dimensions. To simplify this description a bit we'll assume that the matrix

$$A = \begin{bmatrix} a_{11}\,, & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is diagonalizable and that its eigenvalues are not purely imaginary. Then the following five scenarios can occur:

1. The eigenvalues of A are real and positive. In this case the integral curves of v_{lin} in a neighborhood of p look like the curves in Figure 1

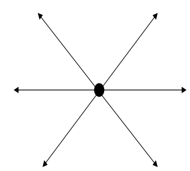

Figure 4.9.1.

and hence, is a small neighborhood of p, the integral sums of v itself look approximately like the curve in Figure 1.

2. The eigenvalues of A are real and negative. In this case the integral curves of v_{lin} look like the curves in Figure 1, but the arrows are pointing into p, rather than out of p, i.e.,

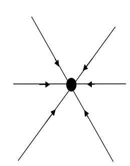

Figure 4.9.2.

3. The eigenvalues of A are real, but one is positive and one is negative. In this case the integral curves of v_{lin} look like the curves in Figure 3.

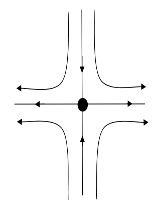

Figure 4.9.3.

4. The eigenvalues of A are complex and of the form, $a \pm \sqrt{-1}b$ with a positive. In this case the integral curves of v_{lin} are spirals going out of p as in Figure 4.

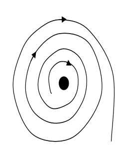

Figure 4.9.4.

5. The eigenvalues of A are complex and of the form, $a \pm \sqrt{-1}b$ with a negative. In this case the integral curves are as in Figure 4 but are spiraling into p rather than out of p.

Definition 4.9.5. *Zeroes of v of types 1 and 4 are called* sources*; those of types 2 and 5 are called* sinks *and those of type 3 are called* saddles.

Thus in particular the two-dimensional version of (4.9.4) tells us

Theorem 4.9.6. *If the vector field, v, on D has only non-degenerate zeroes then* $\mathrm{ind}(v, D)$ *is equal to the number of sources and sinks minus the number of saddles.*