# CHAPTER 4: FORMS ON MANIFOLDS

JUSTIN M. CURRY

## Contents

It is our agenda in this chapter to extend to manifolds the results of Chapters 2 and 3 and to reformulate and prove manifold versions of two of the fundamental theorems of integral calculus: Stokes' theorem and the divergence theorem. In this first section we aim to introduce the necessary background to understand the term "manifold". In doing so, we will be able to say much more sophisticated things than "a manifold is something which locally looks like $\mathbb{R}^n$". Other results in differential topology beyond integral calculus will also become much easier to handle once this first section is understood.

## 1. Manifolds

Before we can state the abstract definition of a manifold, we must first restate some background material from topology, most notably, the definition of a topological space.

### 1.1. **The Notion of a Topological Space.**

**Definition 1.1.** Let $X$ be a set and $\mathcal{J}$ a collection of subsets of $X$. If $\mathcal{J}$ satisfies the following properties, then we say that $\mathcal{J}$ defines a *topology* on $X$ or $(X, \mathcal{J})$ is a *topological space.* The properties are:

(a) $X$ and $\varnothing$ are in $\mathcal{J}$.
(b) For every (finite or infinite) sub-collection $\mathcal{U} = \{U_\alpha, \alpha \in I\}$ of $\mathcal{J}$, $\bigcup_{\alpha\in I} U_\alpha$ is in $\mathcal{J}$.
(c) For every finite sub-collection $\mathcal{U} = \{U_i, i = 1, \dots, n\}$ of $\mathcal{J}$, $\bigcap_{i=1}^n U_i$ is in $\mathcal{J}$.

**Definition 1.2.** We say that a set $U \in \mathcal{J}$ is an *open subset* of $X$.

Many of you are most likely familiar with the concept of an open set from an introductory analysis course, albeit defined in a different manner than above. The open sets defined in analysis provide a nice example of the *metric space topology.*

**Example 1.3.** Let $(X, d)$ be a non-empty metric space. Recall the *open ball* of radius $\epsilon$ around a point $x_0 \in X$ is defined as the set $B_\epsilon(x_0) = \{x \in X, d(x, x_0) < \epsilon\}$. We then say that $U \subseteq X$ is open, i.e. belongs to the metric space topology $\mathcal{J}_d$ if and only if for every $x_0 \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon(x_0) \subset U$.

For those of you who haven't had a course in topology and only a course in analysis, one can assume that "topological space"="metric space", that is, the metric space $(X, d)$ induces a topology $\mathcal{J}$ on the set $X$ to make a topological space $(X, \mathcal{J})$ according to the above example. However, topological spaces are more general than metric spaces as there are topologies that don't arise from any metric. It is important to remember that a topological space is a set with extra data specified. That extra data specifies what the open sets are. For more information on topological spaces, please review Chapter 2 of James Munkres' *Topology*.

1.2. **The Hausdorff Axiom.** One of the nice properties that metric space topologies satisfy is known as the *Hausdorff* axiom. (It is an exercise to show that all metrizable topologies are Hausdorff.)

**Definition 1.4.** $(X, \mathcal{J})$ is a *Hausdorff* topological space if for every $p_1, p_2 \in X$ $p_1 \neq p_2$ there exist open sets $U_i$ $i = 1, 2$ with $p_i \in U_i$ with $U_1 \cap U_2 = \varnothing$.

1.3. **Paracompactness.** Another nice property that metrizable topologies have is known as *paracompactness*. First we offer a preliminary definition.

**Definition 1.5.** A collection of compact sets $C_i \subseteq X$ is an *exhaustion* of $X$ if $C_i \subseteq \mathrm{Int}(C_{i+1})$ and $\bigcup C_i = X$.

**Definition 1.6.** A space $X$ is *paracompact* if there exists an exhaustion of $X$ by compact sets.

**Example 1.7.** The space $X = \mathbb{R}^n$ is paracompact. Simply build an exhaustion via $n$-balls of increasing integer radii, i.e. $C_i = B_i^n(0)$.

**Lemma 1.8.** *Closed subsets of paracompact sets are paracompact.*

*Proof.* If a space $X$ is paracompact then there exists an exhaustion $\{C_i\}$ of $X$ be compact sets. Suppose $S \subseteq X$ is a closed subset. Since closed subsets of compact sets are compact, we have that $S$ is exhausted by $\{C_i \cap S\}$, which are compact. $\square$

1.4. **Manifolds Defined.** Recall that a homeomorphism is a map between topological spaces that is continuous and has a continuous inverse. Since continuity is defined in terms of open sets, we can think of two homeomorphic topological spaces as topologically equivalent. As a convention, we will just say $X$ is a topological space instead of $(X, \mathcal{J})$. We are now in a position to define what a manifold is.

**Definition 1.9.** Let $X$ be a paracompact, Hausdorff space. $X$ is an *$n$-dimensional manifold* (or *$n$-manifold* for short) if for every $p \in X$ there exists an open neighborhood $U$ of $p$ in $X$, an open set $V$ in $\mathbb{R}^n$ and a homeomorphism $\varphi : U \to V$.

**Definition 1.10.** The triple $(U, V, \varphi)$ is called a *chart*.

The homeomorphism between $U \subseteq X$ and $V \subseteq \mathbb{R}^n$ is a rigorous way of saying that a manifold "locally looks like" $\mathbb{R}^n$.

**Definition 1.11.** A chart $(U, V, \varphi)$ is *centered* at $p$ if $\varphi(p) = 0 \in \mathbb{R}^n$.

Note that since we usually consider only one chart at a time, we can always center it via translation.

1.5. **Differentiable Manifolds.** The above definition is the definition of what is called a *topological manifold*. For the purposes of this book, we are interested almost exclusively with *differentiable manifolds*. Detailing this extra differentiability condition requires a little extra work. Just as we began with a set and provided extra data to define a topological space, we will have to start with a manifold and specify extra structure to define a differentiable manifold. This extra structure details how various charts interact with each other.

Let $X$ be a topological manifold and $(U_i, V_i, \varphi_i)$ $i = 1, 2$ be two charts. We use the following *overlap diagram*

$$\begin{tikzcd} & U_1 \cap U_2 \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ V_{1,2} \arrow[rr, "\varphi_{1,2}"] & & V_{2,1} \end{tikzcd}$$

to detail how these charts interact. Here $V_{1,2} = \varphi_1(U_1 \cap U_2)$ and $V_{2,1} = \varphi_2(U_1 \cap U_2)$. These sets are open subsets of $V_1$ and $V_2$ and $\varphi_{1,2} := \varphi_2 \circ \varphi_1^{-1}|_{V_{1,2}}$.

*Remark* 1.12. The map $\varphi_{1,2}$ is a homeomorphism of $V_{1,2}$ and $V_{2,1}$. It's inverse is $\varphi_{2,1} = \varphi_1 \circ \varphi_2^{-1}|_{V_{2,1}}$.

We now state one of the crucial conditions involved in defining a differentiable manifold.

**Definition 1.13.** The charts $(U_i, V_i, \varphi)$ $i = 1, 2$ are *compatible* if $\varphi_{1,2}$ is a diffeomorphism. In which case $\varphi_{2,1} = (\varphi_{1,2})^{-1}$ is also a diffeomorphism.

**Definition 1.14.** An *atlas* is a collection of charts

$$\mathcal{A} = \{(U_\alpha, V_\alpha, \varphi_\alpha), \alpha \in I\}$$

such that any two charts in the collection are compatible and $\bigcup_{\alpha \in I} U_\alpha = X$.

**Definition 1.15.** Let $\mathcal{A}$ be an atlas and $(U, V, \varphi)$ a chart. $(U, V, \varphi)$ is said to be *$\mathcal{A}$-compatible* if it is compatible with all charts in $\mathcal{A}$.

**Lemma 1.16.** *If $(U_i, V_i, \varphi_i)$ $i = 1, 2$ are $\mathcal{A}$-compatible, then they are compatible with each other.*

*Proof.* The proof is left as an exercise. Hint: Let $(U_i, V_i, \varphi_i)$ $i = 1, 2, 3$ be charts in $\mathcal{A}$. Look at overlap diagrams of the form:

$$\begin{tikzcd}
& U_1 \cap U_2 \cap U_3 \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\
V_{1,2,3} \arrow[rr, "\varphi_{1,2}"] & & V_{2,1,3}
\end{tikzcd}$$

Show that in this diagram $\varphi_{1,2} = \varphi_{2,3} \circ \varphi_{1,3}$. □

**Definition 1.17.** An atlas $\mathcal{A}$ is *complete* or *maximal* if every $\mathcal{A}$-compatible chart is already in $\mathcal{A}$.

**Theorem 1.18.** *Given any atlas $\mathcal{A}$ there exists a complete atlas $\hat{\mathcal{A}}$ containing $\mathcal{A}$.*

*Proof.* Take $\hat{\mathcal{A}}$ to be the atlas consisting of all charts that are compatible with the charts in $\mathcal{A}$. By the above lemma, any two charts in $\hat{\mathcal{A}}$ are compatible with each other, which, by definition, makes $\hat{\mathcal{A}}$ an atlas as well. By construction, $\hat{\mathcal{A}}$ is a complete atlas as well. □

Thus if we're given an atlas, we can always avail ourselves of the completion process to turn it into a complete atlas. We are now in a position to state the additional "differentiable structure" that we must put on a topological manifold to make it a differentiable manifold.

**Definition 1.19.** An *$n$-dimensional differentiable manifold* is a pair $(X, \mathcal{A})$ where $X$ is an $n$-dimensional topological manifold with a complete atlas $\mathcal{A}$.

One of the simplest examples of a manifold of this type is the unit circle $\mathbb{S}^1$.

**Example 1.20** (The Unit Circle)**.** Let $X = \mathbb{S}^1 = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 = 1\}$ and $U_1 = \mathbb{S}^1 - (0, 1)$ and $U_2 = \mathbb{S}^1 - (0, -1)$. We would like to show that the circle is a one-dimensional manifold. First we must provide homeomorphisms to open subsets of $\mathbb{R}$. This is accomplished via *stereographic projection*. We first project from $(0, 1)$ – also known as the "north pole" – via $\varphi_1$ by taking a line passing through $(0, 1)$ and any point $p \in U_1$ and defining the image $\varphi(p)$ to be the intersection point of the line with the $x$-axis. Similarly, we can define $\varphi_2$ to be stereographic projection from the south pole $(0, -1)$.

Exercise: Write down explicit formulas for $\varphi_i$ $i = 1, 2$ and show that they define homeomorphisms onto $\varphi_i(U_i) =: V_i = \mathbb{R}$. Then show that $V_{1,2} := \varphi_1(U_1 \cap U_2)$ and $V_{2,1} := \varphi_2(U_1 \cap U_2)$ both equal $\mathbb{R} - 0$ and $\varphi_{1,2} := \varphi_2 \circ \varphi_1^{-1} = \frac{1}{x}$. Conclude that the pair $(U_i, V_i, \varphi_i)$ $i = 1, 2$ define an atlas on $\mathbb{S}^1$. This is known as **The Mercator Atlas**.

From this atlas we get a complete atlas by our completion process. This makes $\mathbb{S}^1$ into a one-dimensional differentiable manifold.

**Example 1.21** (The $n$-Sphere)**.** As an exercise, see if you can extend this construction to $\mathbb{S}^n$, $n \geq 2$. Hint: The construction is virtually the same but the overlap map $\varphi_{1,2} : \mathbb{R}^{n-1} \setminus 0 \to \mathbb{R}^{n-1} \setminus 0$ is $\frac{x}{||x||^2}$.

The process of giving a manifold a differentiable or *smooth* structure, may seem a little haphazard, as there are several different atlases that a given manifold could have. The following example illustrates an alternative atlas for $\mathbb{S}^n$.

**Example 1.22** (The Coordinate Atlas)**.** In the previous exercise you were asked to extend the Mercator construction to the $n$-sphere. That atlas is in some ways the most efficient in that it requires only two charts. In this example we construct an atlas that requires $2n + 2$ charts. Our initial data is as follows:

$$\begin{aligned} X &= \mathbb{S}^n = \{x \in \mathbb{R}^{n+1}, ||x|| = 1\} \\ U_i^+ &= \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n, x_i > 0\} \\ U_i^- &= \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n, x_i < 0\} \\ V &= \mathbb{B}^n = \{y \in \mathbb{R}^n, ||y|| < 1\} \end{aligned}$$

$\varphi_i^\pm : U_i^\pm \subset \mathbb{S}^n \to V \subset \mathbb{R}^n$ to be the projection map defined by

$$\varphi_i^\pm(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}).$$

In the case of $\mathbb{S}^2 \subset \mathbb{R}^3$, the maps would be $\varphi_1^\pm(x, y, z) = (y, z)$, $\varphi_2^\pm(x, y, z) = (x, z)$ and $\varphi_3^\pm(x, y, z) = (x, y)$. The inverse in the general case is

$$(\varphi_i^\pm)^{-1}(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) = (x_1, \ldots, \pm\sqrt{1 - \sum_{j\neq i} x_j^2}, \ldots, x_{n+1}).$$

Exercise: Compute $\varphi_i^\pm \circ (\varphi_j^\pm)^{-1}$ and show that they are smooth. Deduce that $\mathbb{S}^n$ is a differentiable manifold.

The previous several exercises may seem like a great deal of effort to show that the familiar $n$-sphere is a manifold – it is by definition a subset of Euclidean space! The next example exhibits a manifold that is not easily visualized as a subset of a larger ambient space.

**Example 1.23** (Real Projective Space)**.** $\mathbb{RP}^n$, $n$-dimensional real projective space, is defined as the space of one-dimensional subspaces (lines) in $\mathbb{R}^{n+1}$. Define the map

$$\mathbb{S}^n \xrightarrow{\pi} \mathbb{RP}^n$$

that sends a vector $v$ to its span $< v >$.

Exercise: Check this is a two-to-one map. Check that $\pi : U_i^+ \hookrightarrow \mathbb{RP}^n$ is injective and onto a set $U_i$. We then have the charts defined with homeomorphisms

$$\varphi_i : U_i \xrightarrow{\pi^{-1}} U_i^+ \xrightarrow{\varphi_i^+} V.$$

These form an atlas, making $\mathbb{RP}^n$ into a manifold. More details about this example can be found in the exercises at the end of the section.

**Example 1.24.** If $(X, \mathcal{A})$ is a manifold and $U$ is an open subset of $X$, then $U$ is a manifold in its own right.

*Proof.* Let $(U_\alpha, V_\alpha, \varphi_\alpha), \alpha \in I$ be the elements of an atlas $\mathcal{A}$. Let

$$\begin{aligned} U'_\alpha &= U_\alpha \cap U, \\ V'_\alpha &= \varphi_\alpha(U_\alpha \cap U), \\ \varphi'_\alpha &= \varphi|_{U_\alpha \cap U}. \end{aligned}$$

Observe that $(U'_\alpha, V'_\alpha, \varphi'_\alpha), \alpha \in I$ define an atlas $\mathcal{A}'$ on $U$. $\square$

1.6. **Smooth Mappings between Manifolds.**

**Notation 1.25** (Conventions for the Chapter)**.** From now on we will adopt the convention that "manifolds" will actually mean "differentiable manifold". Furthermore, we will simply refer to "a manifold $X$", assuming that $(X, \mathcal{A})$ is implicitly understood. Also in this spirit when we refer to a chart $(U, V, \varphi)$ we'll assume without saying that $(U, V, \varphi)$ is in $\mathcal{A}$.

**Definition 1.26.** Let $X$ be an $n$-dimensional manifold and $f : X \to \mathbb{R}^m$ a continuous map. We say that $f$ is $\mathcal{C}^\infty$ if for every chart $(U, V, \varphi)$ the map $f \circ \varphi^{-1} : V \to \mathbb{R}^m$ is $\mathcal{C}^\infty$.

*Remark* 1.27. The assumption that $f$ is continuous is not necessary to the definition, because the definition of smoothness implies continuity; however, it is convenient to have that the pre-image of open sets is open before doing further work.

The map $f \circ \varphi^{-1}$ is sometimes called the *coordinate representation* of $f$ as it is the function of the coordinates on the open set, $V \subseteq \mathbb{R}^n$. Thus the above definition says that a function on a manifold is smooth if and only if it is smooth in every coordinate representation of that function. A similar result is true for maps between manifolds.

**Definition 1.28.** Let $X$ and $Y$ be $n$ and $m$-dimensional manifolds respectively and $f : X \to Y$ a continuous map. We say $f$ is $\mathcal{C}^\infty$ if for every $p \in X$ and for every chart $(U, V, \varphi)$ on $Y$ $\varphi_Y \circ f : f^{-1}(U) \to \mathbb{R}^m$ is $\mathcal{C}^\infty$.

Note that since $f$ is continuous $f^{-1}(U)$ is open in $X$ and is thus a manifold in its own right. Putting together the above two definitions yields the following, more general, definition.

**Definition 1.29.** Let $X$ and $Y$ be $n$ and $m$-dimensional manifolds respectively and $f : X \to Y$ a continuous map. We say $f$ is $\mathcal{C}^\infty$ if for every $p \in X$ there exist charts $(U_X, V_X, \varphi_X)$ and $(U_Y, V_Y, \varphi_Y)$ about $p$ and $f(p)$ respectively so that $\varphi_Y \circ f \circ \varphi_X^{-1}$ is $\mathcal{C}^\infty$.

*Remark* 1.30 (For Every vs. There Exists). Traditionally the difference between "for every" and "there exists" in mathematical definitions is night and day, but in the case of smooth manifold theory this is not true. In particular, in the above several definitions, we could have simply required that there exist *one* chart where the required function is $\mathcal{C}^\infty$. The fact that all the charts in the atlas are assumed to be compatible then *gives us for free* that the function is $\mathcal{C}^\infty$ in *every chart!*

**Lemma 1.31** (Composition of smooth maps is smooth)**.** *Assume $X_i$ for $i = 1, 2, 3$ are manifolds. If $f_i : X_i \to X_{i+1}$ for $i = 1, 2$ are $\mathcal{C}^\infty$ maps, then $f_2 \circ f_1$ is a $\mathcal{C}^\infty$ map.*

*Proof.* Left as an exercise for the reader. □

1.7. **Smooth Mappings between Arbitrary Subsets.** Assume $M_i$ $i = 1, 2$ are manifolds and $X_i$ a subset of $M_i$ with $f : X_1 \to X_2$ a continuous map.

**Definition 1.32.** $f$ is a $\mathcal{C}^\infty$ map if for every $p \in X_1$ there exists a neighborhood $\mathcal{O}$ of $p$ in $M_1$ and a $\mathcal{C}^\infty$ map $\tilde{f} : \mathcal{O} \to M_2$ such that $f = \tilde{f}$ on $X_1 \cap \mathcal{O}$

**Theorem 1.33.** *Assume $M_i$ $i = 1, 2, 3$ are manifolds with subsets $X_i$. If $f_i : X_i \to X_{i+1}$ $i = 1, 2$ are $\mathcal{C}^\infty$ maps, then $f_2 \circ f_1 : X_1 \to X_3$ is $\mathcal{C}^\infty$ as well.*

**Exercise 1.34.** Prove the preceding Theorem.

**Definition 1.35.** For $X_i \subseteq M_i$ $i = 1, 2$ $f : X_1 \to X_2$ is a *diffeomorphism* if $f$ is $1 - 1$, onto and $f$ and $f^{-1}$ are $\mathcal{C}^\infty$ maps.

1.8. **Submanifolds.**

**Definition 1.36.** Let $M = M^n$ be an $n$-dimensional manifold and $X$ a subset of $M$. $X$ is a *$k$-dimensional submanifold* of $M$ if for every $p \in X$ there exists an open neighborhood $U$ of $p$ in $X$, an open set $V$ in $\mathbb{R}^k$ and a diffeomorphism $\varphi : U \to V$.

**Claim 1.37.** *The collection of all triples $(U, V, \varphi)$ where $U$ is an open set in $X$, $V$ an open set in $\mathbb{R}^k$ and $\varphi : U \to V$ a diffeomorphism, defines an atlas $\mathcal{A}$ on $X$.*

*Proof.* Given the charts $(U_i, V_i, \varphi_i)$ $i = 1, 2$

$$\begin{tikzcd} & U_1 \cap U_2 \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ V_{1,2} \arrow[rr, "\varphi_{1,2}"] & & V_{2,1} \end{tikzcd}$$

we see that if $\varphi_1$ and $\varphi_2$ are diffeomorphisms and $\varphi_{1,2} := \varphi_2 \circ \varphi_1^{-1}|_{V_{1,2}}$, then $\varphi_{1,2}$ is automatically a diffeomorphism. $\square$

1.9. **A More Intuitive Picture.** Although manifolds such as $\mathbb{RP}^n$ are not easily visualized as subsets of Euclidean space – thus lending support to the "intrinsic" viewpoint of manifolds – many manifolds that we will encounter later on are naturally submanifolds of $\mathbb{R}^N$ for some $N$.

In fact we will describe below (see §2.7) a result which says that *every* manifold is a submanifold of $\mathbb{R}^N$ for some $N$. In other words no generality is lost by regarding manifolds as being submanifolds of $\mathbb{R}^N$. This viewpoint we will call the "subset" or "extrinsic" picture of manifolds to contrast it with the intrinsic picture of manifolds that we've been exposing in this chapter. A clear exposition of this "extrinsic" picture can be found in Spivak's *Calculus on Manifolds* or Munkres' *Analysis on Manifolds.* One of its advantages is that their definition of "manifold" completely avoids the complications of charts and atlases: If $M_1$ and $M_2$ are Euclidean spaces, all one needs to make sense of the definitions (1.37) and (1.41) in the previous section is elementary freshman calculus. On the other hand this viewpoint also has its disadvantages. For instance it is explained in §5.4 how to realize $\mathbb{R}P^2$ as a submanifold of $\mathbb{R}^6$, but the "extrinsic" picture of $\mathbb{R}P^2$ as a submanifold of $\mathbb{R}^6$ is not nearly illuminating as the intrinsic picture of $\mathbb{R}P^2$ described in §1. (This remark applies to a lot of other examples as well, for instance the example discussed in §1.10, exercise 7.)

1.10. **Exercises.**

(1) **Products.** Assume that $M_1, \ldots, M_k$ are topological manifolds of dimensions $n_1, \ldots, n_k$ respectively. Prove that the product manifold $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $(n_1 + \cdots + n_k)$. Assume that the $M_i$ are smooth manifolds and prove that the resulting product is a smooth manifold as well.

(2) **Graphs.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^k$ be a continuous function. Define the *graph* of $f$ to be the set

$$G_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k, x \in U, y = f(x)\}$$

and prove that it is a manifold of dimension $n$. Hint – How does this set come equipped with its own chart? Now assume that $f$ is $\mathcal{C}^\infty$. Prove that the graph is a smooth manifold.

(3) **Spheres and Atlases.** Review the construction of the Mercator atlas and the coordinate atlas for the $n$-sphere. Show that the two atlases are compatible.

(4) **Atlas Revisited.** Let M be a set and let $\mathcal{U} = \{U_i, i \in I\}$ be a collection of subsets of $X$. Suppose that for each $i \in I$ one is given an open set $V_i$ in $\mathbb{R}^n$ and a bijective map $\varphi_i : U_i \to V_i$. We will call $\mathcal{U}$ an *atlas* if
- $\bigcup U_i = M$
- $\varphi_i(U_i \cap U_j) =: V_{i,j}$ is an open subset of $V_i$ for all $i, j \in I$
- In the overlap diagram below, the bottom arrow is a diffeomorphism.

$$\begin{tikzcd} & U_i \cap U_j \arrow[dl, "\varphi_i"'] \arrow[dr, "\varphi_j"] & \\ V_{i,j} \arrow[rr, "\varphi_{i,j}"] & & V_{j,i} \end{tikzcd}$$

(a) Prove that $M$ can be equipped with a topology by decreeing that $U \subseteq M$ is *open* if and only if $\varphi_i(U \cap U_i)$ is open for all $i$.

(b) Show that if $M$ is equipped with this topology then the atlas above really is an atlas in the sense defined in the previous section. Deduce that $M$ is a manifold.

(5) **Real Projective Space.** $\mathbb{RP}^n$ – Let $\pi : \mathbb{R}^{n+1} \smallsetminus 0 \to \mathbb{RP}^n$ be the map that sends a vector to its span, i.e.

$$\pi(x_1, \ldots, x_{n+1}) = [x_1, \ldots, x_{n+1}]$$

where the brackets denote *homogeneous coordinates*. Thus, for a given $x \in \mathbb{R}^{n+1} \smallsetminus 0$, $\pi(x) = [x]$ denotes the equivalence class of vectors contained in the same one-dimensional subspace generated by $x$.

(a) Define $U_i$ to be the set of projected points $[x_1, \ldots, x_{n+1}]$ where $x_i \neq 0$. Show that the map

$$\varphi_i : U_i \xrightarrow{\cong} \mathbb{R}^n$$

defined by

$$\varphi_i([x_1, \ldots, x_{n+1}]) = (\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i})$$

is indeed a homeomorphism.

(b) Check that $\mathcal{U}$ is an atlas in the sense of Exercise 4.

(6) **Real Projective Space Re-visited.** Another approach to defining the manifold structure on $\mathbb{RP}^n$. The $2-1$ map $\mathbb{S}^n \to \mathbb{RP}^n$.

(7) **Grassmannians** Let $\mathrm{Gr}(k, n)$ denote the space of all $k$-dimensional subspaces of an $n$-dimensional vector space, i.e $\mathbb{R}^n$. Note that this is a generalization of projective space, since $\mathrm{Gr}(1, n+1) \cong \mathbb{RP}^n$.

Let $e_i$ for $i = 1, \ldots, n$ be the standard basis vectors of $\mathbb{R}^n$. For every $k$-element subset, $I$, of $\{1, \ldots, n\}$ let $V_I = \mathrm{span}\{e_j, i \notin I\}$. Let $U_I$ be the set of all $k$-dimensional subspaces of $V$ of $\mathbb{R}^n$, i.e. $V \in \mathrm{Gr}(k, n)$, with $V \cap V_I = \{0\}$.

We will sketch below a proof of the following result:

**Theorem 1.38.** *The $U_I$'s define an atlas in the sense of Exercise 4.*

(a) Suppose $V$ is a $k$-dimensional subspace of $\mathbb{R}^n$. Let $w_r = (a_{1,r}, \ldots, a_{n,r})$ for $r = 1, \ldots, k$ be a basis of $V$ and let $A$ be the $k \times n$ matrix whose row vectors are the $w_r$'s. Let $A_I$ be the $k \times k$ minor of $A$ whose columns are the vectors $v_i = (a_{i,1}, \ldots, a_{i,k})$ $i \in I$. Show that $\det A_I \neq 0$ if and only if $V \cap V_I = \{0\}$.

Hint: Show that if $\det A_I = 0$ there exists a vector $(c_1, \ldots, c_k) = c \in \mathbb{R}^k \setminus \{0\}$ with $A_I c = 0$ and conclude that $\sum c_r w_r = \sum_{i \notin I} \lambda_j e_j$.

(b) Let $V \cap V_I = \{0\}$. Show that there is a *unique* basis, $w_r = (a_{1,r}, \ldots, a_{n,r})$ for $r = 1, \ldots, k$, of $V$ for which the $k \times k$ matrix $A_I$ is the identity matrix.

(c) For each $V \in U_I$ choose a basis $w_r = (a_{1,r}, \ldots, a_{n,r})$ $r = 1, \ldots, k$ such that $A_I$ is the identity matrix. Let $A_V$ be the $k \times (n-k)$ dimensional matrix obtained from the $k \times n$ matrix $A = [a_{i,j}]$ by deleting the minor $A_I$. Show that the map $V \in U_I \to A_V$ is a bijective map

$$\phi_I : U_I \xrightarrow{\cong} M_{k,n-k}$$

(d) Show that, equipped with these maps, the collection $\{U_I, I \subseteq \{1, \ldots, n\}\}$ is an atlas in the sense of Exercise 4.

Hints:

(i) Suppose $V \in U_I \cap U_J$ and let $w_r = (a_{1,r}, \ldots, a_{n,r})$ $r = 1, \ldots, k$ be a basis of $V$. Show that $\phi_I(V) = M_{k,n-k}$ is the matrix $A_I^{-1}A$ with its $I$-minor deleted and $\phi_J(V)$ is the matrix $A_J^{-1}A$ with its $J$-minor deleted.

(ii) For each $A \in M_{k,n}$ let $\rho_I(A) \in M_{k,n-k}$ be the matrix $A$ with its $I$-th minor deleted and for each $B \in M_{k,n-k}$ let $\gamma_I(B)$ be the matrix obtained from $B$ by inserting the $k \times k$ identity matrix as the $I$-th minor of $\gamma_I(B)$. Show that $B \in \phi_I(U_I \cap U_J)$ if and only if the $J$-th minor of $\gamma_I(B)$ is non-singular and show that if $\gamma_I(B) = A$, then $\phi_J \circ \phi_I^{-1}(B) = \rho_J(A_J^{-1}A)$. Conclude that $\phi_J \circ \phi_I^{-1}$ is a $\mathcal{C}^\infty$ map.

## 2. Tangent Spaces

The entire point of introducing differentiable manifolds is to have manifolds that one can "do Calculus on". In order for this to make sense we need a concept of a derivative. Differentiation, as understood in elementary calculus courses, involves tangent-making. Essentially the same is true for the manifold case with a few subtleties. In particular, if we adopt the intrinsic viewpoint of manifolds, how does one make sense of a space "tangent" to a manifold when there isn't a larger ambient space for the manifold and tangent space to live in? Just as we have presented two pictures of manifolds, we must now develop their corresponding versions of tangent spaces.

**Notation 2.1.** We will now use $M$ and $N$ to refer to differentiable manifolds instead of the previously used $X$ and $Y$. Occasionally $M^n$ will be used to denote an $n$-dimensional differentiable manifold $M$.

2.1. **Curves and Tangents.**

**Definition 2.2.** A *curve* in $M$ is a pair $(\gamma, I)$ where $I \subseteq \mathbb{R}$ is an open interval and $\gamma : I \to M$ is a smooth map. We will denote the set of curves passing through a point $p \in M$ by

$$C_p(M) := \{(\gamma, I), 0 \in I, \gamma(0) = p\}.$$

*Remark* 2.3. Notice that in the above definition we have given the point $p \in M$ the "special" position of $\gamma(t)$ for $t = 0$. Note that as long as a curve $\gamma$ passes through the point $p$ for some $t_p \in I$, we can then re-parametrize and obtain $\tilde{\gamma} = \gamma(t - t_p)$.

As a convention, we consider $(\gamma_1, I_1) \equiv (\gamma_2, I_2)$ if and only if $\gamma_1 = \gamma_2$ on a subinterval $I$ of $I_1 \cap I_2$ with $0 \in I$.

*Remark* 2.4. For the initiated reader $C_p(M)$ is called the set of "germs of $\mathcal{C}^\infty$ maps $\gamma : (\mathbb{R}, 0) \to (M, p)$."

Since the tangent space only depends locally on the manifold, we will focus on the set of curves passing through neighborhood $U$ of $p \in M$, and, via the equivalence relation "$\equiv$" identify $C_p(U)$ with $C_p(M)$. Furthermore, we want to capture the notion of a tangent space in terms of *vectors tangent to curves*. In order to do so we will temporarily assume that $U \subseteq \mathbb{R}^n$ — translating back to the intrinsic manifold viewpoint later.

**Definition 2.5.** Assume $(\gamma_i, I_i) \in C_p(U)$, $i = 1, 2$ for $U \subseteq \mathbb{R}^n$. We say that $(\gamma_1, I_1)$ and $(\gamma_2, I_2)$ are *tangent* at $p$ if $\frac{d\gamma_1}{dt}(0) = \frac{d\gamma_2}{dt}(0)$.

Note that in this setup $\gamma_i : I_i \to U \subseteq \mathbb{R}^n$ are vector-valued functions and so their derivatives can be thought of as tangent vectors.

**Definition 2.6.** Two curves in $C_p(U)$ are equivalent if they are tangent at $p$, i.e.

$$(\gamma_1, I_1) \sim_p (\gamma_2, I_2) \Leftrightarrow \frac{d\gamma_1}{dt}(0) = \frac{d\gamma_2}{dt}(0).$$

**Exercise 2.7.** Check that this does indeed define an equivalence relation. Hint – Check that $\sim_p$ is reflexive, symmetric and transitive.

**Definition 2.8.** The *tangent space* to a point $p \in U$ consists of the set of equivalence classes of curves, i.e.

$$T_pU := C_p(U) \big/ \sim_p$$

*Remark* 2.9 (Functoriality). The map

$$(\gamma, I) \to \frac{d\gamma}{dt}(0)$$

induces a bijective map

$$T_p(U) \to \mathbb{R}^n.$$

Now let $V$ be an open subset of $\mathbb{R}^m$ and $f : (U, p) \to (V, q = f(p))$ a $\mathcal{C}^\infty$ map. This induces a map

$$\begin{array}{rcl} \tilde{f} : C_p(U) & \to & C_q(V) \\ (\gamma, I) & \mapsto & (f \circ \gamma, I) \end{array}$$

which, in turn, induces a map on tangent spaces, i.e. gives rise to a diagram

$$\begin{tikzcd} C_p(U) \arrow[r, "\tilde{f}"] \arrow[d] & C_q(V) \arrow[d] \\ T_pU \arrow[r, "df_p"] \arrow[d] & T_qV \arrow[d] \\ \mathbb{R}^n \arrow[r, "J_f(p)"] & \mathbb{R}^m \end{tikzcd}$$

We now would like to translate these definitions to the intrinsic manifold setting. In particular, two curves on a manifold are tangent if and only if their coordinate representations are tangent.

**Definition 2.10.** Assume $M$ is a manifold, with chart $(U, V, \varphi)$ centered at $p \in M$, then $(\gamma_i, I_i) \in C_p(M)$, $i = 1, 2$ are *tangent* at $p$ if $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ are tangent at $\varphi(p) = 0 \in \mathbb{R}^n$.

**Exercise 2.11.** Show that this definition is independent of the choice of chart. Hint – Use overlap diagrams and the diagram from the previous remark.

As before, tangency is an equivalence relation $\sim_p$, thus allowing us to define the tangent space to a manifold in an identical manner.

**Definition 2.12.** $T_p(M) := C_p(M) / \sim_p$

2.2. **Mappings between Tangent Spaces.**

**Claim 2.13.** *If $M^m, N^n$ are manifolds and $f : (M, p) \to (N, q)$ a smooth map, then the map $\tilde{f} : C_p(M) \to C_p(N)$ preserves tangency.*

*Proof.* Suppose $\gamma_i \in C_p(M) \quad i = 1,2$ are tangent. By definition, in some chart $(U_M, V_M, \varphi_M)$ centered at $p$, $d(\varphi_M \circ \gamma_1)/dt(0) = d(\varphi_M \circ \gamma_2)/dt(0)$. If $(U_N, V_N, \varphi_N)$ is a chart on $N$ such that $f(U_M) \subseteq U_N$, we see that

$$d(\varphi_N \circ f \circ \gamma_i)/dt(0) = d(\varphi_N \circ f \circ \varphi_M^{-1})|_0 d(\varphi_M \circ \gamma_i)/dt(0)$$

are equal for $i = 1,2$ so tangency is preserved. Note that $\hat{\gamma}_i := \varphi_M \circ \gamma_i : \mathbb{R} \to \mathbb{R}^m$ so these derivatives are well-defined. The coordinate representative of $f$ is $\hat{f} = \varphi_N \circ f \circ \varphi_M^{-1}$ so that derivative is also well-defined. □

*Remark* 2.14 (Functoriality for Mappings of Manifolds). Let $(U, V, \varphi)$ be a chart centered at $p$ and $(U', V', \varphi')$ at chart centered at $f(p) = q$, with $f(U) \subseteq U'$.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"', "\cong"] & U' \arrow[d, "\cong"', "\varphi'"] \\ V \arrow[r, "g"] & V' \end{tikzcd}$$

Here $g = \varphi' \circ f \circ \varphi^{-1}$, which, as before, induces a map on the set of curves:

$$\begin{tikzcd} C_p(U) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\varphi}"', "\cong"] & C_q(U') \arrow[d, "\cong"', "\tilde{\varphi}'"] \\ C_0(V) \arrow[r, "\tilde{g}"] & C_0(V') \end{tikzcd}$$

Applying the equivalence relation from before then induces a map on the tangent spaces:

$$\begin{tikzcd} T_p(U) \arrow[r, "df_p"] \arrow[d, "d\varphi_p"', "\cong"] & T_q(U') \arrow[d, "\cong"', "d\varphi'_p"] \\ T_0(V) \arrow[r, "dg_0"] & T_0(V') \end{tikzcd}$$

**Claim 2.15.** *The map, $df_p$,is independent of the choice of chart.*

*Proof.* For any $n$-dimensional manifold $M$ we get the overlap diagram

$$\begin{tikzcd} & U_1 \cap U_2 \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ V_{1,2} \arrow[rr, "\varphi_{1,2}"] & & V_{2,1} \end{tikzcd}$$

and

$$\begin{tikzcd} & T_pM \arrow[dl, "(d\varphi_1)_p"', "\cong"] \arrow[dr, "(d\varphi_2)_p", "\cong"'] & \\ T_0\mathbb{R}^n \arrow[rr, "(d\varphi_{1,2})_0"] \arrow[d, "\cong"] & & T_0\mathbb{R}^n \arrow[d, "\cong"] \\ \mathbb{R}^n \arrow[rr, "J_{\varphi_{1,2}}(0)"] & & \mathbb{R}^n \end{tikzcd}$$

Since the maps $\varphi_i$ $i = 1, 2$ are diffeomorphisms, their derivatives are bijections and hence so is $(d\varphi_1, 2)$. $\square$

Consider now the case where $M^n$ is a manifold and $(U, V, \varphi)$ is a chart centered at $p$. The map $d\varphi_p$ allows us to put coordinates on the tangent space, $T_pM$, just as $\varphi$ does for the underlying manifold. Moreover by claim 2.15 this vector space structure is independent of charts. In other words

**Definition 2.16.** $T_pM$ becomes a vector space by requiring $d\varphi_p$ to be linear.

We will leave for you to check the following two claims.

**Claim 2.17.** *If $f : (M, p) \to (N, q)$ is a* smooth *map, then $df_p : T_pM \to T_qN$ is a* linear *map of vector spaces.*

**Claim 2.18.** *If $f : (M, p) \to (N, q)$ is a* diffeomorphism*, then $df_p : T_pM \to T_qN$ is an* isomorphism *of vector spaces.*

This latter claim actually has a stronger cousin known as the *inverse function theorem*. Studying maps of tangent spaces reveals important properties of the underlying maps and vice-versa.

2.3. **The Inverse Function Theorem.** Assume $U_1$, $U_2$ are open sets in $\mathbb{R}^n$ and $f : (U_1, p_1) \to (U_2, p_2)$ is a $\mathcal{C}^\infty$ map. The Jacobi mapping $J_f(p) : \mathbb{R}^n \to \mathbb{R}^n$ $J_f(p_1) = [\frac{\partial f_i}{\partial x_j}(p_1)]$ is the derivative of $f$ at $p_1$, which is the best linear approximation to the map. We have the following result:

**Theorem 2.19** (Inverse Function Theorem for $\mathbb{R}^n$)**.** *The map $J_f(p_1)$ is bijective if and only if $f$ maps a neighborhood of $p_1$ in $U_1$ diffeomorphically onto a neighborhood of $p_2$ in $U_2$.*

From the diagram

$$\begin{tikzcd} T_{p_1}U_1 \arrow[r, "df_{p_1}"] \arrow[d, "\cong"'] & T_{p_2}U_2 \arrow[d, "\cong"] \\ \mathbb{R}^n \arrow[r, "J_f(p_1)"] & \mathbb{R}^n \end{tikzcd}$$

we know that $J_f(p_1)$ bijective $\Leftrightarrow df_{p_1}$ bijective.

**Theorem 2.20** (Inverse Function Theorem for Manifolds)**.** *Assume that $M_1^n, M_2^n$ are manifolds and $f : (M_1, p_1) \to (M_2, p_2)$ is a smooth map. $df_{p_1} : T_{p_1}M_1 \to T_{p_2}M_2$ is bijective if and only if $f$ maps a neighborhood of $p_1$ diffeomorphically onto a neighborhood of $p_2$.*

*Proof.* Assume $(U_i, V_i, \varphi_i)$ $i = 1, 2$ are charts centered at $p_i$ with $f(U_1) \subset U_2$. Setting $g := \varphi_2 \circ f \circ \varphi_1^{-1}$ we get the commutative diagram

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\cong"'] & U_2 \arrow[d, "\cong"] \\ V_1 \arrow[r, "g"] & V_2 \end{tikzcd}$$

which induces the corresponding diagram on the tangent spaces

$$\begin{tikzcd} T_{p_1}M_1 \arrow[r, "df_{p_1}"] \arrow[d, "\cong"'] & T_{p_2}M_2 \arrow[d, "\cong"] \\ T_0V_1 \arrow[r, "dg_0"] & T_0V_2 \end{tikzcd}$$

We know that the vertical arrows are bijections by the chain rule. Accordingly, $df_{p_1}$ is bijective if and only if $dg_0$ bijective. If $dg_0$ is bijective than $g$ is a local diffeomorphism at 0 by the inverse function theorem for $\mathbb{R}^n$ and thus $f$ is a local diffeomorphism at $p_i$. $\square$

2.4. **Submersions.** The Inverse Function Theorem provides useful information about mappings by looking at their closest linear approximation – the derivative. The assumption that *df* is bijective is a very strong assumption and we should consider ways of weakening this assumption. In this subsection we consider the case when *df* is merely onto (*submersions*). In the next subsection we consider when *df* is $1-1$ (*immersions*).

Since the derivative is a linear mapping, it is natural to assume that some linear algebra is involved. Let's review some basic facts.

**Definition 2.21** (Canonical Submersion)**.** For $k < n$ we define the *canonical submersion*

$$\pi : \mathbb{R}^n \to \mathbb{R}^k$$

to be the map $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

We then have the following nice linear algebra result regarding submersions:

**Proposition 2.22.** *If $A : \mathbb{R}^n \to \mathbb{R}^k$ is onto, there exists a bijective linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $AB = \pi$.*

*Proof.* The proof is an exercise for the reader. Hint: Show that one can choose a basis, $v_1, \ldots, v_n$ of $\mathbb{R}^n$ such that

$$Av_i = e_i\,, \quad i = 1, \ldots, k$$

is the standard basis of $\mathbb{R}^k$ and

$$Av_i = 0\,, \quad i > k\,.$$

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ and set $Be_i = v_i$. □

Now let us return to the manifold setting.

**Definition 2.23.** Assume $M_1^m, M_2^n$ are manifolds with $m \geq n$ and $f : (M_1, p_1) \to (M_2, p_2)$ is a smooth map. $f$ is a *submersion* at $p_1$ if and only if $df_{p_1}$ is onto.

**Theorem 2.24.** *If $f$ is a submersion at $p_1$ then there exist charts $(U_i, V_i, \varphi_i)$ centered at $p_i$ such that $f(U_1) \subseteq U_2$ and the following diagram commutes*

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\cong"'] & U_2 \arrow[d, "\cong"] \\ V_1 \arrow[r, "\pi"] & V_2 \end{tikzcd}$$

*Proof.* Choose any charts $(U_i, V_i, \varphi_i)$ centered at $p_i$ such that $f(U_1) \subset U_2$. Setting $g := \varphi_2 \circ f \circ \varphi_1^{-1}$ as before

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\cong"'] & U_2 \arrow[d, "\cong"] \\ V_1 \arrow[r, "g"] & V_2 \end{tikzcd}$$

Check that if $df_{p_1}$ is onto then $dg_0$ is onto. Hint – Chain Rule. Now we can make a linear change of coordinates in $\mathbb{R}^m$ such that $J_g(0) = \pi$. Let $G : (V_1, 0) \to (\mathbb{R}^m, 0)$ be the map sending $x = (x_1, \ldots, x_m)$ to $G(x) = (g_1(x), \ldots, g_n(x), x_{n+1}, \ldots, x_m)$. Recognize that

$$J_G(0) = \begin{bmatrix} J_g(0) & 0 \\ 0 & I \end{bmatrix}$$

and since $J_g(0) = \pi$ we have that $J_G(0) = I$ and $\pi \circ G = g$. By the inverse function theorem we know that $G$ is a diffeomorphism of $V_1$

onto a neighborhood $W_1$ of 0 in $\mathbb{R}^m$. This results in the diagram

$$\begin{tikzcd}
U_1 \arrow[r, "f"] \arrow[d, "\varphi_1"'] & U_2 \arrow[d, "\varphi_2"] \\
V_1 \arrow[r, "g"] \arrow[d, "G"', "\cong"] & V_2 \\
W_1 \arrow[ur, "\pi"'] &
\end{tikzcd}$$

Replacing $\varphi_1$ with $G \circ \varphi_1$ provides the desired diagram:

$$\begin{tikzcd}
U_1 \arrow[r, "f"] \arrow[d, "\cong"'] & U_2 \arrow[d, "\cong"] \\
V_1 \arrow[r, "\pi"] & V_2
\end{tikzcd}$$

$\square$

2.5. **Submanifolds and Regular Values.** One important class of submanifolds are "pre images of regular values of mappings". More explicitly:

**Definition 2.25.** Assume $M_1^n, M_2^m$ are manifolds with $n > m$ and $f : M_1 \to M_2$ is a smooth map. We say $q \in M_2$ is a *regular value* of $f$ if for every $p \in M_1$, $p \in f^{-1}(q)$, $f$ is a submersion at $p$.

**Theorem 2.26.** *Let $k = n - m$ and let $X = f^{-1}(q)$, then $X$ is a $k$-dimensional submanifold of $M_1$.*

*Proof.* Putting $p_1 = p$ and $p_2 = q$, the canonical submersion theorem says that there exist charts $(U_i, V_i, \varphi_i)$ centered at $p_i$ such that $f(U_1) = U_2$ and

$$\begin{tikzcd}
U_1 \arrow[r, "f"] \arrow[d, "\cong"'] & U_2 \arrow[d, "\cong"] \\
V_1 \arrow[r, "\pi"] & V_2
\end{tikzcd}$$

is a commutative diagram and $\pi(x_1, \ldots, x_n) = (x_{k+1}, \ldots, x_n)$. Thus

$$\begin{aligned}
\pi^{-1}(0) &= \{x \in V_1, x_{k+1} = \cdots = x_n = 0\} \\
&= V_1 \cap \mathbb{R}^k
\end{aligned}$$

and $\varphi$ maps $U_1 \cap f^{-1}(p_2)$ bijectively onto $V_1 \cap \mathbb{R}^k$. Since $X = f^{-1}(p_2)$ $X \cap U_1 = U$ is diffeomorphic to an open subset $V = V_1 \cap \mathbb{R}^k$ of $\mathbb{R}^k$. $\square$

**Example 2.27** (The n-Sphere)**.** If $M = \mathbb{R}^{n+1}$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ $f(x) = x_1^2 + \cdots + x_{n+1}^2$, then $f^{-1}(1) = \mathbb{S}^n$. Check that $1 \in \mathbb{R}$ is a regular value of $f$ thus verifying that $\mathbb{S}^n$ is an $n$-dimensional submanifold of $\mathbb{R}^{n+1}$.

**Example 2.28** (Orthogonal Matrices)**.** We begin with the following two manifolds

$$\begin{aligned} M_1 = &\quad \mathcal{M}_n &= \mathbb{R}^{n^2} \\ M_2 = &\quad \mathcal{S}_n &= \mathbb{R}^{n(n+1)} \end{aligned}$$

where $\mathcal{M}_n$ denotes the space of all $n \times n$ real matrices and $\mathcal{S}_n$ denotes the space of real symmetric matrices. We then have the map $f : M_1 \to M_2$ defined by $f(A) = A^t A$. Check that the $n \times n$ identity matrix $I \in \mathcal{S}_n$ is a regular value of $f$. Conclude that the space of $n \times n$ orthogonal matrices $\mathcal{O}_n = f^{-1}(I)$ is an $n(n-1)/2$ dimensional submanifold of $\mathcal{M}_n$.

2.6. **Immersions.** Again, we recall what is meant by immersions in the linear algebra sense.

**Definition 2.29** (Canonical Immersion)**.** For $k < n$ we define the *canonical immersion*

$$\iota : \mathbb{R}^k \to \mathbb{R}^n$$

to be the map $\iota(x_1, \ldots, x_k) = (x_1, \ldots x_k, 0, \ldots 0)$.

**Exercise 2.30.** Verify that the inclusion map defined above is the transpose of the projection map, i.e. $\pi^t = \iota$.

Checking the previous exercise leads to a quick proof that every injective map can be put into its canonical form.

**Proposition 2.31.** *If $A : \mathbb{R}^k \to \mathbb{R}^n$ is one–one, there exists a bijective linear map $C : \mathbb{R}^n \to \mathbb{R}^n$ such that $CA = \iota$.*

*Proof.* The rank of $[a_{i,j}]$ is equal to the rank of $[a_{j,i}]$, so if if $A$ is one–one, there exists a bijective linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $A^t B = \pi$. Letting $C = B^t$ and taking transposes we get $\iota = \pi^t = CB$ $\square$

**Definition 2.32.** Let $f : (M_1, p_1) \to (M_2, p_2)$ be a smooth map as before. $f$ is an *immersion* at $p_1$ if and only if $df_{p_1}$ is injective, i.e. $1-1$.

**Theorem 2.33.** *If $f$ is a immersion at $p_1$ then there exist charts $(U_i, V_i, \varphi_i)$ centered at $p_i$ such that $f(U_1) \subseteq U_2$ and the following diagram commutes*

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\cong"'] & U_2 \arrow[d, "\cong"] \\ V_1 \arrow[r, "\iota"] & V_2 \end{tikzcd}$$

*Proof.* Choose any charts $(U_i, V_i, \varphi_i)$ centered at $p_i$ such that $f(U_1) \subset U_2$. Setting $g := \varphi_2 \circ f \circ \varphi_1^{-1}$ as before

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\cong"'] & U_2 \arrow[d, "\cong"] \\ V_1 \arrow[r, "g"] & V_2 \end{tikzcd}$$

Check that if $df_{p_1}$ is injective then $dg_0$ is injective. Now we can make a linear change of coordinates in $\mathbb{R}^n$ such that $J_g(0) = \iota$. Let $\tilde{V} \subseteq \mathbb{R}^n$ be the open set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n, (x_1, \ldots, x_m) \in V\}$ and let $G : \tilde{V} \to \mathbb{R}^n$ be the map sending $x = (x_1, \ldots, x_n)$ to $G(x) = g(x_1, \ldots, x_m) + (0, \ldots, 0, x_{m+1}, \ldots, x_n)$. Check that $J_G(0) = I$ and $G \circ \iota = g$.

By the inverse function theorem $G$ maps a neighborhood, $W$, of 0 in $\tilde{V}$ diffeomorphically onto a neighborhood of 0 in $V_2$. In the diagram above lets replace $V_2$ by $(V_2)_{new} = G(W)$ and $V_1$ by $(V_1)_{new} = g^{-1}(G(W))$. Shrinking $U_1$ and $U_2$ if necessary we get the diagram:

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\varphi_1"'] & U_2 \arrow[d, "\varphi_2"] \\ V_1 \arrow[r, "g"] \arrow[dr, "\iota"] & V_2 \\ & W \arrow[u, "\cong"', "G"] \end{tikzcd}$$

Replacing $\varphi_2$ by $G^{-1} \circ \varphi_2$ and letting $(V_2)_{new} = W$ provides the desired diagram:

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\cong"'] & U_2 \arrow[d, "\cong"] \\ V_1 \arrow[r, "\iota"] & V_2 \end{tikzcd}$$

□

## 2.7. **Immersions and the Subset Picture.**

**Theorem 2.34.** *Assume $M$ is an $n$-dimensional manifold. If $X$ a $k$-dimensional submanifold then the inclusion map $\iota : X \hookrightarrow M$ is a $1-1$ proper immersion.*

*Proof.* Left as an exercise for the reader. Here are some hints: Let $p \in X$ and $U$ an open neighborhood of $p$ in $X$, $V$ an open neighborhood of 0 in $\mathbb{R}^k$ and $\varphi : U \to V$ a diffeomorphism. Shrinking $U$ if necessary, we can extend $\varphi$ to a $\mathcal{C}^\infty$ map $\tilde{\varphi} : \tilde{U} \to V$ where $\tilde{U}$ is an open neighborhood of $p$ in $M$ and $\tilde{\varphi}$ is a $\mathcal{C}^\infty$ map. Show that $\varphi = \tilde{\varphi} \circ \iota$ and deduce the injectivity of $d\iota_p$ from the chain rule and bijectivity of $d\varphi_p$. $\square$

A corollary of this result and the canonical immersion theorem is the existence of *adapted charts* for submanifolds. These charts give us the nice coordinate property that a $k$-dimensional submanifold only requires $k$ coordinates of the original $n$-dimensional chart.

**Corollary 2.35.** *Let $M^n$ be a manifold and $X \subset M$ a closed subset of $M$. $X$ is an $k$-dimensional submanifold of $M$ if for every $p \in X$ there exists a chart $(U, V, \varphi)$ centered at $p$ and an open neighborhood $W$ of* $0$ *in $\mathbb{R}^k$ such that*

$$\varphi(U \cap X) = \iota(W). \tag{2.36}$$

*By projecting onto the first $k$-coordinates, we get a homeomorphism $\psi : U \cap X \to W$ with the defining property*

$$\begin{tikzcd} U \cap X \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & V \\ & W \arrow[ur, "\iota"'] & \end{tikzcd}$$

**Definition 2.37.** A chart $(U, V, \varphi)$ with the property 2.36 is called an *adapted chart* for the submanifold $X$.

We alluded in §1.9 to the "intrinsic" definition of manifolds via charts and atlases and contrasted this with the "extrinsic" definition in which manifolds are, by assumption, "submanifolds of $\mathbb{R}^N$". A celebrated theorem of Hassler Whitney, the *Whitney imbedding theorem* asserts that every $n$-dimensional manifold can be imbedded diffeomorphically into $\mathbb{R}^{2n+1}$ as an $n$-dimensional submanifold, thus showing that these two definitions are compatible. We won't attempt to prove Whitney's theorem; however in §5 we will sketch a (weak form of) a special case of this theorem, and leave for you, as an exercise, to fill in the details. We'll ask you to show that every compact $n$-dimensional manifold is, up to diffeomorphism, a submanifold of some $\mathbb{R}^N$. (However, we won't require you to show that $N = 2n + 1$.)

*Remark* 2.38 (Tangent Space is a Vector Subspace). Assume that $M$ is an $n$-dimensional manifold and $X$ is a $k$-dimensional submanifold with $\iota : X \hookrightarrow M$ the inclusion map. For every $p \in X$ the map

$$d\iota_p : T_pX \to T_pM$$

is injective so by identifying $T_pX$ with its image under this map we can think of $T_pX$ as a vector subspace of $T_pM$.

The above remark is visually helpful when $M = \mathbb{R}^n$ and so $T_pM \cong \mathbb{R}^n$. In this case we can think of the tangent space to $X^k$ at $p$, $T_pX$, as a $k$-dimensional plane sitting inside a copy of $\mathbb{R}^n$ centered at $p \in X \subset \mathbb{R}^n$.

For manifolds $X$ realized as inverse images of regular values under a mapping $f$, the vector subspace structure of $T_pX$ can be defined in terms of the linear mapping $df_p$.

**Claim 2.39.** *Assume $M_1$, $M_2$ are manifolds, $f : M_1 \to M_2$ a map with regular value $q \in M_2$ defining a submanifold $X = f^{-1}(q)$. If $p \in X$ then*

$$T_pX = \ker df_p.$$

*Proof.* Composing $f$ with the inclusion map defines a map $f \circ \iota : X \to M_2$. Since for all points $p \in X$, $f(p) = q$ it defines a constant map on $X$, thus $d(f \circ \iota)_p = df_p \circ d\iota_p = 0$. Consequently, $\operatorname{Im} d\iota_p \subseteq \ker df_p$ proving that $T_pX \subseteq \ker df_p$. To prove equality, we use a dimension argument. Since

$$\begin{aligned} \dim X &= \dim M_1 - \dim M_2 \\ &= \dim T_pM_1 - \dim T_qM_2 \\ &= \dim \ker df_p \end{aligned}$$

and $\dim X = \dim T_pX$, we have that $\dim T_pX = \dim \ker df_p$ and thus $T_pX = \ker df_p$. $\square$

**Example 2.40** (Skew-Symmetric Matrices)**.** Recall that we showed that $\mathcal{O}_n = f^{-1}(I)$ where $f : \mathcal{M}_n \to \mathcal{S}_n$ is the map $f(A) = A^tA$. Show that at the identity matrix $I \in \mathcal{M}_n$ $df_I(A) = A + A^t$. Conclude that

$$T_I\mathcal{O}_n = \{A \in \mathcal{M}_n, A^t = -A\}.$$

2.8. **Exercises.**

(1) Let $f : X \to Y$ be a $\mathcal{C}^\infty$ map and define $Z := \operatorname{graph} f$. Show that the tangent space to $Z$ at a point $(x, y)$ is

$$T_{x,y}Z = \{v, df_x(v), v \in T_x\}.$$

(2) Let $X$ and $Y$ be manifolds and $X \times Y$ their product. Show that:

$$T_{x,y}X \times Y \cong T_xX \times T_yY$$

## 3. Vector Fields on Manifolds

**Definition 3.1.** Assume $M_n$ is an $n$-dimensional manifold. A *vector field* on $M$ is a function $v$ that associates with every point $p$ a vector $v(p) \in T_pM$.

3.1. **Push-Forwards and Pull-Backs.**

**Definition 3.2.** Let $X$ and $Y$ be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ mapping. Given a vector field, $v$, on $X$ and a vector field, $w$, on $Y$, we'll say that $v$ and $w$ are *$f$-related* if, for all $p \in X$ and $q = f(p)$

$$(df)_p v(p) = w(q)\,. \tag{3.3}$$

**Definition 3.4** (Push-Forward for Vector Fields)**.** Assume $X$ and $Y$ are manifolds and $f : X \to Y$ is a diffeomorphism. The *push-forward* of a vector field $v$ on $X$ to a vector field $w =: f_* v$ on $Y$ is defined point-wise by equation 3.3.

**Definition 3.5** (Pull-Back for Vector Fields)**.** Similarly, given a vector field, $w$, on $Y$ we can define a vector field, $v$, on $X$ by applying the same construction to the inverse diffeomorphism, $f^{-1} : Y \to X$. We will call the vector field $(f^{-1})_* w$ the *pull-back* of $w$ by $f$ and denote it by $f^* w$.

*Remark* 3.6 (Functoriality). Let $X$, $Y$ and $Z$ be manifolds and $f : X \to Y$ and $g : Y \to Z$ diffeomorphisms. If $v$ is a vector field on $X$, then

$$(g \circ f)_* v = g_*(f_* v)$$

is a vector field on $Z$. Similarly, if $w$ is a vector field on $Z$, then

$$(g \circ f)^* w = f^*(g^* w)$$

is a vector field on $X$.

**Exercise 3.7.** Describe why the assumption that $f : X \to Y$ is a diffeomorphism is necessary for the definition of push-forwards and pull-backs for vector fields. Later we will see that pull-backs for $k$-forms only require $\mathcal{C}^\infty$ maps.

3.2. **Smooth Vector Fields.** With the push-forward/pull-back terminology in place, we now have a convenient language for discussing *smooth vector fields*. If a vector field $v$ is defined on an open subset $U$ of $X$ then we know, by definition, that for every $p \in U$ there exists an open neighborhood $\mathcal{O}$ of $p$ and a diffeomorphism $\varphi : \mathcal{O} \to V \subseteq \mathbb{R}^n$. Since $\mathbb{R}^n$ is a manifold, and $V \subseteq \mathbb{R}^n$ a submanifold, we can just set $Y = V$ and $f = \varphi$ in the definition of a push-forward to obtain the vector field $\varphi_* v$ on $V$. Following our previous strategy, we will define a smooth vector field as one whose push-forward on $\mathbb{R}^n$ is smooth.

**Definition 3.8.** A vector field $v$ on $U \subseteq X$ is $\mathcal{C}^\infty$ if $\varphi_* v$ is $\mathcal{C}^\infty$.

In similar fashion we can define a smooth vector field globally if it is smooth locally for all points on the manifold.

**Definition 3.9.** A vector field $v$ on a manifold $X^n$ is $\mathcal{C}^\infty$ if, for every point $p \in X$, $v$ is $\mathcal{C}^\infty$ on a neighborhood of $p$.

**Exercise 3.10.** Show that these definitions are independent of the choice of chart.

**Proposition 3.11.** *Assume $v$ is a $\mathcal{C}^\infty$ vector field on $X$ and $f : X \to Y$ is a diffeomorphism, then $f_* v$ is $\mathcal{C}^\infty$. Similarly, if $w$ is a $\mathcal{C}^\infty$ vector field on $Y$ then $f^* w$ is a $\mathcal{C}^\infty$ vector field on $X$.*

3.3. **Integral Curves.** Assume $M = M^n$ is an $n$-dimensional manifold. Recall that a smooth curve passing through $p \in M$ is a $\mathcal{C}^\infty$ map $\gamma : I \subset \mathbb{R} \to M$ satisfying $\gamma(0) = p$. The derivative at 0, or tangent map at 0, is $d\gamma_0 : T_0\mathbb{R} \to T_pM$. If we apply $d\gamma_0$ to the unit vector $u = 1$, we obtain

$$\frac{d\gamma}{dt}(0) = d\gamma_0(1).$$

Recall that a vector field $v$ just assigns to points on the manifold to vectors in the corresponding tangent space, i.e. $v : M \to TM$. Similarly, by fixing the unit vector as we have done above and letting the point of evaluation $t$ to vary we obtain a map $d\gamma_t : t \in I \subset \mathbb{R} \to T_{\gamma(t)}M$. If a vector field and a curve agree on this map, we call the curve an *integral curve.*

**Definition 3.12.** $\gamma : I \to M$ is an *integral curve* of $v$ if for all $a \in I$

$$\frac{d\gamma}{dt}(a) = v(\gamma(a)).$$

3.4. **ODE Theory on Manifolds.**

**Proposition 3.13.** *Let $X$ and $Y$ be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. If $v$ and $w$ are vector fields on $X$ and $Y$ which are $f$-related, then integral curves of $v$ get mapped by $f$ onto integral curves of $w$.*

*Proof.* If the curve, $\gamma : I \to X$ is an integral curve of $v$ we have to show that $f \circ \gamma : I \to Y$ is an integral curve of $w$. If $\gamma(t) = p$ and $q = f(p)$ then by the chain rule

$$\begin{aligned} w(q) &= df_p(v(p)) = df_p(d\gamma_t(\vec{u})) \\ &= d(f \circ \gamma)_t(\vec{u})\,. \end{aligned}$$

□

From this result it follows that the local existence, uniqueness and "smooth dependence on initial data" results about vector fields that we described in §2.1 of Chapter 2 are true for vector fields on manifolds. More explicitly, let $U \subseteq X$ be the domain of some chart and $(U, V, \varphi)$.

Since $V \subseteq \mathbb{R}^n$ these results are true for the vector field $w = \varphi_* v$ and hence, since $w$ and $v$ are $\varphi$-related, they are true for $v$. In particular, we have recaptured the following important theorems for differential equations on manifolds:

**Proposition 3.14** (Local Existence)**.** *For every $p \in U$ there exists an integral curve, $\gamma(t)$ , $-\epsilon < t < \epsilon$, of $v$ with $\gamma(0) = p$.*

**Proposition 3.15** (Local Uniqueness)**.** *Let $\gamma_i : I_i \to U$ $i = 1, 2$ be integral curves of $v$ and let $I = I_1 \cap I_2$. If $\gamma_2(a) = \gamma_1(a)$ for some $a \in I$ then $\gamma_1 \equiv \gamma_2$ on $I$. Furthermore, there exists a unique integral curve, $\gamma : I_1 \cup I_2 \to U$ with $\gamma = \gamma_1$ on $I_1$ and $\gamma = \gamma_2$ on $I_2$.*

**Proposition 3.16** (Smooth Dependence on Initial Data)**.** *For every $p \in U$ there exists a neighborhood, $\mathcal{O}$ of $p$ in $U$, an interval $(-\epsilon, \epsilon)$ and a $\mathcal{C}^\infty$ map, $h : \mathcal{O} \times (-\epsilon, \epsilon) \to U$ such that for every $p \in \mathcal{O}$ the curve*

$$\gamma_p(t) = h(p, t)\,, \quad -\epsilon < t < \epsilon\,,$$

*is an integral curve of $v$ with $\gamma_p(0) = p$.*

### 3.5. **Complete Vector Fields.**

**Definition 3.17.** A $\mathcal{C}^\infty$ curve $\gamma : [0, b) \to M$ is a *maximal integral curve* of $v$ if it can't be extended to an interval $[0, b']$ where $b' > b$, i.e. there is no integral curve $\gamma' : [0, b'] \to M$ such that $\gamma = \gamma'|_{[0,b)}$.

**Definition 3.18.** A sequence of points $p_i \in M$ $i = 1, 2, \ldots$ *tends to infinity* in $M$ if for every compact set $C \subseteq M$ there exists an $i_0$ such that $p_i \notin C$ for $i > i_0$.

**Theorem 3.19.** *If $\gamma : [0, b) \to M$ is a maximal integral curve of $v$ then either of the following is true:*

(a) $b = +\infty$
(b) *There exists a sequence $\{t_i\} \in [0, b)$ with $t_i \to b$ and the image sequence $\{\gamma(t_i)\}$ tends to infinity in $M$.*

*Proof.* (See Birkhoff and Rota page 172.) Suppose not. Then there exists a compact set $C$ such that for all $t \in [0, b)$ $\gamma(t) \in C$ and hence there exists a sequence $t_i \in [0, b)$ and a point $p \in C$ such that $t_i \to t$ and $\gamma(t_i) \to p$. Let $U$ be a open neighborhood of $p$ and

$$h : U \times (-\epsilon, \epsilon) \to M$$

a $\mathcal{C}^\infty$ map such that for all $q \in U$ $\gamma_q(t) = h(q, t)$ $-\epsilon < t < \epsilon$ is an integral curve of $p$ with $\gamma_q(0) = q$. Since $\gamma(t_i) \to p$ and $t_i \to b$ there exists a $t_i$ such that $t_i > b - \frac{\epsilon}{2}$ and $\gamma(t_i) \in U$. Set $c = t_i$ and $q = \gamma(t_i)$. Then the curve $\tilde{\gamma}(t) : [0, b + \frac{\epsilon}{2}) \to M$ defined by

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & 0 \leq t \leq c \\ \gamma_q(t + c) & c \leq t \leq b + \frac{\epsilon}{2} \end{cases}$$

is an integral curve extending $\gamma$. □

If $M$ is compact then we can, in fact, say more. Because an integral curve cannot, by definition, leave the manifold, then if the manifold is compact, it can never tend to infinity. By the previous theorem, this curve exists for all future time. Vector fields that have integral curves for all time are called *complete*.

**Definition 3.20.** A vector field $v$ is *complete* if for every $p \in M$ there exists an integral curve $\gamma_p(t)$ $-\infty < t < \infty$ satisfying $\gamma_p(0) = p$.

**Theorem 3.21.** *If $M$ is compact, then, for any vector field $v$ on $M$, $v$ is complete.*

*Proof.* By the theorem above there exists for every $p \in M$ an integral curve $\gamma_p : [0, \infty) \to M$ with $\gamma_p(0) = p$. Noting that integral curves of $-v$ are just integral curves of $v$ reparametrized by setting $t \to -t$. Accordingly, by the theorem above, for every $p \in M$ an integral curve $\gamma_p : (-\infty, 0] \to M$ of $v$ with $\gamma_p(0) = p$ and putting these two results together an integral curve of $v$ $\gamma_p : (-\infty, \infty) \to M$ with $\gamma_p(0) = p$. □

Complete vector fields naturally give rise to a map that allows us to "flow" open sets on a manifold based on how the individual integral curves evolve in time for each point. Such a map $f : M \times \mathbb{R} \to M$ is obtained by setting

$$f(p, t) := \gamma_p(t)$$

and then defining the associated map $f_t : M \to M$ to be a function of points rather than time, i.e.

$$f_t(p) := f(p, t).$$

One can visualize this map as placing a tracer particle in some deterministic fluid flow and watching the particle, originating at some point $p$, get carried after some time $s$ to a point $f_s(p)$. Allowing the map to begin with $f_s(p)$ and flow for some time $t$ should obtain the same result as starting at $p$ and flowing for time $t+s$, since the flow is deterministic.

**Claim 3.22.** *If $v$ is complete and $f_t : M \to M$ is defined as above, then $f_t \circ f_s = f_{t+s}$.*

*Proof.* Let $q = f_s(p)$. Then the curves $\gamma_1(t) = f_{s+t}(p)$ $-\infty < t < \infty$ and $\gamma_2(t) = f_t(q)$ $-\infty < t < \infty$ are both integral curves of $v$ with initial point $\gamma_1(0) = \gamma_2(0) = q$ and hence they coincide for all time. $\square$

**Theorem 3.23.** *If $M$ is compact then $f : M \times \mathbb{R} \to M$ is $\mathcal{C}^\infty$.*

*Proof.* By smooth dependence on initial data there exists for every $p \in M$ a neighborhood $U_p$ of $p$ and an $\epsilon_p > 0$ such that

$$f : U_p \times (-\epsilon_p, \epsilon_p) \to M$$

is $\mathcal{C}^\infty$. The sets $U_p$ for $p \in M$ are a covering of $M$, which is compact. There exists a finite subcover $U_{p_i}$ $i = 1, \dots, k$ of $M$. Now let $\epsilon = \min \epsilon_{p_i}$. Since the sets $U_{p_i} \times (-\epsilon, \epsilon)$ cover $M \times (-\epsilon, \epsilon)$ and $f$ restricted to each of these sets is $\mathcal{C}^\infty$,

$$f : M \times (-\epsilon, \epsilon) \to M$$

is a $\mathcal{C}^\infty$ map. Now note that for any $T > 0$ there exists a large positive integer $N$ such that $T/N < \epsilon$. Using the fact that

$$f_T = f_{T/N} \circ \cdots \circ f_{T/N}$$

and since each $f_{T/N}$ is $\mathcal{C}^\infty$, the composition $f : M \times (-T, T) \to M$ is $\mathcal{C}^\infty$. Since $T$ is arbitrary, we conclude that $f : M \times \mathbb{R} \to M$ is $\mathcal{C}^\infty$. $\square$

*Remark* 3.24. With a little effort, the assumption that $M$ is compact in Theorem 3.23 can be weakened. See Exercises 12 and 13 at the end of the section for more details.

3.6. **One-Parameter Groups of Diffeomorphisms.** You may have observed from Claim 3.22 that the functions $f_t$ are parameterized by the additive group $(\mathbb{R}, +)$. In particular, $f_0 = \mathrm{Id}_M$ is the identity mapping and for any $t \in \mathbb{R}$ there exists a $-t \in \mathbb{R}$ so that $f_t \circ f_{-t} = f_{t-t} = f_0 = \mathrm{Id}_M$, so the set $f_t$ has an identity element and inverses – making it a group. The result stating that $f : M \times \mathbb{R} \to M$ is $\mathcal{C}^\infty$ gives us that $f_t : M \to M$ is $\mathcal{C}^\infty$ for any $t$ as well. Consequently, both $f_t$ and its inverse $(f_t)^{-1} = f_{-t}$ are $\mathcal{C}^\infty$ – making it a diffeomorphism.

**Definition 3.25.** The set of functions

$$\{f_t, -\infty < t < \infty\}$$

is called a *one-parameter group of diffeomorphisms* generated by $v$.

If the vector field $v$ is not complete then it does *not* generate a one-parameter group of diffeomorphisms. We have shown that if the underlying manifold $M$ is compact, then any vector field is complete, but if $M$ is not compact, then it admits a great deal of incomplete vector fields.

**Example 3.26.** Let $M = \mathbb{R}$ and $v = x^2 \frac{\partial}{\partial x}$. Exercise – What are the integral curves? Where are they undefined?

**Proposition 3.27.** *If a vector field $v$ is compactly supported then it is complete – and thus generates a one-parameter group of diffeomorphisms.*

*Proof.* The proof is identical to the case when $M$ is compact. See above. □

3.7. **Exercises.**

(1) Let $X \subseteq \mathbb{R}^3$ be the paraboloid, $x_3 = x_1^2 + x_2^2$ and let $w$ be the vector field

$$w = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_3} .$$

(a) Show that $w$ is tangent to $X$ and hence defines by restriction a vector field, $v$, on $X$.
(b) What are the integral curves of $v$?

(2) Let $S^2$ be the unit 2-sphere, $x_1^2 + x_2^2 + x_3^2 = 1$, in $\mathbb{R}^3$ and let $w$ be the vector field

$$w = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} .$$

(a) Show that $w$ is tangent to $S^2$, and hence by restriction defines a vector field, $v$, on $S^2$.
(b) What are the integral curves of $v$?

(3) As in problem 2 let $S^2$ be the unit 2-sphere in $\mathbb{R}^3$ and let $w$ be the vector field

$$w = \frac{\partial}{\partial x_3} - x_3 \left( x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right)$$

(a) Show that $w$ is tangent to $S^2$ and hence by restriction defines a vector field, $v$, on $S^2$.
(b) What do its integral curves look like?

(4) Let $S^1$ be the unit circle, $x_1^2 + x_2^2 = 1$, in $\mathbb{R}^2$ and let $X = S^1 \times S^1$ in $\mathbb{R}^4$ with defining equations

$$f_1 = x_1^2 + x_2^2 - 1 = 0$$

$$f_2 = x_3^2 + x_4^2 - 1 = 0 .$$

(a) Show that the vector field

$$w = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + \lambda \left( x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} \right) ,$$

$\lambda \in \mathbb{R}$, is tangent to $X$ and hence defines by restriction a vector field, $v$, on $X$.
(b) What are the integral curves of $v$?
(c) Show that $L_w f_i = 0$.

(5) For the vector field, $v$, in problem 4, describe the one-parameter group of diffeomorphisms it generates.
(6) Let $X$ and $v$ be as in problem 1 and let $f : \mathbb{R}^2 \to X$ be the map, $f(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^2)$. Show that if $u$ is the vector field,
$$u = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\,,$$
then $f_* u = v$.
(7) Let $X$ be a submanifold of $X$ in $\mathbb{R}^N$ and let $v$ and $w$ be the vector fields on $X$ and $U$. Denoting by $\iota$ the inclusion map of $X$ into $U$, show that $v$ and $w$ are $\iota$-related if and only if $w$ is tangent to $X$ and its restriction to $X$ is $v$.
(8) Let $X$ be a submanifold of $\mathbb{R}^N$ and $U$ an open subset of $\mathbb{R}^N$ containing $X$, and let $v$ and $w$ be the vector fields on $X$ and $U$. Denoting by $\iota$ the inclusion map of $X$ into $U$, show that $v$ and $w$ are $\iota$-related if and only if $w$ is tangent to $X$ and its restriction to $X$ is $v$.
(9) An elementary result in number theory asserts

**Theorem 3.28.** *A number, $\lambda \in \mathbb{R}$, is irrational if and only if the set*
$$\{m + \lambda n\,, \quad m \text{ and } n \text{ integers}\}$$
*is a* dense *subset of* $\mathbb{R}$.

Let $v$ be the vector field in problem 4. Using the theorem above prove that if $\lambda/2\pi$ is irrational then for every integral curve, $\gamma(t)$, $-\infty < t < \infty$, of $v$ the set of points on this curve is a dense subset of $X$.

(10) Let $v$ be a vector field on $X$ and $\varphi : X \to \mathbb{R}$, a $\mathcal{C}^\infty$ function. Show that if the function
$$L_v \varphi = \iota(v)\, d\varphi \tag{3.29}$$
is zero $\varphi$ is constant along integral curves of $v$.
(11) Suppose that $\varphi : X \to \mathbb{R}$ is proper. Show that if $L_v \varphi = 0$, $v$ is complete.
*Hint:* For $p \in X$ let $a = \varphi(p)$. By assumption, $\varphi^{-1}(a)$ is compact. Let $\rho \in \mathcal{C}_0^\infty(X)$ be a "bump" function which is one on $\varphi^{-1}(a)$ and let $w$ be the vector field, $\rho v$. By Theorem 3.27, $w$ is complete and since

$$L_w\varphi = \iota(\rho v)\, d\varphi = \rho\iota(v)\, d\varphi = 0$$

$\varphi$ is constant along integral curves of $w$. Let $\gamma(t)$, $-\infty < t < \infty$, be the integral curve of $w$ with initial point, $\gamma(0) = p$. Show that $\gamma$ is an integral curve of $v$.

(12) Show that in Theorem 3.23 the hypothesis that $M$ is compact can be replaced with $v$ compactly supported.

(13) Let $C_i$ $i = 1, 2, \ldots$ be an exhaustion of $M$ by compact sets and $\rho_i \in \mathcal{C}_o^\infty(M)$ a bump function which is 1 on $C_i$. By 12, Theorem 3.23 is true for each of the vector fields $v_i := \rho_i v$. Conclude that Theorem 3.23 is true for $v$ itself. Here are a few hints:
   (a) Let $\gamma : [0, T] \to M$ be an integral curve of $v$. Show that for some $i$ $\gamma([0, T])$ is contained in Int$C_i$.
   (b) Conclude that $\gamma$ is an integral curve of $v_i$.
   (c) Let $(f_i)_t$, $-\infty < t < \infty$ be the one-parameter group of diffeomorphisms associated with $v_i$. Show that there exists a neighborhood, $U$, of $\gamma(0)$ with the property

   $$(f_i)_t(U) \subseteq \mathrm{Int}C_i$$

   for $0 \le t \le T$.
   (d) Conclude that, on the set $U \times [0, T)$, $f_i = f$.
   (e) Conclude from the foregoing that $f : U \times [0, T] \to M$ is a $\mathcal{C}^\infty$ map.
   (f) Finally, noting that $p = \gamma(0)$ can be an arbitrary point on $M$ and $T$ an arbitrary positive real number, conclude that

   $$f : M \times (-\infty, \infty) \to M$$

   is $\mathcal{C}^\infty$.

## 4. Differential Forms on Manifolds

**Definition 4.1.** Let $X$ be a manifold. A *differential $k$-form* is a function $\omega$ that assigns to each $p \in X$ an element $\omega(p)$ of $\Lambda^k(T_p^* X)$.

4.1. **Push-Forwards and Pull-Backs.** Recall that in the previous section we defined the terms *push-forward* and *pull-back* for vector fields via a diffeomorphism $f$. For differential forms the situation is even nicer. Just as in §2.5 we can define the pull-back operation on forms for *any* $\mathcal{C}^\infty$ map $f : X \to Y$. Specifically: Let $\omega$ be a $k$-form on $Y$. For every $p \in X$, and $q = f(p)$ the linear map

$$df_p : T_p X \to T_q Y$$

induces by (1.8.2) a pull-back map

$$(df_p)^* : \Lambda^k(T_q^*) \to \Lambda^k(T_p^*)$$

and, as in §2.5, we'll define the pull-back, $f^*\omega$, of $\omega$ to $X$ by defining it at $p$ by the identity

$$(f^*\omega)(p) = (df_p)^*\omega(q)\,. \tag{4.2}$$

Notice that, in contrast to vector fields where we defined push-forwards first, pull-backs of $k$-forms are defined first for any $\mathcal{C}^\infty$ map, independent of push-forwards. However, in both scenarios, we can get push-forwards as pull-backs of the inverse map. To summarize:

**Definition 4.3** (Pull-backs and Push-forwards)**.** Suppose $X$ and $Y$ are manifolds, $\omega$ a $k$-form in $Y$, with $f : X \to Y$ a $\mathcal{C}^\infty$ map where $f(p) = q$. The *pull-back* of $\omega$ is defined point-wise as

$$(f^*\omega)(p) = (df_p)^*\omega(q).$$

Now suppose that $\eta$ is a $k$-form on $X$ and $g : X \to Y$ is a diffeomorphism, $g(p) = q$ then the *push-forward* of $\eta$ is

$$(g_*\eta)(q) = (g^{-1*}\eta)(q) = (dg_q^{-1})^*\eta(p).$$

*Remark* 4.4 (Functoriality). Let $X$, $Y$ and $Z$ be manifolds and $f : X \to Y$ and $g : Y \to Z$ $\mathcal{C}^\infty$ maps. Then if $\omega$ is a $k$-form on $Z$

$$f^*(g^*\omega) = (g \circ f)^*\omega\,.$$

Now assume that $\omega$ is a $k$-form on $X$ and $f : X \to Y$, $g : Y \to Z$ are diffeomorphisms. Observe that

$$\begin{aligned}(g \circ f)_*\omega &= [(g \circ f)^{-1}]^*\omega \\ &= (f^{-1} \circ g^{-1})^*\omega \\ &= g^{-1*}(f^{-1*}\omega) \\ &= g_*(f_*\omega).\end{aligned}$$

4.2. **Smooth $k$-forms.** As with vector fields, *smooth $k$-forms* are $k$-forms whose local coordinate representations are smooth.

**Definition 4.5.** Assume $U \subseteq X$ is the domain of some chart $(U, V, \varphi)$ and $\omega$ is a $k$-form on $U$. $\omega$ is $\mathcal{C}^\infty$ if $\varphi_*\omega$ is $\mathcal{C}^\infty$ on $V$.

Extending this definition globally provides us with the definition:

**Definition 4.6.** A $k$-form $\omega$ on $X$ is $\mathcal{C}^\infty$ if, for every point $p \in X$, $\omega$ is $\mathcal{C}^\infty$ on some neighborhood of $p$.

**Exercise 4.7.** As with vector fields, show that these definitions are independent of choice of chart.

**Proposition 4.8.** *Assume $X_i$ $i = 1, 2$ are manifolds and $f : X_1 \to X_2$ is a $\mathcal{C}^\infty$ map. If $\omega$ is a $\mathcal{C}^\infty$ $k$-form on $Y$, then $f^*\omega$ is a $\mathcal{C}^\infty$ $k$-form on $X$.*

*Proof.* For $p_i \in X_i$ $i = 1, 2$ and $p_2 = f(p_1)$ let $U_i$ be neighborhoods of $p_i$, shrinking $U_1$ if necessary so that $f(U_1) \subseteq U_2$. Assuming $(U_i, V_i, \varphi_i)$ $i = 1, 2$ are charts in their respective atlases, then the map $g := \varphi_2 \circ f \circ \varphi_1^{-1} : V_1 \to V_2$ is $\mathcal{C}^\infty$. Since $\omega$ is a $\mathcal{C}^\infty$ $k$-form on $Y$, $\varphi_{2*}\omega$ is $\mathcal{C}^\infty$, so

$$\varphi_{1*}(f^*\omega) = (f \circ \varphi^{-1})^*\omega = (\varphi_2^{-1} \circ g)^* = g^*(\varphi_{2*}\omega)$$

is $\mathcal{C}^\infty$. □

### 4.3. The Exterior Differentiation Operation.

**Definition 4.9.** Let $\omega$ be a $\mathcal{C}^\infty$ $k$-form on a manifold $X$. For $p \in X$ take a chart $(U, V, \varphi)$ and define the *exterior derivative* of $\omega$, $d\omega$, on $X$ by the formula

$$d\omega := (\varphi^{-1})_* \, d\varphi_*\omega \,. \tag{4.10}$$

We now wish to show that this definition is independent of chart. Assume $p \in X$ is contained in two charts $(U_i, V_i, \varphi_i)$ $i = 1, 2$ and define $\psi := \varphi_2 \circ \varphi_1^{-1}$. Observe that

$$\begin{aligned} d\varphi_{2*}\omega &= d(\psi \circ \varphi_1)_*\omega \\ &= d\psi_*\varphi_{1*}\omega \\ &= \psi_* d\varphi_{1*}\omega \\ &= \varphi_{2*}\varphi_{1*}^{-1} d\varphi_{1*}\omega \end{aligned}$$

hence

$$\varphi_{2*}^{-1} \, d\varphi_{2*}\omega = \varphi_{1*}^{-1} \, d\varphi_{1*}\omega \,.$$

Note that we have used the fact that $\psi_* d = d\psi_*$, i.e. the push-forward commutes with the exterior derivative $d$. This follows from the fact that the pull-back commutes with the exterior derivative and defining the push-forward as the pull-back via the inverse map. Recall that for $\omega = f dx_I$ and a $\mathcal{C}^\infty$ map $G$, $G^*d\omega = G^*(df \wedge dx_I) = d(f \circ G) \wedge d(x_I \circ G) = dG^*(f dx_I)$.

We can therefore, define the exterior derivative, $d\omega$, globally by defining it to be equal to (4.10) on every chart.

It's easy to see from the Definition 4.9 that this exterior differentiation operation inherits from the exterior differentiation operation on open subsets of $\mathbb{R}^n$ the properties from Section 2.3 of Chapter 2:

I. For $\omega_1$, $\omega_2$ in $\Omega^k(U)$ $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.

II. For $\omega_1 \in \Omega^k(U)$ and $\omega_2 \in \Omega^\ell(U)$ $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$.
III. For $\omega \in \Omega^k(U)$ $d(d\omega) = 0$.

Note that for zero forms, i.e., $\mathcal{C}^\infty$ functions, $f : X \to \mathbb{R}$, $df$ is the "intrinsic" $df$ defined in Section 2.1, i.e., for $p \in X$ $df_p$ is the derivative of $f$

$$df_p : T_pX \to \mathbb{R}$$

viewed as an element of $\Lambda^1(T_p^*X)$.

4.4. **The Interior Product and Lie Derivative Operation.** Given a $k$-form, $\omega \in \Omega^k(X)$ and a $\mathcal{C}^\infty$vector field, $w$, we will define the interior product

$$\iota(v)\omega \in \Omega^{k-1}(X)\,, \tag{4.11}$$

as in §2.4, by setting

$$(\iota(v)\omega)_p = \iota(v_p)\omega_p$$

and the Lie derivative

$$L_v\omega = \Omega^k(X) \tag{4.12}$$

by setting

$$L_v\omega = \iota(v)\,d\omega + d\iota(v)\omega\,. \tag{4.13}$$

It's easily checked that these operations satisfy the identities (2.4.2)–(2.4.8) and (2.4.12)–(2.4.13) (since, just as in §2.4, these identities are deduced from the definitions (4.11) and (3.3) by purely formal manipulations). Moreover, if $v$ is complete and

$$f_t : X \to X\,, \quad -\infty < t < \infty$$

is the one-parameter group of diffeomorphisms of $X$ generated by $v$ the Lie derivative operation can be defined by the alternative recipe

$$L_v\omega = \left(\frac{d}{dt} f_t^*\omega\right)(t = 0) \tag{4.14}$$

as in (2.5.22). (Just as in §2.5 one proves this by showing that the operation (4.14) has the properties (2.12) and (2.13) and hence that it agrees with the operation (4.13) provided the two operations agree on zero-forms.)

4.5. **Exercises in Symplectic Geometry: The Cotangent Bundle.** The following is an extremely important example of a manifold. It arises naturally in classical mechanics, by considering an $N$-particle system constrained to a certain *configuration space* $X$ ($X$ is typically a submanifold of $\mathbb{R}^{3N}$). A concrete example is the pendulum, whose configuration space is simply $X = \mathbb{S}^1$. The *phase space* – or position-momentum space – is then described by the cotangent bundle, which we will define below.

**Definition 4.15.** Let $M = M^{2n}$ be an even dimensional manifold and $\omega \in \Omega^2(M)$. We say $\omega$ is a *symplectic form* if $d\omega = 0$ ($\omega$ is closed) and $(\omega \wedge \cdots \wedge \omega)_p = \omega_p^{\wedge n} \neq 0$ for all $p \in M$ ($\omega$ is non-degenerate).

**Definition 4.16.** If $M = M^{2n}$ and $\omega$ is a symplectic form, then we call $(M, \omega)$ a *symplectic manifold*.

(1) **Cotangent Bundle as a Manifold.** Let $X$ be an $n$-dimensional manifold,
$$T^*X = \{(p, \zeta), p \in X, \zeta \in T_p^*X\}$$
the cotangent bundle, and $\pi : T^*X \to X$ the projection map $(p, \zeta) \mapsto p$. Show that if $(U, V, \varphi)$ is a chart on $X$ and $T^*U = \pi^{-1}(U)$ is the subset of $T^*X$ "sitting over $U$" then one gets a bijective map
$$\tilde{\varphi} : \tilde{U} := T^*U \to V \times \mathbb{R}^{n*} =: \tilde{V}$$
mapping $(p, \zeta)$ to $(q, \eta)$ where $\varphi(p) = q$ $\eta \in T_q^*\mathbb{R}^n \cong \mathbb{R}^{n*}$ $\zeta = (d\varphi_p)^*\eta$. Show that the collection $(\tilde{U}, \tilde{V}, \tilde{\varphi})$ is an atlas in the sense of Exercise 4. Conclude that the cotangent bundle of a manifold, $T^*X$, is a manifold.

(2) **Canonical One Form.** Let $M = T^*X$ be the cotangent bundle of a manifold. For each point $q = (p, \zeta) \in M$, let $\alpha_q$ be the element of $T_q^*M$ defined by the identity $\alpha_q = (d\pi)_q^*\zeta$. Note that $d\pi_q$ is a map from $T_qM \to T_pX$ so its transpose goes in the opposite direction. Show that the assignment $q \in M \mapsto \alpha_q \in T_q^*M$ defines a one-form. (This form is called the *canonical* one-form on $M$.)

(3) **Functoriality.** Show that if $X_i$ are manifolds, $f : X_1 \to X_2$ a diffeomorphism, and $M_i = T^*X_i$ $i = 1, 2$ are the respective cotangent bundles, then $f$ gives rise to a diffeomorphism $g : M_1 \to M_2$ where
$$g(p_1, \zeta_1) = (p_2, \zeta_2) \Leftrightarrow \begin{cases} p_2 &= f(p_1) \\ \zeta_1 &= (df_{p_1})^*\zeta_2 \end{cases}$$

Show that if $\alpha_1$ and $\alpha_2$ are the canonical one-forms on $M_1$ and $M_2$ then $g^*\alpha_2 = \alpha_1$. Hint: Look at

$$\begin{array}{ccc} T_{q_1}M & \xrightarrow{dg} & T_{q_2}M \\ {\scriptstyle d\pi_{q_1}}\downarrow & & \downarrow{\scriptstyle d\pi_{q_2}} \\ T_{p_1}X_1 & \xrightarrow{df} & T_{p_2}X_2 \end{array}$$

which after applying transposes reverses the vertical and horizontal arrows.

(4) Show that if $X = \mathbb{R}^n$ and $M = T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^{n*}$, with linear coordinates, $x_1, \ldots, x_n$, $\xi_1, \ldots, \xi_n$, then the canonical one-form $\alpha$ is equal to $\sum_i \xi_i dx_i$. Conclude that $\alpha$ is smooth.

(5) Let $M = T^*X$ and let $\omega = -d\alpha$ be the exterior derivative of the canonical one-form. Show that $\omega$ is symplectic. Hint: By functoriality it suffices to check this for $X = \mathbb{R}^n$.

(6) **Hamiltonian Vector Fields.** Let $H$ be a $\mathcal{C}^\infty$ function on $M$. Show that there is a unique vector field $v_H$ on $M$ with the defining property

$$\iota(v_H)\omega = dH.$$

Hint: By linear algebra, check that $\omega_p^n \neq 0$ if and only if the map $T_pM \to T_p^*M$ $v \mapsto \iota(v)_{\omega_p}$ is bijective. Alternative Hint: By functoriality, check the special case $X = \mathbb{R}^n$.

(7) **Conservation of Energy.** Suppose $v_H$ is complete. Let $f_t : M \to M$ be the one-parameter group of diffeomorphisms. From $\iota(v_H)\omega = dH$ conclude that $f_t^*H = H$, i.e. the Hamiltonian – "energy" – is conserved. Hint: Show $L_{v_H}H = 0$, $f_t^*\omega = \omega$.

(8) **Conservation of Volume.** Following the previous exercise, from $\iota(v_H)\omega = 0$ conclude that $f_t^*\omega^n = \omega^n$.

## 5. Some Topological Results

### 5.1. The Support of a Function.

**Definition 5.1.** Assume that $U$ is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a $\mathcal{C}^\infty$ map. Define the *support* of a function $f$ to be the closed set $\mathrm{supp}(f) := \overline{\{p \in U, f(p) \neq 0\}}$.

**Definition 5.2.** A function is said to be *compactly supported* if its support is compact. For $U \subseteq \mathbb{R}^n$, the space of compactly supported $\mathcal{C}^\infty$ functions defined on $U$ is denoted $\mathcal{C}_o^\infty(U)$.

*Remark* 5.3. Assume $X$ is a manifold and $U \subseteq X$. If $f \in \mathcal{C}_o^\infty(U)$ then we can regard $f$ as being in $\mathcal{C}_o^\infty(X)$, i.e. we can extend $f$ to a function on all of $X$ by setting $f \equiv 0$ on $X \smallsetminus U$.

5.2. **Partitions of Unity.** We are now poised to introduce the use of partitions of unity. Requisite to their definition is the existence of so-called *bump functions.*

**Lemma 5.4.** *Let $X$ be an $n$-dimensional manifold. For every point $p \in X$ and open set $U \subseteq X$ containing $p$ there exists a function $f \in \mathcal{C}_o^\infty(U)$ such that $f \equiv 1$ on a neighborhood of $p$ and $0 \leq f \leq 1$.*

*Proof.* Shrinking $U$ if necessary, we can assume $U$ is the domain of a chart $\varphi : U \to V$. Let $q = \varphi(p)$. The result is true for $V, q$ and hence for $U, p$.

□

**Theorem 5.5.** *If $X$ is a manifold with an open cover $\{U_\alpha, \alpha \in I\}$, then there exist functions $\rho_i \in \mathcal{C}_o^\infty(X), i \in \mathbb{N}$ such that*

(1) $0 \leq \rho_i \leq 1$
(2) *for every compact set $C$ there exists an $i_0$ such that* $\operatorname{supp}(\rho_i) \cap C = \varnothing$ *for $i > i_0$*
(3) $\sum_i \rho_i = 1$
(4) *For every $i$ there exists an $\alpha_i \in I$ such that* $\operatorname{supp}(\rho_i) \subseteq U_{\alpha_i}$.

*Proof.* Let $C_1 \subseteq C_2 \subseteq \cdots$ be a compact exhaustion of $X$. Pick a point $p \in C_i \smallsetminus \operatorname{Int}(C_{i-1})$. $p \in U_\alpha$ for some $\alpha \in I$ and there exists a bump function $f_p \in \mathcal{C}_o^\infty((\operatorname{Int}(C_{i+1}) - C_{i-2}) \cap U_\alpha)$ with $f_p \equiv 1$ on a neighborhood $U_p$ of $p$. The sets $U_p, p \in C_i \smallsetminus \operatorname{Int}(C_{i-1})$ cover $C_i \smallsetminus \operatorname{Int}(C_{i-1})$. Since $C_i \smallsetminus \operatorname{Int}(C_{i-1})$ is compact there exists a finite subcover $U_{p_k}, 1 \leq k \leq N_i$. Let $f_{k,i} = f_{p_k}$, i.e. $f_k = f_{k,1} \quad 1 \leq k \leq N_1$, $f_{k+N_1} = f_{k,2} \quad 1 \leq k \leq N_2$, etc. The functions $f_i$ have all the properties desired above except the third property, and to arrange that they have this property simply renormalize, setting $\rho_i = \frac{f_i}{\sum f_i}$. □

5.3. **Proper Mappings.**

**Lemma 5.6.** *If $M_i$ $i = 1, 2$ are manifolds and $f : M_1 \to M_2$ is a proper $\mathcal{C}^\infty$ map, then*

(a) *$f(M_1)$ is a closed subset of $M_2$*
(b) *Given $p_1 \in M_1$ and $U_1$ a neighborhood of $p_1$ in $M_1$ there exists a neighborhood $U_2$ of the image point $p_2 = f(p_1)$ in $M_2$ such that $f^{-1}(U_2) \subset U_1$*

**Theorem 5.7.** *If $f : M_1 \to M_2$ is a $1-1$ proper immersion and $X = f(M_1)$, then $X$ is a closed submanifold of $M_2$ and $f$ is a diffeomorphism of $M_1$ onto $X$.*

*Proof.* Let $p_1 \in M_1$ and $p_2 = f(p_1)$. By the canonical immersion theorem, there exists a chart $(U_i, V_i, \varphi_i)$ centered at $p_i$ such that $U_1 \subset f^{-1}(U_2)$ and

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\cong"'] & U_2 \arrow[d, "\cong"] \\ V_1 \arrow[r, "\iota"] & V_2 \end{tikzcd}$$

where $\iota : \mathbb{R}^k \to \mathbb{R}^n$ is the canonical immersion

$$\iota(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0)$$

($k$ and $n$ are the dimensions of $M_1$ and $M_2$). By the above lemma we can shrink $U_2$ such that $f^{-1}(U_2) = U_1$ and hence $\iota(V_1) = \varphi_2(U_2 \cap X)$. Thus $\iota^{-1} \circ \varphi_2 : U_2 \cap X \to V_1$ is a diffeomorphism of the open neighborhood $U = U_2 \cap X$ of $p_2$ in $V$ onto the open subset $V_1$ of $\mathbb{R}^k$. $\square$

5.4. **The Whitney imbedding theorem.** The following exercises outline a proof of the Whitney Embedding Theorem for compact manifolds.

(1) **Partitions of Unity on Compact sets.** Let $X$ be a compact set and $U_\alpha$ $\alpha = 1, \ldots m$ an open cover. If $\rho_i$ $i = 1, \ldots k$ is a partition of unity subordinate to the open cover, show that this partition of unity can be chosen so that $m = k$ and the support of $\rho_i$ is contained in $U_i$.

(2) **Whitney Embedding for Compact Manifolds.** Let $X$ be a compact $n$-manifold. Suppose $(U_i, V_i, \varphi_i)$ are charts on $X$ whose domains are a cover of $X$. Show that there is a $1-1$ immersion $f : X \to \mathbb{R}^{nN+N}$. Hint: Let $\rho_i$ $i = 1, \ldots, N$ be a partition of unity as in Exercise 1. Show that the map $f : X \to \mathbb{R}^{nN+N}$ defined by

$$f(x) = (\rho_1\varphi_1(x), \ldots, \rho_N\varphi_N(x), \rho_1, \ldots, \rho_N)$$

is such a $1-1$ immersion. Sub-Hint: Show that if $p \in U_i$ and $df_p(v) = 0$ then $(d\rho_i)_p(v) = 0$ and $\rho_i(p)(d\varphi_i)_p(v) = 0$.

(3) **Whitney Embedding for $\mathbb{RP}^2$.** Consider the map $\gamma : \mathbb{S}^2 \to \mathbb{R}^6$ defined by

$$(x_1, x_2, x_3) \to (x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2, x_3).$$

(a) Show that this map is an immersion.
(b) Show that it is $2-1$ i.e. that

$$\gamma(x_1, x_2, x_3) = \gamma(-x_1, -x_2, -x_3).$$

(c) Conclude that there is a $1-1$ immersion $\tilde{\gamma} : \mathbb{RP}^2 \to \mathbb{R}^6$ with the property

$$\tilde{\gamma}([x_1, x_2, x_3]) = \gamma(x_1, x_2, x_3)$$

for $(x_1, x_2, x_3) \in \mathbb{S}^2$.